Integer Programming
Basic Concepts

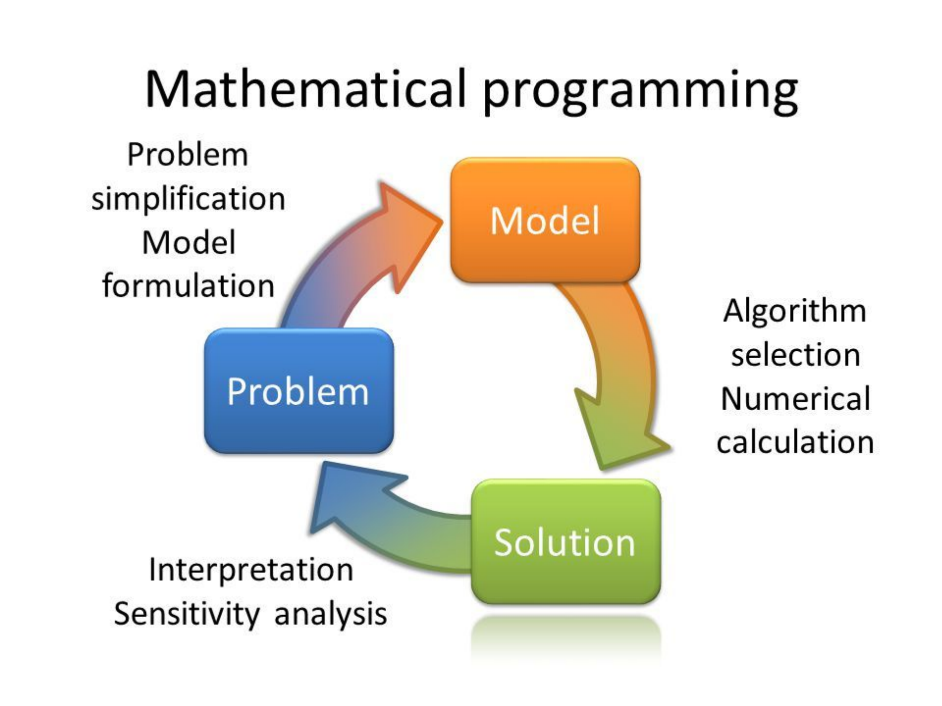
Outline

- What is it?
- Problem formulations.
- Binary variables.
- Modeling nonlinear functions

What is it? [1]

- Mathematical programming is the branch of mathematics concerned with the theory and methods for solving problems on finding the extrema of functions on sets defined by linear and non-linear constraints (equalities and inequalities) in a finite-dimensional vector space.

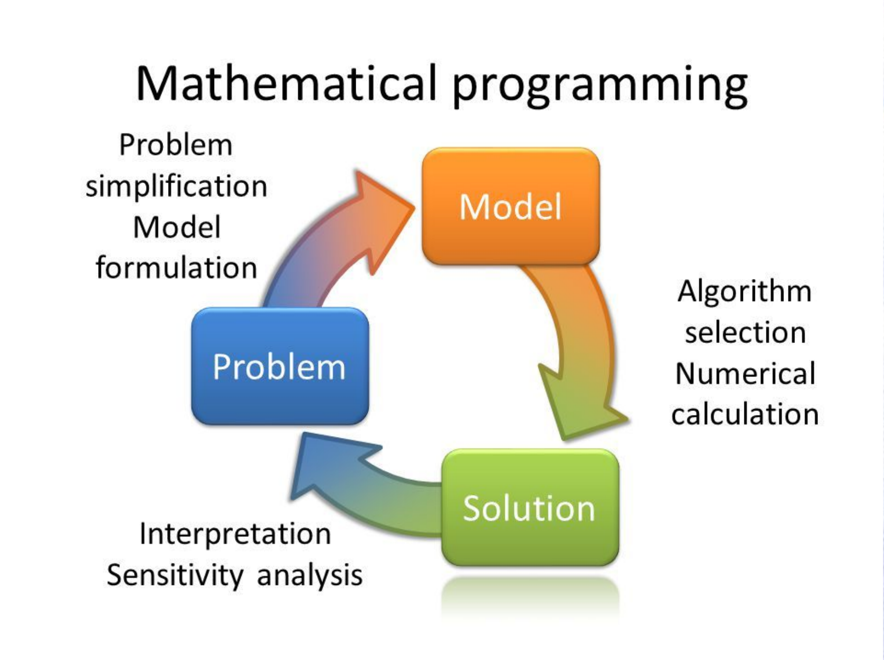
- Mathematical programming is a branch of operations research, which comprises a wide class of control problems



What is it? [1]

- The problems of mathematical programming find applications in various areas of human activity where it is necessary to choose one of the possible ways of action, e.g. in solving numerous problems of **control** and **planning** of production processes as well as in problems of design and long-term planning.

- The term "mathematical programming" is connected with the fact that the goal of solving various problems is choosing **programs of action**.



What is it?

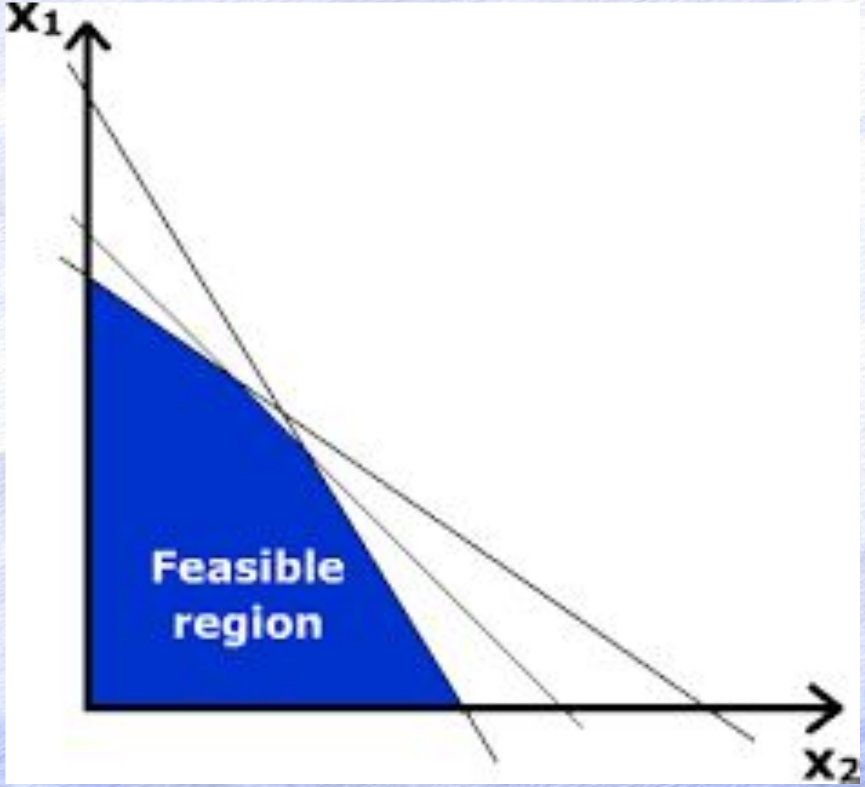
Linear Program (LP)

