Autovalores e Autovetores de $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

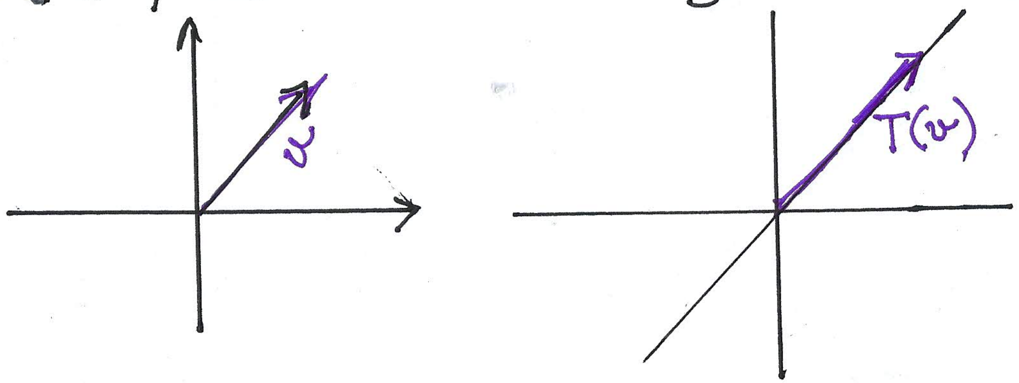
DEF: Um número real λ é um AUTOVALOR de $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ se existe um vetor $v \in \mathbb{R}^n$, $v \neq 0$ tal que $T(v) = \lambda v$.

Nesse caso, v é chamado de AUTOVETOR associado ao autovalor λ.

$$\mathbb{R}^n(\lambda) = \{ v \in \mathbb{R}^n \mid T(v) = \lambda v \}$$

$\mathbb{R}^n(\lambda)$ é um SUBESPAÇO DE $\mathbb{R}^n$.

$\mathbb{R}^n(\lambda) \overset{def}{=}$ AUTO ESPAÇO associado ao autovalor λ.



Seja $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$A = [T]_{can}$. e então $T = T_A$

Se $v = (x_1, \dots, x_n)$, então $T(v) = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Seja λ um autovalor de T e $v \neq 0$ um autovetor associado a λ.

$$T(v) = \lambda v \iff A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \iff$$

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \lambda I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff (A - \lambda I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbb{R}^n(\lambda) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; (A - \lambda I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

AUTO ESPAÇO ASSOCIADO a λ

Mas se $M = A - \lambda I_n$

$\mathbb{R}^n(\lambda) = \mathrm{Ker}\, T_M$

Logo, λ é um autovalor de $T \iff \mathrm{Ker}\, T_M \neq 0$.

$\mathrm{Ker}\, T_M \neq 0 \iff T_M$ não é injetora $\iff T_M$ não é bijetora $\iff T_M$ não é inversível $\iff$ M não é inversível.

Mas M não é inversível $\iff \det M \neq 0$.

Definimos então $p_A(x) = (-1)^n \det(A - x I_n)$ polinômio característico de A.

λ é um autovalor de A (ou T_A) $\iff p_A(\lambda) = 0$.

Exemplos

(1) $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$$p_A(x) = \det \begin{bmatrix} 3-x & 1 \\ 1 & 2-x \end{bmatrix} = (3-x)(2-x) - 1$$
$$= 6 - 5x + x^2 - 1$$
$$= x^2 - 5x + 5$$

$$x^2 - 5x + 5 = 0 \iff x = \frac{5 \pm \sqrt{25-20}}{2} \iff$$

$$x = \frac{5+\sqrt{5}}{2} \quad \text{ou} \quad x = \frac{5-\sqrt{5}}{2}$$

Os autovalores de A são $\lambda_1 = \frac{5+\sqrt{5}}{2}$ e $\lambda_2 = \frac{5-\sqrt{5}}{2}$.

Projeção Ortogonal em $u = (3,4)$

$$A = \left[\operatorname{proj}_u v\right] = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$p_A(x) = \det(A - xI) = \det \begin{bmatrix} \frac{9}{25} - x & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} - x \end{bmatrix}$$

$$= \left(\frac{9}{25} - x\right)\left(\frac{16}{25} - x\right) - \frac{144}{625}$$

$$= x^2 - x + \frac{9 \times 16}{625} - \frac{144}{625} = x^2 - x$$

Os autovalores de $\operatorname{proj}_u v$ são $\lambda_1 = 0$ e $\lambda_1 = 1$.

