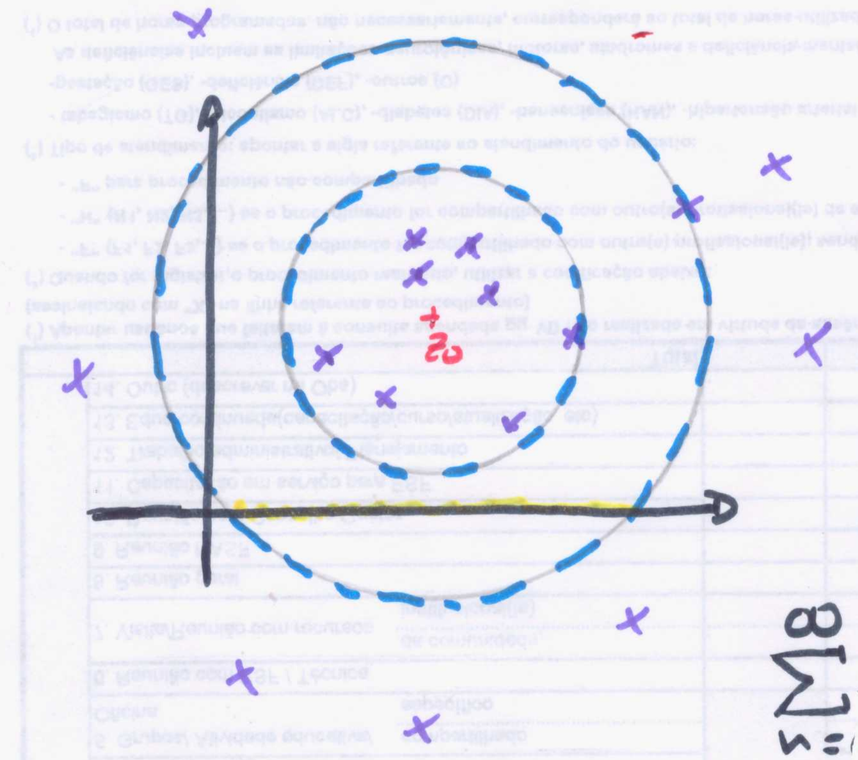
12/11/2020



× singularities (isolated)

× z_0

- - annular region of convergence for the Laurent series

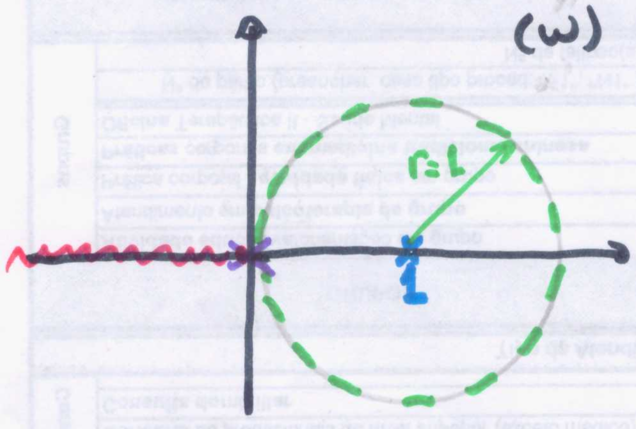
$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

P7. (Page 165)

$$-\pi < \theta < \pi$$

$$|w-1| < 1$$

$$w = re^{i\theta}$$



$$\frac{1}{(1-z)} = \sum_{k=0}^{\infty} z^k \quad \forall z \,\middle|\, |z|<$$

$$w = 1 - z \Rightarrow z = 1 - w = z = -(w-1)$$

Convergence disk: $|z|<1 \Rightarrow |1-w|<1 \Leftrightarrow |w-1|<1$

$$\int_1^w \frac{1}{s}\,ds = \sum_{k=0}^{\infty} (-1)^k \int_1^w (s-1)^n\,ds \quad \Bigg| \quad \frac{1}{w} = \sum_{k=0}^{\infty} (-1)^k (w-1)^k$$

$$Log(w) - \underbrace{\cancel{Log(1)}}_{0} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{(s-1)^{n+1}}{(n+1)}\right]_1^w$$

$$\operatorname{Log} W = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p}(w-1)^{p} \quad \forall W \in \mathbb{C} \,|\, |w-1|<$$

x x x

Residues and Poles — Chapter 6.