max cx

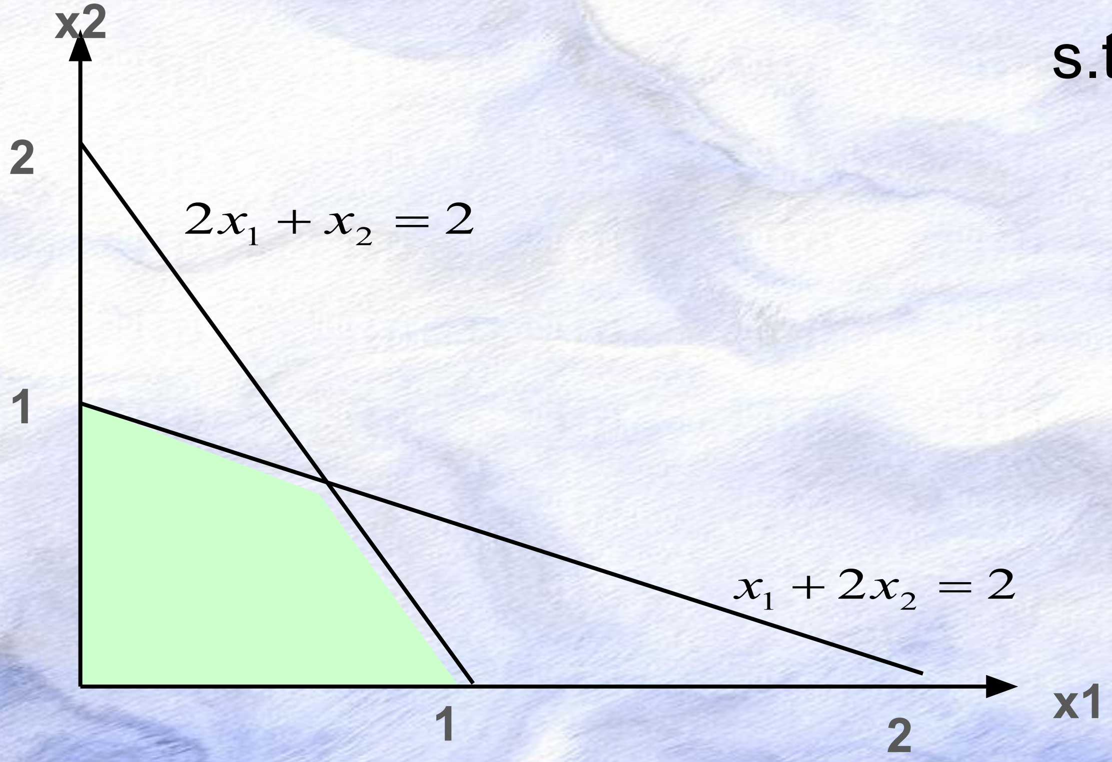
st.

$Ax \leq b$

$x \geq 0$



What is it?
Linear Program (LP)



Max x_1+x_2

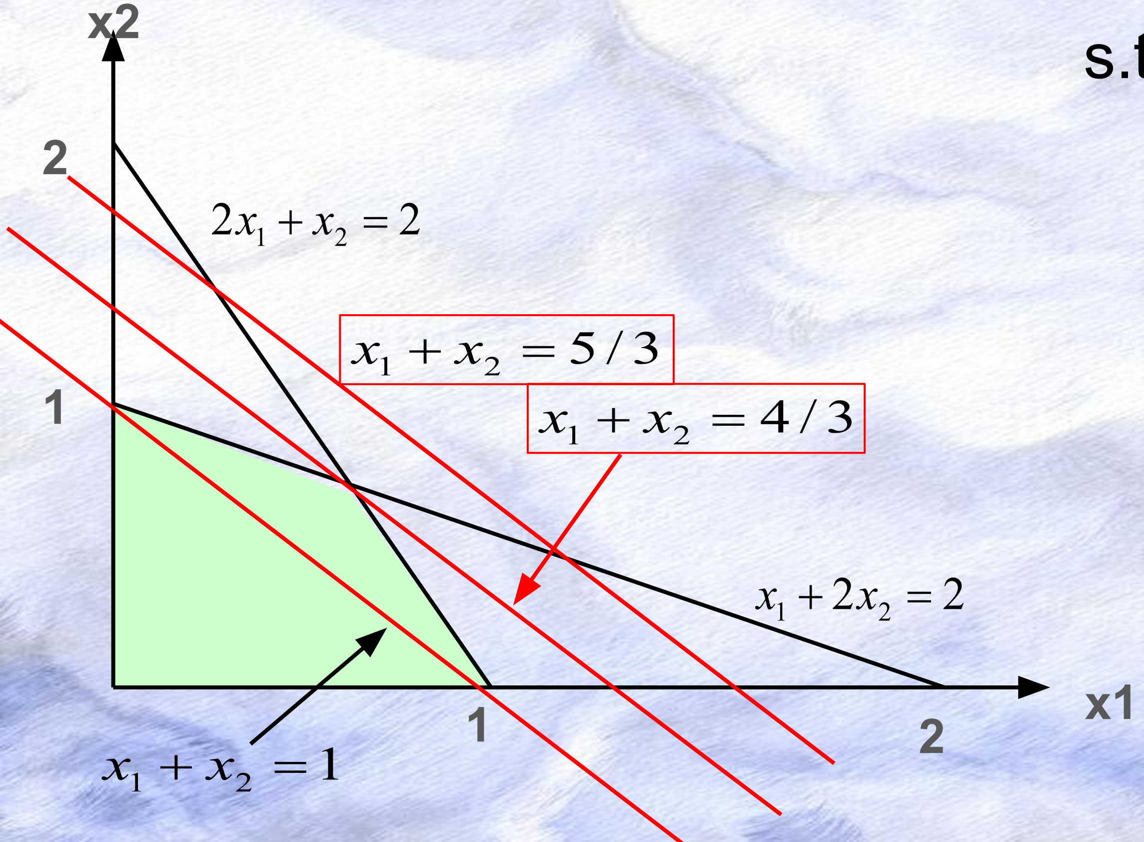
s.t.