Queremos achar

$$\mathbb{R}^2(\lambda_1) \quad \text{e} \quad \mathbb{R}^2(\lambda_2)$$

$\lambda_1 = 0$

$$\mathbb{R}^2(\overset{=0}{\lambda_1}) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 9/25 - 0 & 12/25 \\ 12/25 & 16/25 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} \xrightarrow[25/4 L_2]{25/3 L1} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \xrightarrow{L2 - L1} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

$3x + 4y = 0$

$x = -\frac{4}{3} y$

$$\mathbb{R}^2(0) = \{ (-\tfrac{4}{3} y, y) \mid y \in \mathbb{R} \} = [(-4, 3)]$$

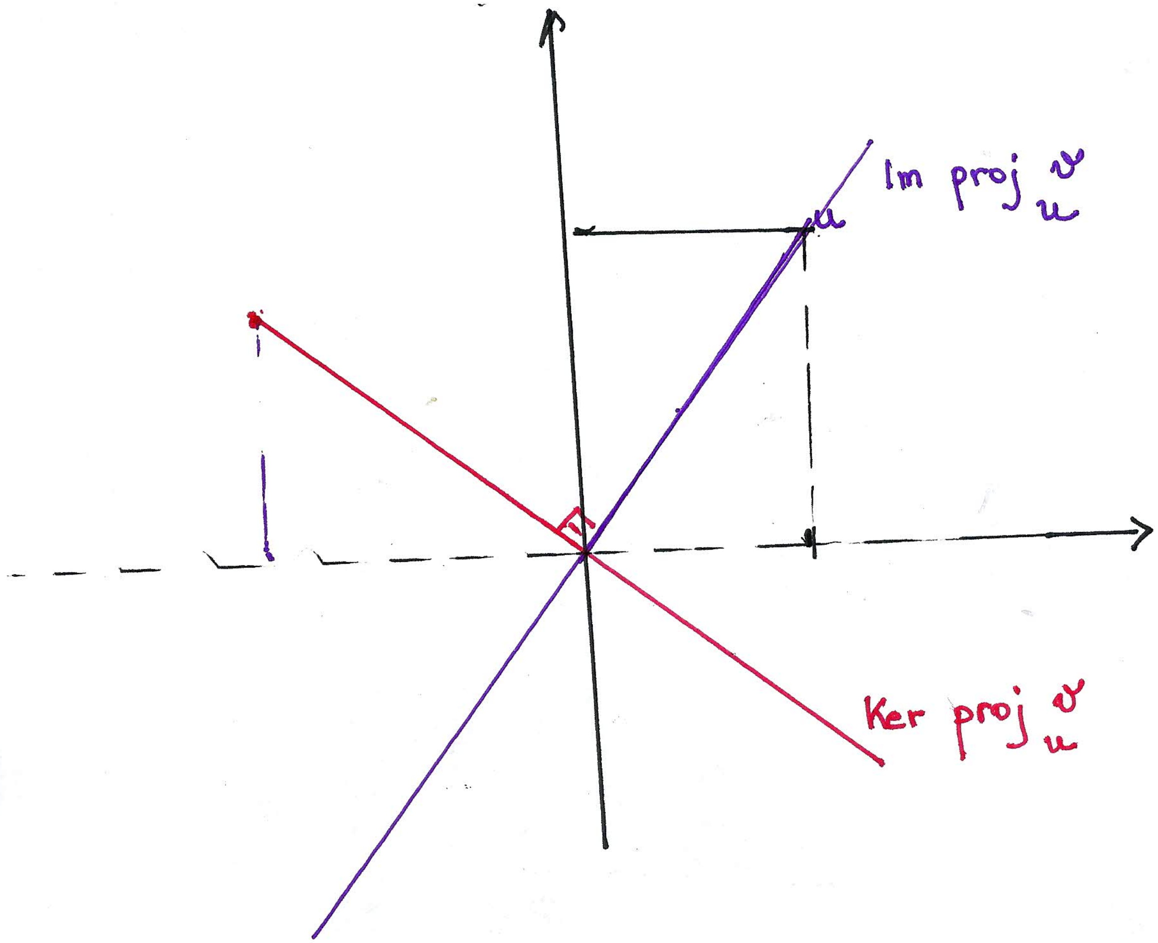
$$\mathbb{R}^2(1) = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 9/25 - 1 & 12/25 \\ 12/25 & \frac{16}{25} - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -16/25 & 12/25 \\ 12/25 & -9/25 \end{bmatrix} \xrightarrow[25/3 L_2]{25/4 L_1} \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \xrightarrow[L_2 + L_1]{L_2 \leftrightarrow} \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}$$