two important series:

$$\begin{cases} \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^{n} & |z|<1 \\[2ex] e^{z} = \displaystyle\sum_{n=0}^{\infty} \dfrac{z^{n}}{n!} & |z|<\infty \end{cases}$$

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n \quad \forall z \in \mathbb{C} \,|\, R_1 < |z-z_0| < R_2$$

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad n = 0, \pm 1, \pm 2, \dots \; n \in \mathbb{Z}$$

for $n \in \mathbb{N} \Rightarrow$ Cauchy

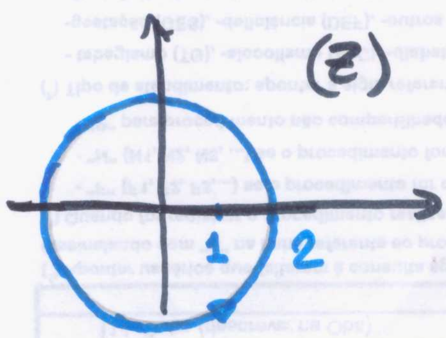
Where C is a simple closed contour within the region $R_1 < |z-z_0| < R_2$

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz \Rightarrow \operatorname*{Res}_{z=z_0}[f(z)] \equiv 2\pi i\, c_{-1}$$

$$\operatorname{Res}[f(z), z_0] \equiv \oint_C f(z)\,dz$$

$$\dots + \frac{c_{-2}}{(z-z_0)^2} + \boxed{\frac{c_{-1}}{(z-z_0)}} + c_0 + c_1(z-z_0) + \dots$$

Example 6



$$\oint_C \exp\left(\frac{1}{z^2}\right)dz = 0$$

$$e^{\xi} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$$

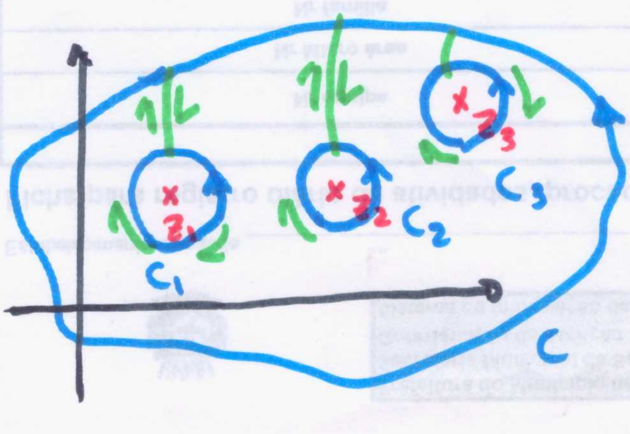
$$e^{1/z^2} = 1 + \frac{1}{1!\,z^2} + \frac{1}{2!\,z^4} + \dots \qquad 0<|z|<\infty$$

Since $C_{-1} = 0 \Rightarrow$ the residue is zero and thus the integral.

The residue theorem:

if C is a positively oriented simple closed contour within and on which f(z) is analytic, except for a finite number of singularities z_k (k=1,2..n) in its interior, then:

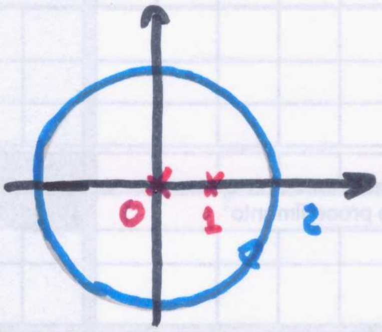
$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z), z_k]$$



$$\oint_C f(z)\,dz - \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z$$

$$\oint_C \frac{5z-2}{z(z-1)}\,dz \qquad C: |z| = 2$$



$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad |z|<1$$

$$\frac{5z-2}{z(z-1)} = \left(\frac{5z-2}{z}\right)\left(\frac{-1}{1-z}\right) =$$

$$= \left(\frac{5z-2}{z}\right)\left(-1 - z - z^2 - \dots\right)$$

$$= \frac{2}{z} - 3 - 3z - \dots$$

$\therefore \boxed{c_{-1} = 2}$ $\qquad \operatorname{Res}[f(z), 0] = 2$ for $0<|z|<1$

Now for $0<|z-1|<1$ we have:

$$\frac{5z-2}{z(z-1)} = \left[\frac{5(z-1)+3}{z-1}\right]\left[\frac{1}{1+(z-1)}\right]$$

$$= \left(5 + \frac{3}{(z-1)}\right)\left[1 - (z-1) + (z-1)^2 - \dots\right]$$

$$\Rightarrow c_{-1} = 3 \Rightarrow \operatorname{Res}[f(z), 1] = 3$$

for $0<|z-1|<1$

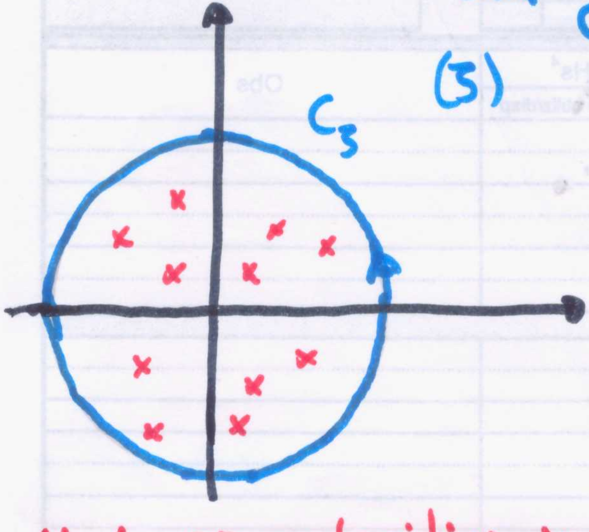
$$\oint_C \frac{(5z-2)}{z(z-1)}\,dz = 2\pi i\,(2+3) = 10\pi i$$

$$\oint_c \frac{5z-2}{z(z-1)}dz = \oint_C \frac{2}{z}dz + \oint_C \frac{3}{(z-1)}dz =$$

$$= 4\pi i + 6\pi i = 10\pi i$$

The residue at infinity. From the book by A. Sveshnikov and A. Tikhonov "Functions of a Complex Variable" (Mir) Pages 128-129

$$\text{Res}[f(\varsigma),\infty] \equiv \frac{1}{2\pi i}\oint_{C_\varsigma^-} f(\varsigma)d\varsigma = -\frac{1}{2\pi i}\oint_{C_\varsigma^+} f(\varsigma)d\varsigma = -\sum_{k=1}^{N-1}\text{Res}[f,\varsigma_k]$$



N-1 singularities inside ($\varsigma_N = \infty$ is the Nth one)