$x_1+2x_2\leq 2$

$2x_1+x_2\leq 2$

$x_1\geq 0,\ x_2\geq 0$

What is it?
Linear Program (LP)



Max x_1+x_2

s.t.

$x_1+2x_2 \le 2$

$2x_1+x_2 \le 2$

$x_1 \ge 0,\ x_2 \ge 0$

Solution: (2/3, 2/3)

What is it?
Integer Program (IP)

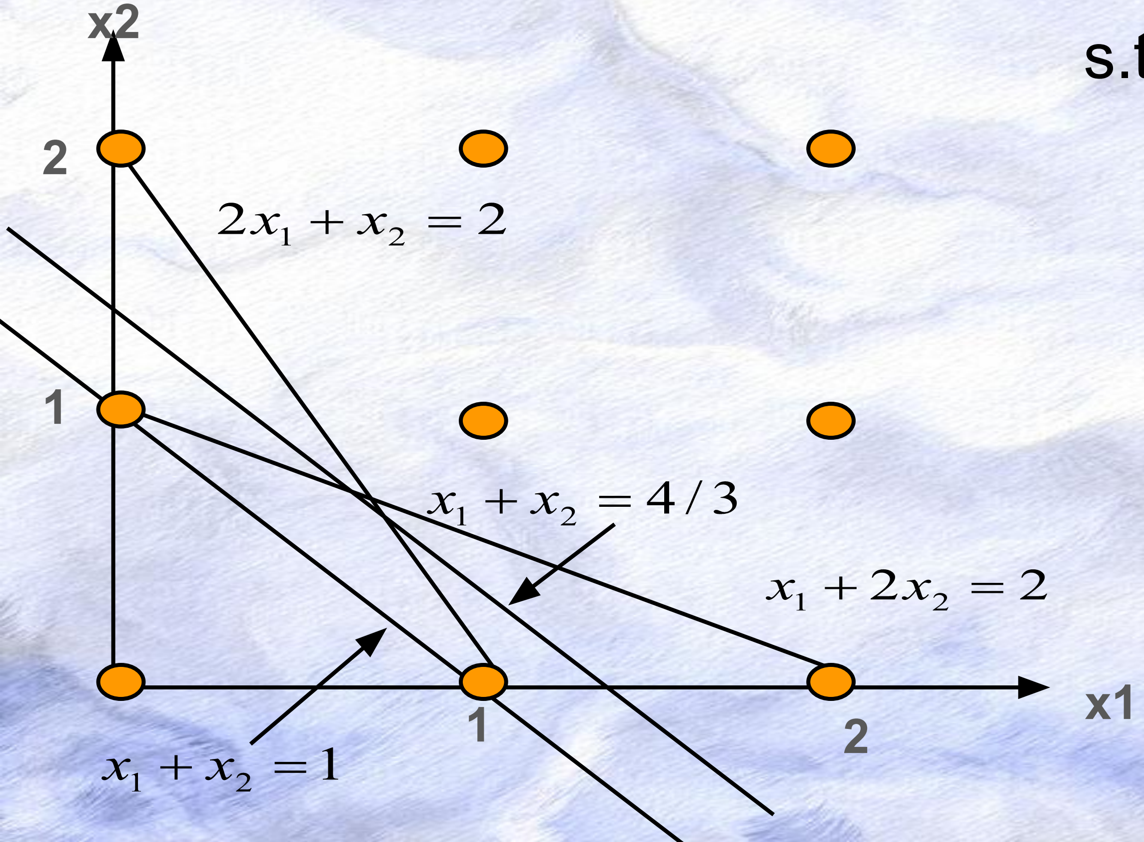
max cx

st.

$Ax \leq b$

$x \in Z_+$ (positive integer)

What is it?
Integer Program (IP)



Max x_1+x_2

s.t.