$-4x + 3y = 0$

$\Rightarrow x = 3/4 y$

$$\mathbb{R}^2(1) = \{ (3/4 y, y) \mid y \in \mathbb{R} \} = [(3, 4)]$$



u
Im proj_u^v
Ker proj_u^v

(2) Reflexão em torno da reta $y = 2x$.

$$[R]_{can,can} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = A$$

$$p_A(x) = \det(A - x I_2) = \det \begin{bmatrix} -3/5 - x & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} - x \end{bmatrix}$$

$$= \left(-\frac{3}{5} - x\right)\left(\frac{3}{5} - x\right) - \frac{16}{25}$$

$$= x^2 - \frac{9}{25} - \frac{16}{25} = x^2 - 1$$

$$x^2 - 1 = 0 \iff x = \pm 1.$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

Achar $\mathbb{R}^2(1)$ e $\mathbb{R}^2(-1)$

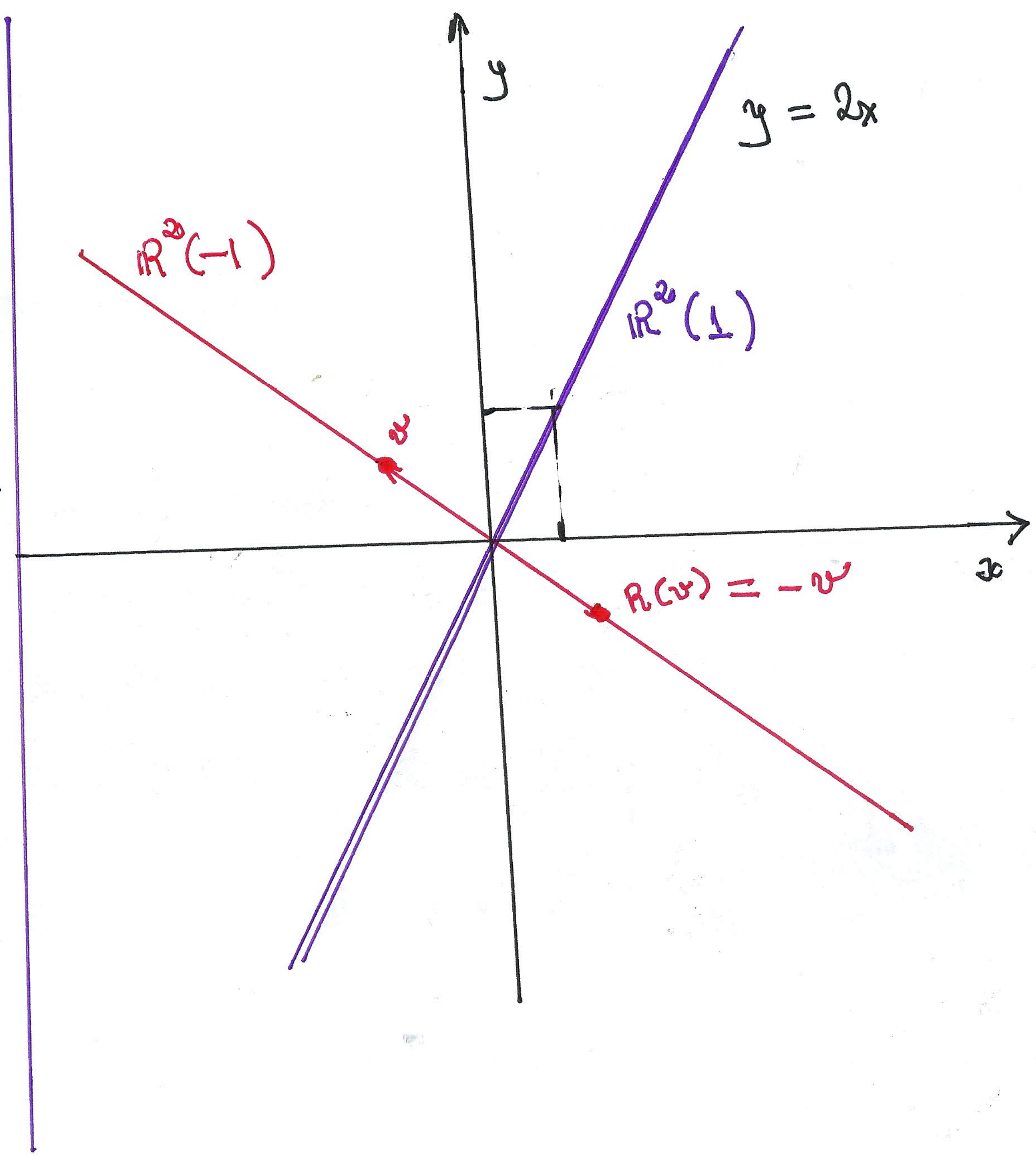
$\mathbb{R}^2(-1)$

$= \{(x,y) \in \mathbb{R}^2 \mid \begin{bmatrix} -\frac{3}{5}+1 & 4/5 \\ 4/5 & \frac{3}{5}+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$