Now we make: $\varsigma = \frac{1}{z}$

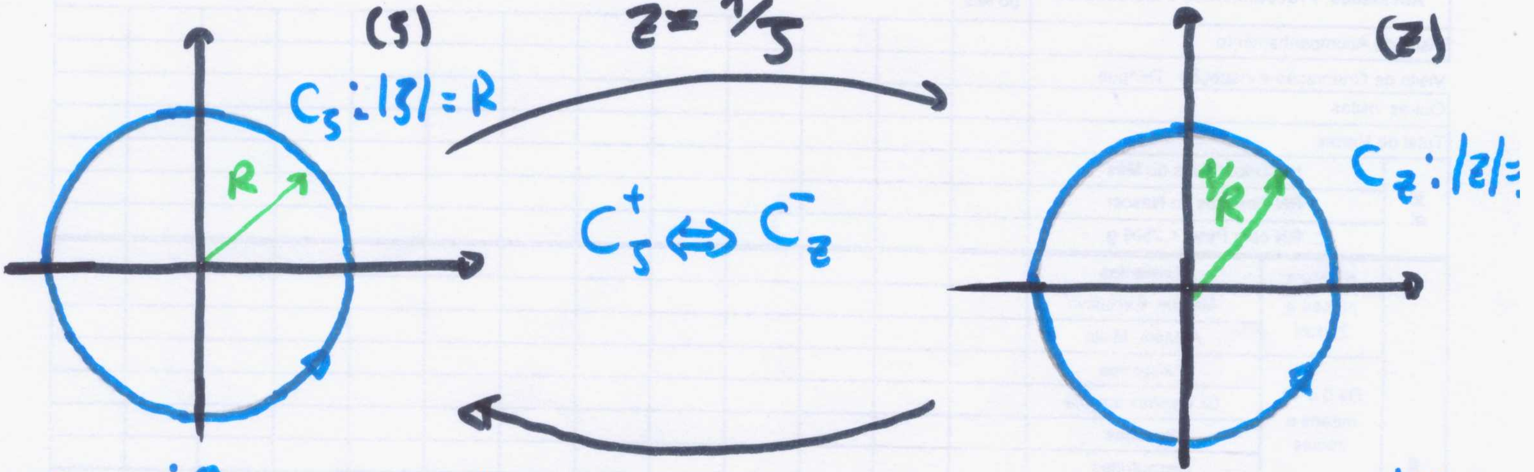
$$d\varsigma = -\frac{dz}{z^2}\begin{cases}\varsigma\to 0 \Rightarrow z\to\infty\\ \varsigma\to\infty \Rightarrow z\to 0\end{cases}$$

$$-\sum_{k=1}^{N-1}\text{Res}[f(\varsigma),\varsigma_k] = -\frac{1}{2\pi i}\oint_{C_z^-} f\left(\frac{1}{z}\right)\frac{dz}{z^2} =$$

$$= \frac{1}{2\pi i}\oint_{C_z^+} f(1/z)\frac{dz}{z^2} =$$

$$\text{Res}\left[\frac{f(1/z)}{z^2}, 0\right] = -\sum_{k=1}^{N-1}\text{Res}[f(\varsigma),\varsigma_k]$$

$$\operatorname{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right] = \operatorname{Res}\left[f(\zeta), \infty\right] = -\sum_{k=1}^{N-1} \operatorname{Res}\left[f(\zeta), \zeta_k\right]$$ 6



$\zeta = 1/z$

$\zeta = R e^{i\theta} \quad 0 \le \theta \le 2\pi$

$d\zeta = R i e^{i\theta} d\theta$

$z = \frac{1}{R} e^{-i\theta} = \rho e^{i\varphi}$

$\rho = 1/R, \ \varphi = -\theta$

$dz = -\frac{1}{R} i e^{-i\theta} d\theta$

$$\frac{1}{2\pi i} \oint_{C^+_\zeta} f(\zeta) d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(R,\theta) R e^{i\theta} d\theta$$

$z^{-2} = \rho^{-2} e^{-i2\varphi} = R^2 e^{i2\theta}$

$$\frac{1}{2\pi i} \oint_{C^+_\zeta} f(\zeta) d\zeta = -\frac{1}{2\pi i} \oint_{C^-_z} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz = \frac{1}{2\pi i} \int_0^{2\pi} R^2 e^{i2\theta} f(R,\theta) \left(\frac{-i}{R}\right) e^{-i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(R,\theta) R e^{i\theta} d\theta$$