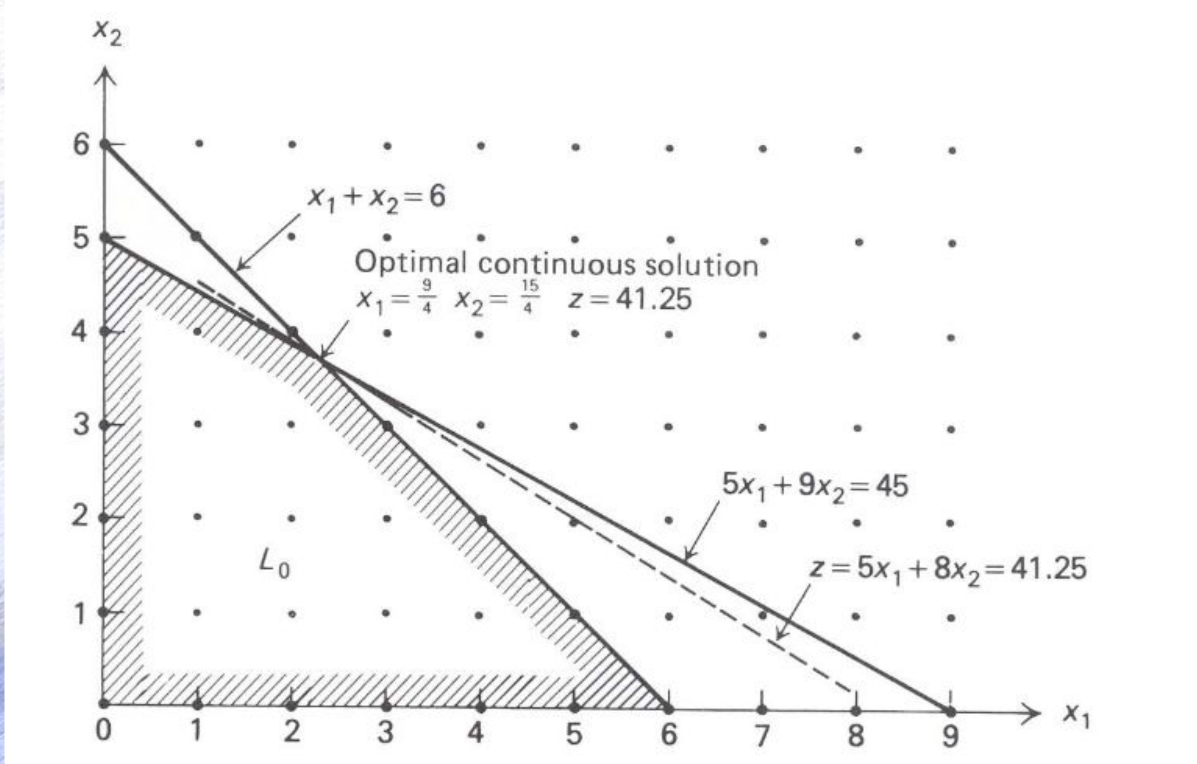
$x_1+2x_2 \le 2$

$2x_1+x_2 \le 2$

$x_1 \ge 0$, $x_2 \ge 0$ integer

Solution: (1, 0) and (0,1)

What is it?
Integer Program (IP)



Adapted from [2]

Max $5x_1+8x_2$

s.t.

$x_1+x_2 \leq 6$

$5x_1+9x_2 \leq 45$

$x_1 \geq 0$, $x_2 \geq 0$ integer

What is it?
Mixed-Integer Program (MIP)

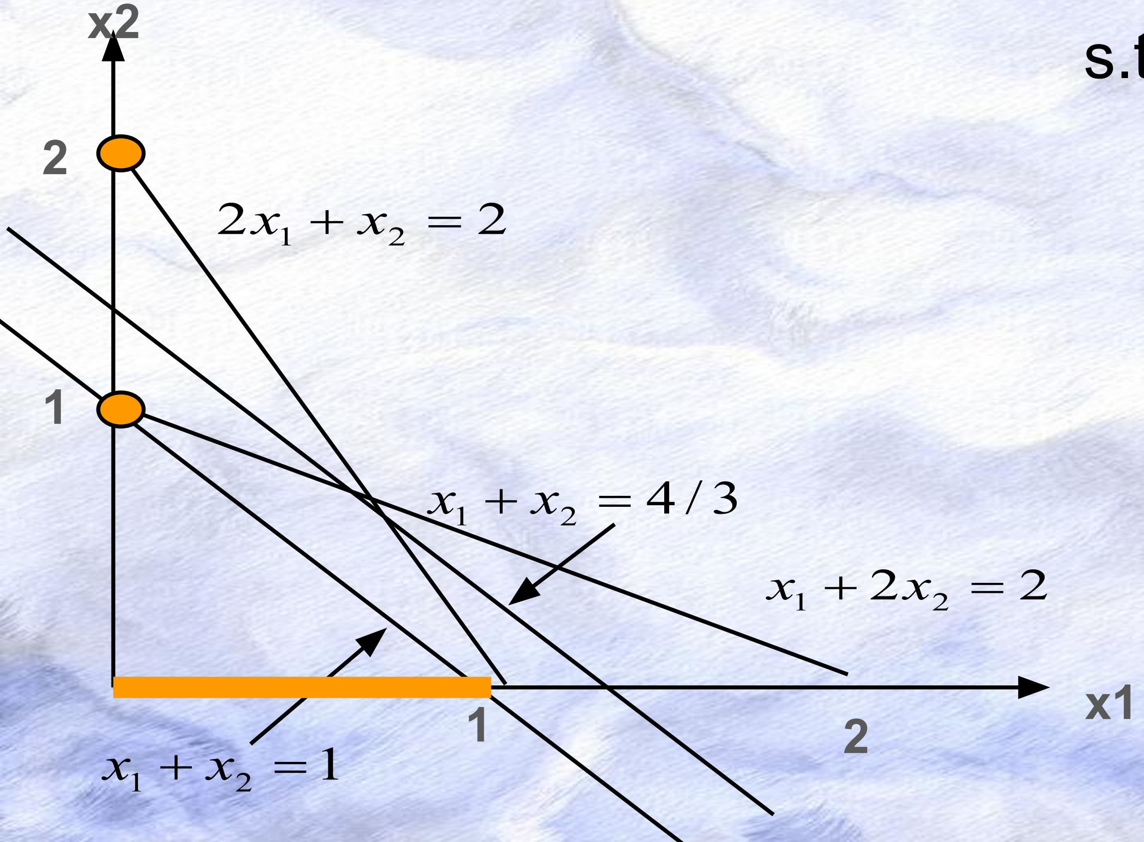
max $cx + hy$

st.

$Ax + Gy \leq b$

$x \geq 0$, $y \in Z_+$ (positive integer)

What is it?
Mixed-Integer Program (MIP)



Max x_1+x_2

s.t.