$= \{ v \in \mathbb{R}^2 \mid Av = -v \}$

Fazendo as contas,

$\mathbb{R}^2(-1) = [(2,-1)]$ //



$\mathbb{R}^2(1) = \{ v \in \mathbb{R}^2 \mid Av = v \}$

$\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\iff \begin{bmatrix} -\frac{3}{5}-1 & 4/5 \\ 4/5 & 3/5-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\iff \begin{bmatrix} -\frac{8}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\iff \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

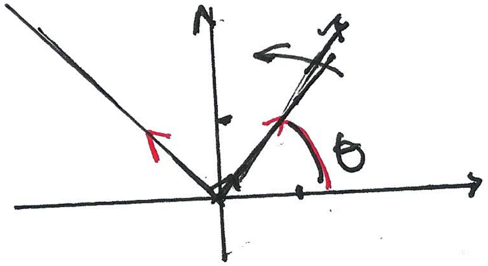
$\iff +2x = y$

$\iff (x,y) = (x,2x)$

$= x(1,2)$

$\mathbb{R}^2(-1) = [(1,2)]$

Rotação de ângulo θ



$$R_\theta = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

$$p_{R_\theta}(x) = \det \begin{bmatrix} \cos\theta - x & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta - x \end{bmatrix} = (\cos\theta - x)^2 + \operatorname{sen}^2\theta$$

$$= \cos^2\theta - 2\cos\theta\, x + x^2 + \operatorname{sen}^2\theta$$

$$p_{R_\theta}(x) = x^2 - 2\cos\theta\, x + 1$$

$$\Delta = 4\cos^2\theta - 4 = 4(\cos^2\theta - 1) \qquad 0 \leq \cos^2\theta \leq 1$$

$$\Delta \leq 0 \qquad \Delta = 0 \iff \cos^2\theta = 1 \iff \theta = k\pi,\ k \in \mathbb{Z}$$

Se $\theta = 2k\pi$, $k \in \mathbb{Z}$, $R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Se $\theta = (2k+1)\pi$, $k \in \mathbb{Z}$, $R_\theta = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

R_θ não tem autovalores se $\theta \neq k\pi$, $k \in \mathbb{Z}$

Pense geometricamente, o que faz a rotação? Por que não tem autovetores?

Seja $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -5 \\ 0 & 1 & 4 \end{bmatrix}$

$$\det(A - x I_3) = \det \begin{bmatrix} -x & 0 & 2 \\ 1 & -x & -5 \\ 0 & 1 & 4-x \end{bmatrix}$$

$$= -x\left(-x(4-x)+5\right) + 2 \cdot 1$$

$$= -x^3 + 4x^2 - 5x + 2$$

$$p_A(x) = x^3 - 4x^2 + 5x - 2$$

$$\begin{array}{l|l} x^3 - 4x^2 + 5x - 2 & x - 1 \\ \hline -x^3 + x^2 & x^2 - 3x + 2 \\ \hline -3x^2 + 5x & \\ +3x^2 - 3x & \\ \hline 2x - 2 & \end{array}$$

$$p_A(1) = 1 - 4 + 5 - 2 = 0$$

$$p_A(x) = (x-1)^2 (x-2)$$

$\lambda_1 = 1 \qquad \lambda_2 = 2$

$$\mathbb{R}^3(1) = \{ v \in \mathbb{R}^3 \mid Av = v \}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -5 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -5 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$y + 3z = 0$

$x - 2z = 0 \qquad x = 2z \qquad y = -3z$

$\mathbb{R}^3(1) = [(2, -3, 1)]$

$\mathbb{R}^3(2) = \{ v \in \mathbb{R}^3 \mid Av = 2v \}$

$$\begin{bmatrix} -2 & 0 & 2 \\ 1 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - z = 0 \implies x = z$

$y + 2z = 0 \implies y = -2z$

$\mathbb{R}^2(2) = [(1, -2, 1)]$