Example 2 : $c: |z| = 2$

$$\oint_c \frac{5z-2}{z(z-1)}\,dz \; ; \quad f(z) = \frac{(5z-2)}{z(z-1)}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{\left(\frac{5}{z}\right) - 2}{\frac{1}{z}\left(\frac{1}{z} - 1\right)} \left(\frac{1}{z^2}\right) = \left(\frac{1}{\cancel{z^2}}\right) \frac{(5-2z)\cancel{z^2}}{z(1-z)} = \frac{5-2z}{z(1-z)}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)} = \left(\frac{5-2z}{z}\right)\left(\frac{1}{1-z}\right)$$

$$= \left(\frac{5-2z}{z}\right)\left[1 + z + z^2 + \dots\right] = \frac{5}{z} + \dots \quad [0 < |z| < 1]$$

$C_{-1} = 5 \Rightarrow 10\pi i$

$$\oint_c \frac{(5-2z)/(1-z)}{z}\,dz = \left[\frac{5-2z}{(1-z)}\right]_{z=0} = 5 \Rightarrow 10\pi i$$

Residues at poles:

$f(z) = \dfrac{\phi(z)}{(z-z_0)}$ where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$

Taylor series of $\phi(z)$ about $z = z_0$

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \dots$$

$$f(z) = \frac{\phi(z)}{(z-z_0)} = \frac{\phi(z_0)}{(z-z_0)} + \frac{\phi'(z_0)}{1!} + \frac{\phi''(z_0)}{2!}(z-z_0)$$

Therefore, we get

$$C_{-1} = \phi(z_0)$$

On the other hand, we may have

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \Rightarrow f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{1!\,(z-z_0)^{m-1}} +$$

$$+ \frac{\phi''(z_0)}{2!\,(z-z_0)^{m-2}} + \dots + \boxed{\frac{\phi^{(m-1)}(z_0)}{(m-1)!\,(z-z_0)}} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)(z-z_0)^{n-m}}{n!}$$

$$\boxed{C_{-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$ $\Rightarrow$ which corresponds to the residue in this case

Zeros and Poles of order "m"

Let's assume that $f(z_0)=0$ and its derivatives vanish at $z=z_0$ up to order (m), that is $f^{(m)}(z_0) \neq 0$:

$$f(z) = (z-z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z-z_0)^n \qquad |z-z_0|<R$$

$f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic at $z=z_0$: Then we say that $f(z)$ has a "m" order critical point at $z=z_0$

19/11/2020

The principal part of a function:

$$f(z) = \underbrace{\sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$0 < |z - z_0| < R$$

Principal part.

The principal part is either zero, finite or infinite.

- Zero terms ⇒ then the function is analytic at $z = z_0$
- if it has a finite number of terms, it has a pole of order "m", which is an isolated singularity, at $z = z_0$

$$\frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{(z-z_0)} + a_0 + \dots$$

- but if it has an infinite number of terms, then $z = z_0$ is called an essential singularity

$$f(z) = \frac{e^z - 1}{z} = \frac{1}{z}\left[\cancel{1} + \frac{z}{1!} + \frac{z^2}{2!} + \cdots - \cancel{1}\right]$$

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \qquad 0<|z|<\infty$$

at $z = 0 \Rightarrow$ the series yields 0, whereas the original expression is <u>not</u> defined.

Hence, on redefining $f(z)$ as: $f(z) = \begin{cases} 1 \; ; z = 0 \\ \frac{e^z - 1}{z} \; ; z \neq \end{cases}$

the new expression has removed the singularity and the function has become entire.

Improper Real Integrals

$$\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx$$ ⇝ where it is assumed that the limit exists.