$x_1+2x_2 \leq 2$

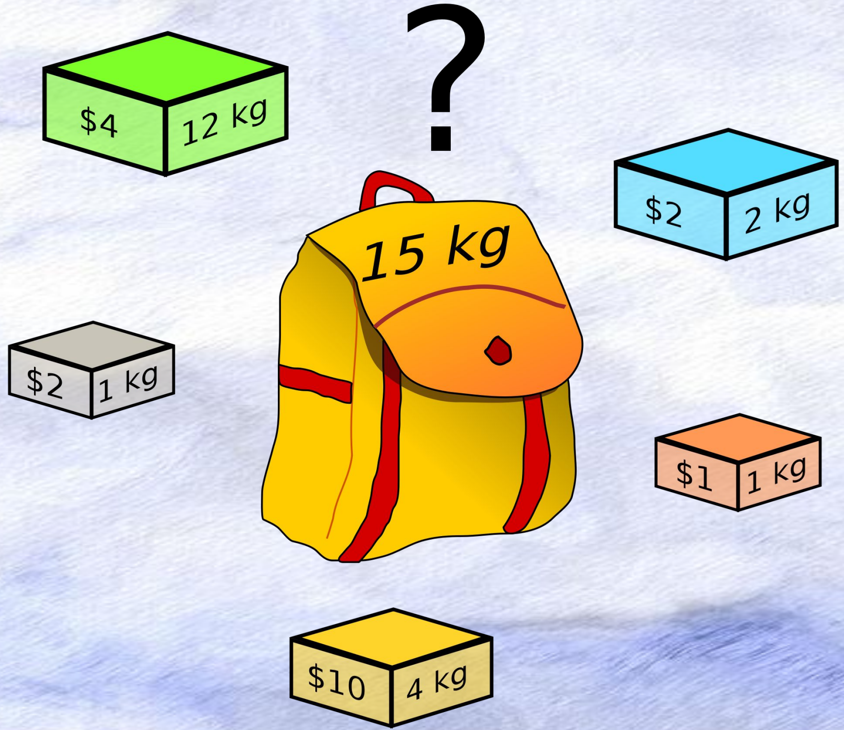
$2x_1+x_2 \leq 2$

$x_1 \geq 0,\ x_2 \in \mathbb{Z}_+$

Solution: (1, 0) and (0,1)

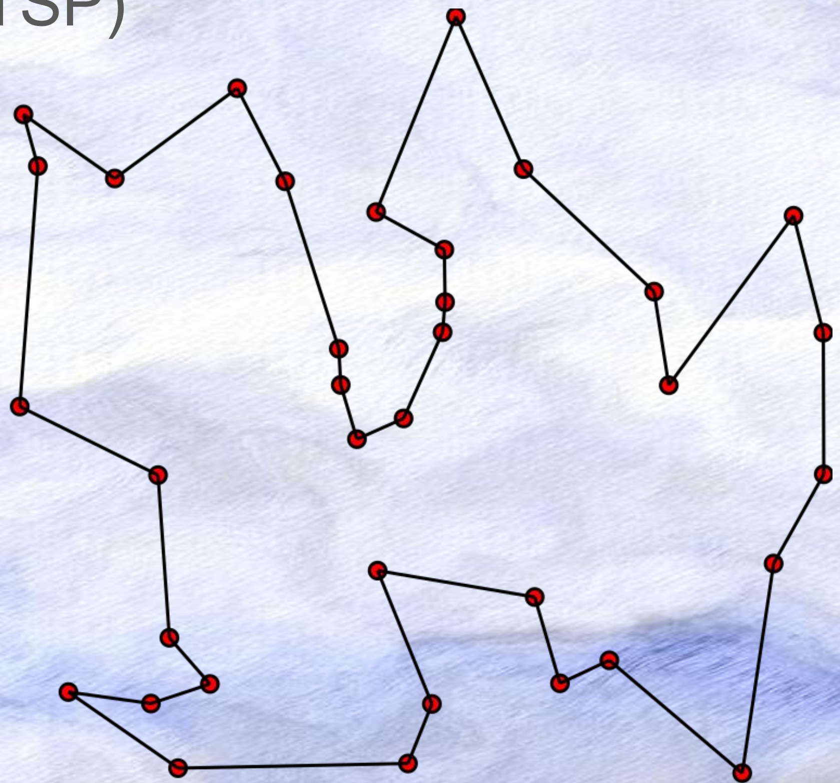
Problem Formulations

Knapsack Problem



Problem Formulations

Traveling Salesman Problem (TSP)



Problem Formulations

Lot-sizing problem

Binary Variables

Logical constraints [2]

- Constraints feasibility: When is the constraint satisfied?

$$f(x1,x2,...,xn) \leq b$$

$$f(x1,x2,...,xn) - By \leq b \quad , y \in \{0,1\}$$

$$y=0 \Rightarrow f(x1,x2,...,xn) \leq b$$

$$y=1 \Rightarrow f(x1,x2,...,xn) \leq b + B$$

Binary Variables

Logical constraints [2]

- Alternative constraints

$$f1(x1,x2,...,xn) \leq b1$$

$$f2(x1,x2,...,xn) \leq b2$$

$$f(x1,x2,...,xn) - By1 \leq b1$$

$$f(x1,x2,...,xn) - By2 \leq b2$$

$$y1+y2 \leq 1, \; y \in \{0,1\}$$

$$f(x1,x2,...,xn) - By \leq b1$$

$$f(x1,x2,...,xn) - B(1-y) \leq b2$$

$$y \in \{0,1\}$$

Binary Variables

Logical constraints [2]

- Conditional constraints:

$$f1(x1,x2,...,xn)>b1 \Rightarrow f2(x1,x2,...,xn) \leq b2$$

$$\Leftrightarrow$$

$$f1(x1,x2,...,xn) \leq b1 \vee f2(x1,x2,...,xn) \leq b2$$

Binary Variables

Logical constraints [2]

- Conditional constraints: k-fold alternatives:

  We must satisfy at least k constraints from

$$f_j(x_1,x_2,...,x_n) \leq b_j \quad \text{for } j=1,....,p$$

$$f_j(x_1,x_2,...,x_n) - B_j(1-y_j) \leq b_j \quad \text{for } j=1,....,p$$

$$y_1+....+y_j \geq k$$

$$y_j \in \{0,1\},\ j=1,...,p$$

Binary Variables

Logical constraints [2]