Now, how about:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$

Cauchy Principal Value

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \fint_{-\infty}^{\infty} f(x)\,dx = \fint_{-\infty}^{\infty} f(x)\,dx \equiv \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

Example:

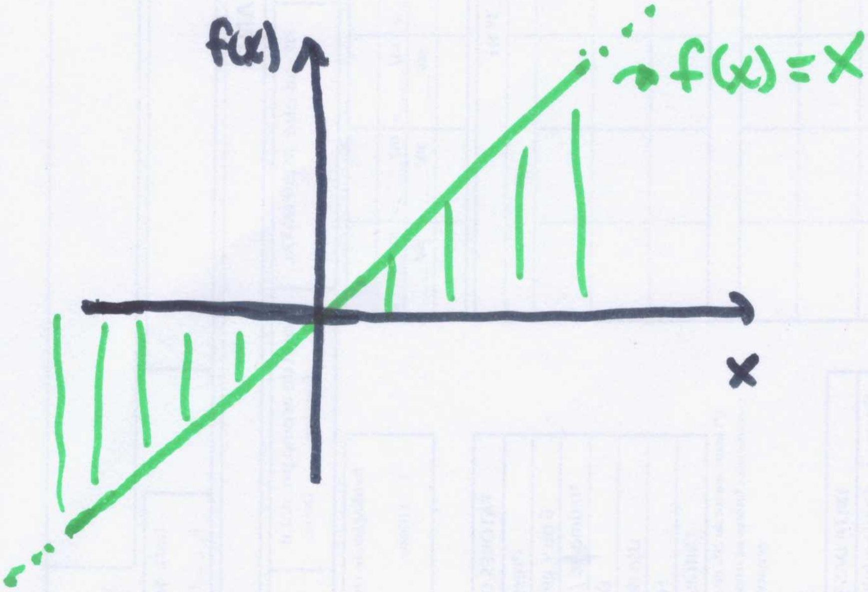
$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} x\,dx$$

$$= \lim_{R_1\to\infty}\left[-\frac{R_1^2}{2}\right] + \lim_{R_2\to\infty}\left[\frac{R_2^2}{2}\right] = -\infty + \infty$$

indeterminate

On the other hand:

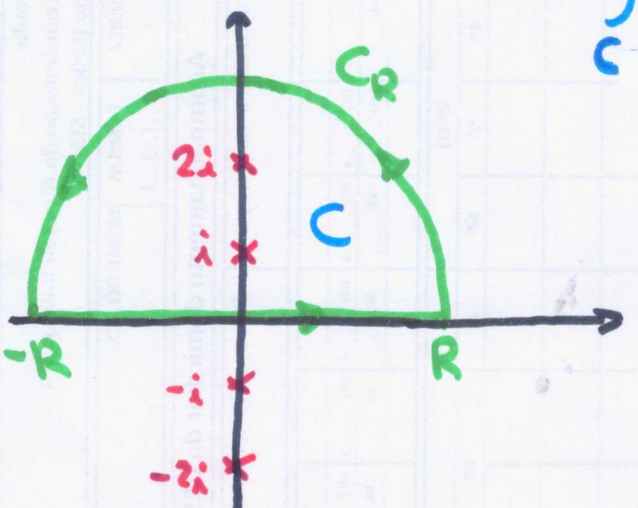
$$\fint_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{+R} x\,dx = \lim_{R\to\infty}\left[\frac{R^2}{2}-\frac{R^2}{2}\right] = 0$$



$$\int_0^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx$$

$$f(x) \longrightarrow f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+1)(z^2+4)}$$

$$f(z) = \frac{2z^2+1}{(z+i)(z-i)(z+2i)(z-2i)}$$



$$\oint_C f(z)\,dz = \int_{-R}^{-R} f(z)\,dz + \int_{C_R} f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i \left[Res(f, i) + Res(f, 2i) \right]$$