- Compound alternatives

Region 1: $f1(x1,x2) - B1y1 \le b1$

$f2(x1,x2) - B2y1 \le b2$

Region 2: $f3(x1,x2) - B3y2 \le b3$

$f4(x1,x2) - B4y2 \le b4$

Region 3: $f5(x1,x2) - B5y3 \le b5$

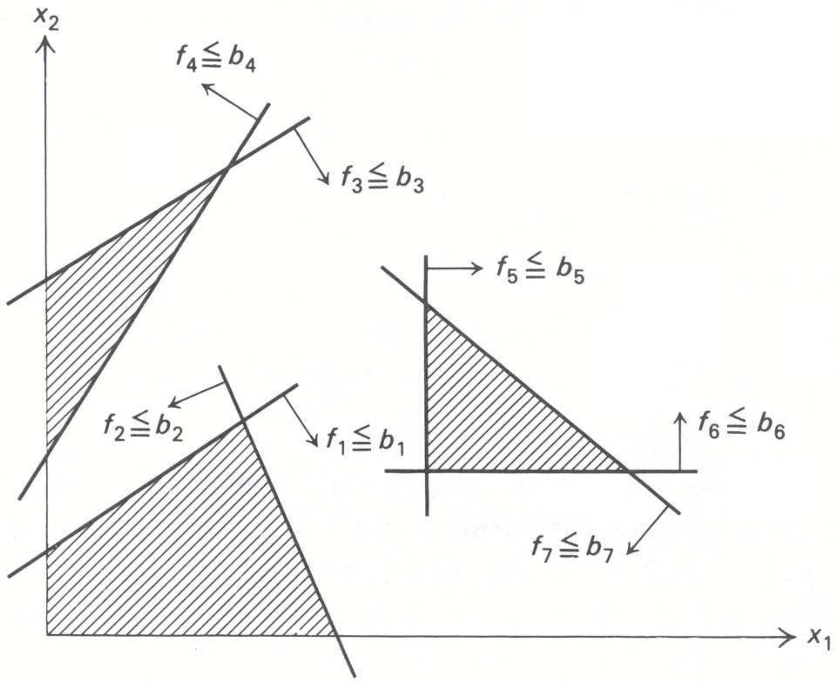
$f6(x1,x2) - B6y3 \le b6$

$f7(x1,x2) - B7y3 \le b7$

$y1 + y2 + y3 \le 2,$

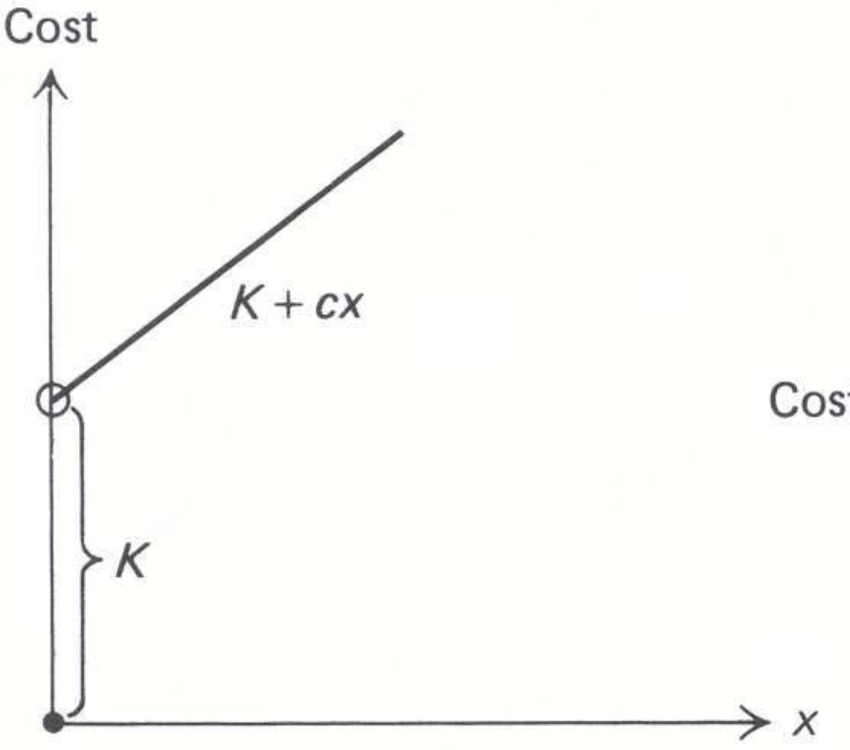
$x1 \ge 0, x2 \ge 0,$

y1, y2, y3 binary.



Adapted from [2]

Modeling Nonlinear Functions



$$\text{Cost} = \begin{cases} 0 & \text{if } x = 0 \\ K + cx & \text{if } x > 0 \end{cases}$$

Cost = Ky + cx

s.t.

$x \le By$,

$x \ge 0$,

$y = 0$ or 1.

Adapted from [2]

Modeling Nonlinear Functions

$x = \delta1 + \delta2 + \delta3,$

$0 \leq \delta1 \leq 4$

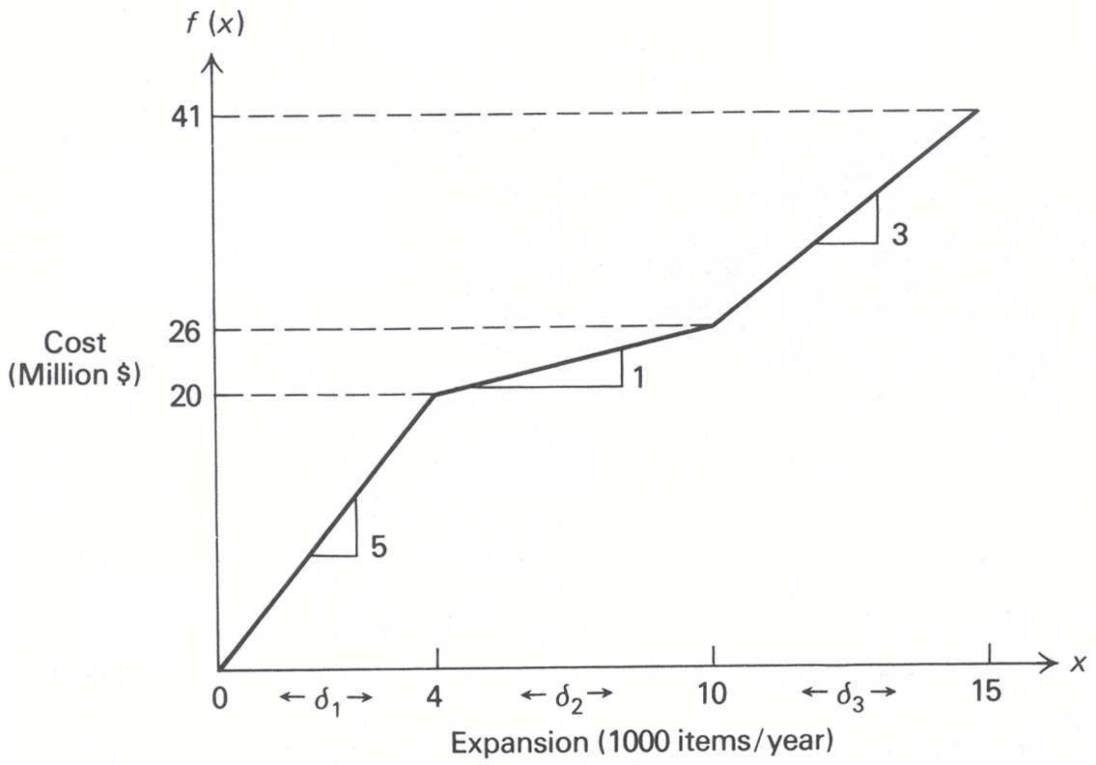
$0 \leq \delta2 \leq 6$

$0 \leq \delta3 \leq 5$

$\text{Cost} = 5\delta1 + \delta2 + 3\delta3.$

$x = 2 \Rightarrow \delta1 = \delta3 = 0$

$\delta2 = 2, \text{Cost}=2$



Adapted from [2]

Modeling Nonlinear Functions

$x = \delta_1 + \delta_2 + \delta_3,$

$4w_1 \le \delta_1 \le 4,$

$6w_2 \le \delta_2 \le 6w_1,$

$0 \le \delta_3 \le 5w_2,$

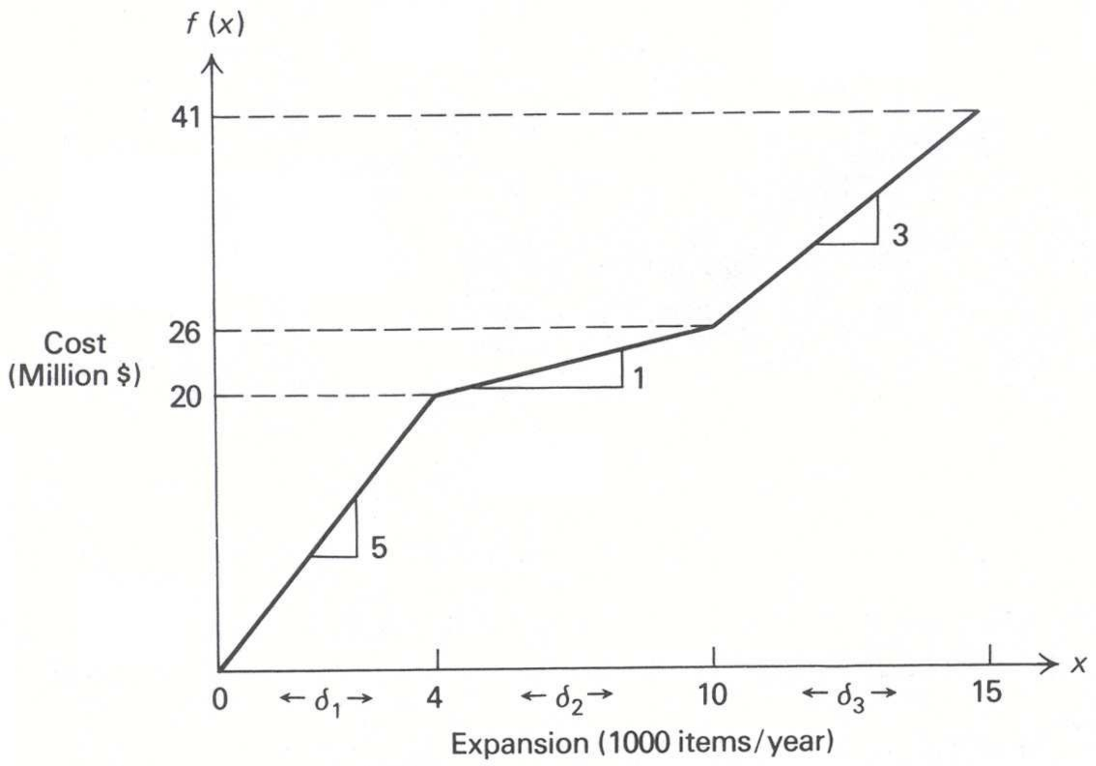
w_1, w_2 binary

$\text{Cost} = 5\delta_1 + \delta_2 + 3\delta_3.$

$x = 2 \Rightarrow \delta_1 = 2, w_1 = 0$

$\delta_3 = \delta_2 = 0, w_2 = 0$

Cost=10



Adapted from [2]

Modeling Nonlinear Functions

$x = \delta 1 + \delta 2 + \delta 3,$

$4w1 \le \delta 1 \le 4,$

$6w2 \le \delta 2 \le 6w1,$

$0 \le \delta 3 \le 5w2,$

w1, w2 binary

- $w1 = 0 \Rightarrow w2 = 0$ with $0 \le \delta 1 \le 4$, $\delta 2 = 0$, and $\delta 3 = 0$.
- $w1 = 1$ e $w2 = 0$ with $\delta 1 = 4$, $0 \le \delta 2 \le 6$, and $\delta 3 = 0$.
- $w1 = 1$ e $w2 = 1$ with $\delta 1 = 4$, $\delta 2 = 6$, and $0 \le \delta 3 \le 5$.

- $w1 = 0, w2 = 0 \Rightarrow 0 \le x \le 4$ since $\delta 2 = \delta 3 = 0$.
- $w1 = 1, w2 = 0 \Rightarrow 4 \le x \le 10$ since $\delta 1 = 4$ and $\delta 3 = 0$.
- $w1 = 1, w2 = 1 \Rightarrow 10 \le x \le 15$ since $\delta 1 = 4$ and $\delta 2 = 6$.

$$L_j w_j \le \delta_j \le Ljw_{j-1},$$

Modeling Nonlinear Functions

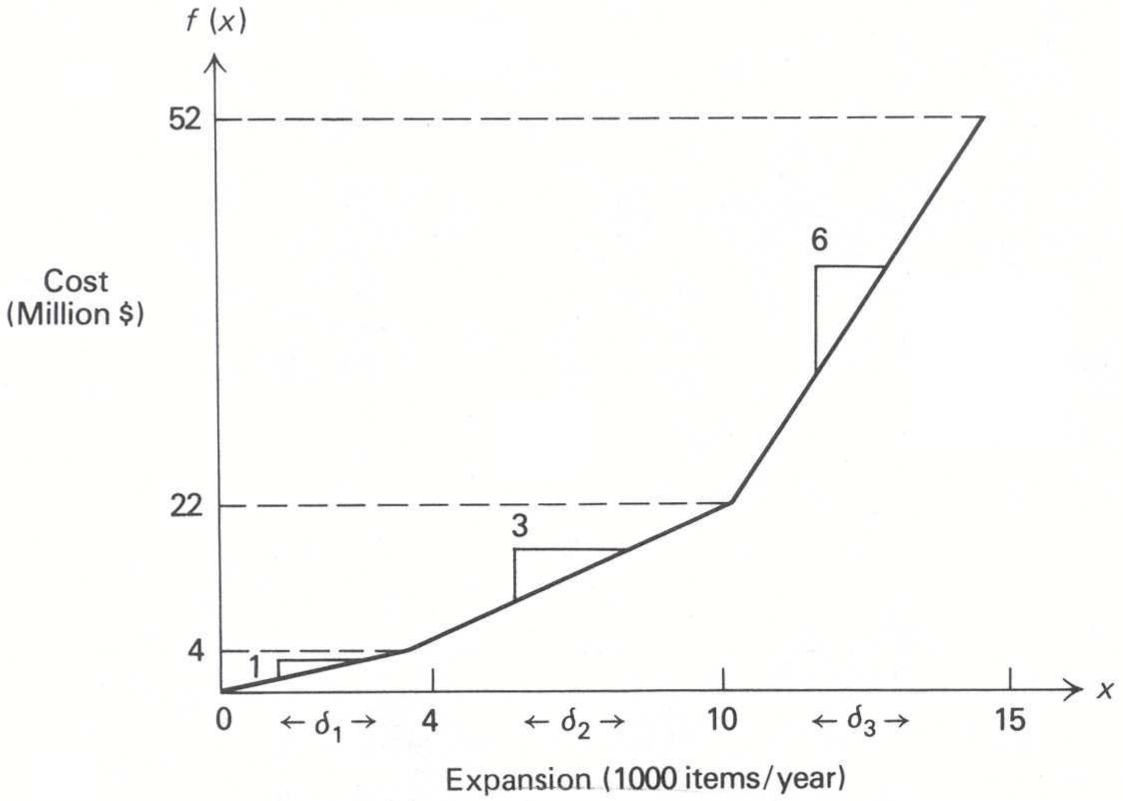
Cost = $\delta_1 + 3\delta_2 + 6\delta_3$

s.t.

$0 \le \delta_1 \le 4$

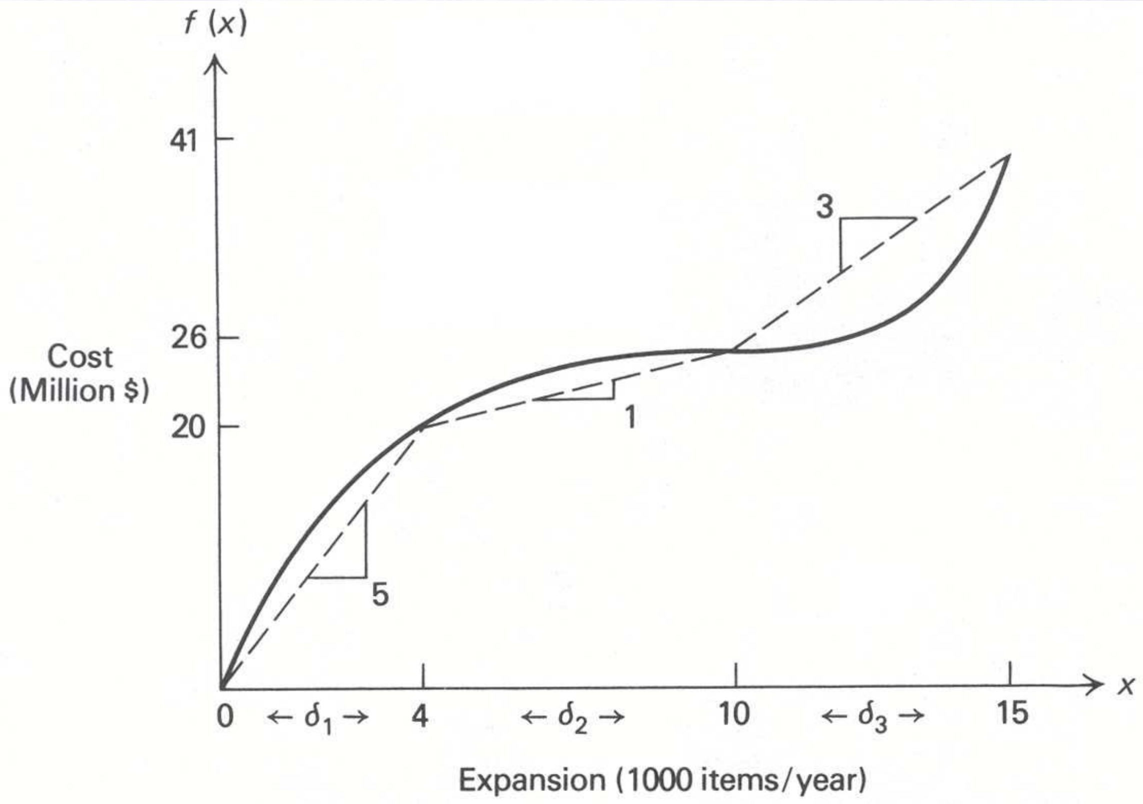
$0 \le \delta_2 \le 6,$

$0 \le \delta_3 \le 5.$



Adapted from [2]

Modeling Nonlinear Functions



Adapted from [2]

Referências

[1] Enciclopedia of Mathematics, https://www.encyclopediaofmath.org/index.php/Mathematical_programming.

[2] MIT - Integer Programming,

http://web.mit.edu/15.053/www/AMP-Chapter-09.pdf.