Computing residues :

at $z = i \Rightarrow \phi(z) = \dfrac{2z^2+1}{(z+i)(z+2i)(z-2i)}$

this $\phi(z)$ is analytic at $z = i$. Hence we can make:

$$\oint \frac{\phi(z)}{(z-i)}\,dz = 2\pi i\,\phi(i)$$

And it implies that: $Res\left[f(z), i\right] = \phi(i) = \dfrac{-1}{2i}$

By the same token: $Res\left[f(z), 2i\right] = \Psi(2i) = \dfrac{3i}{4}$

where $\Psi(z) \equiv \dfrac{2z^2-1}{(z+i)(z-i)(z+2i)}$

$$f(z) = \frac{\Psi(z)}{(z-2i)}$$

The overall result then is: $\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{2i}+\frac{3}{4i}\right)$

$$\oint_C f(z)\,dz = \frac{\pi}{2}$$

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = \frac{\pi}{2}$$

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{\pi}{2} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz$$

$$|2z^2-1| \leq 2|z|^2+1 = 2R^2+1$$

$$|z^4+5z^2+4| = |z^2+1||z^2+4| \geq \left|\,|z^2|-1\right|\left|\,|z^2|-4\right| =$$

$$= (R^2-1)(R^2-4)$$

$$\left|f(z)\right|_{C_R} \leq M_R = \frac{2R^2+1}{(R^2-1)(R^2-4)}, \quad L_{C_R} = \pi R$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \int_{C_R} |f(z)|\,dz \leq M_R\, L_{C_R}$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R\,(2R^2+1)}{(R^2-1)(R^2-4)}$$

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\,dz \right| \leq \lim_{R\to\infty} \frac{\pi R\,(2R^2+1)}{(R^2-1)(R^2-4)} = 0$$

As a result of this the contribution of C_R vanishes in the limit as $R\to\infty$ and Cauchy's Principal value corresponds to:

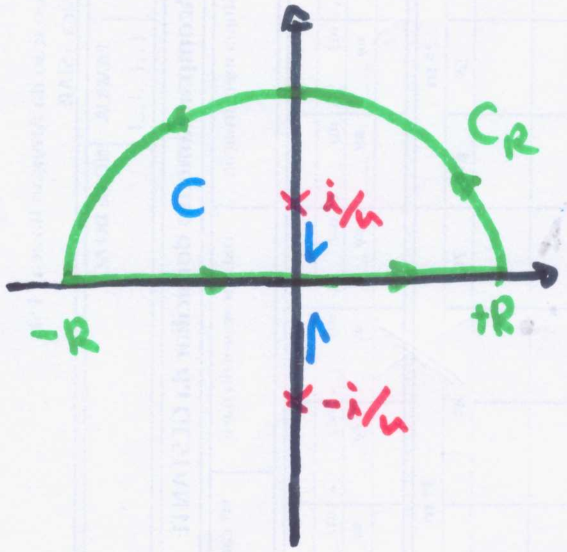
$$\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx = \frac{\pi}{2}$$

Problem 1, page 258 Butkov

Show that $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)\, f(x)\, dx = f(0)$

where $\phi_n(x) = \dfrac{n}{\pi} \dfrac{1}{(1+n^2x^2)} = \dfrac{1}{\pi n\left(x^2+\frac{1}{n^2}\right)}$

$$g(z) = \frac{f(z)}{n\pi(z^2+1/n^2)} = \frac{f(z)}{\pi n(z-i/n)(z+i/n)}$$



$$\oint_C g(z)\,dz = \oint_C \frac{\{f(z)/\pi n(z+i/n)\}}{(z-i/n)}\,dz =$$

$$= \frac{2\pi i\, f(i/n)}{\pi n\,(2i/n)} = f\left(\frac{i}{n}\right)$$

Now, on assuming that $f(z) \xrightarrow[z\to\infty]{} 0$ and, hence, that $f(z)$ is bounded on C_R for $\forall R$:

$$0 \le |f(z)| \le M \quad \forall z \in C_R$$

$$\left|\int_{C_R} g(z)\,dz\right| \le \int_{C_R} \frac{|f(z)|\quad dz}{n\pi|z^2+1/n^2|} \le \frac{M.\pi R}{n\pi\left|\frac{1}{n^2}-R^2\right|}$$

$$\lim_{R\to\infty} \frac{\cancel{R}\, M}{R^{\cancel{2}}\left|\frac{1}{n^2R^2}-1\right|} = 0$$

$$\big|\,|z^2| - |1/n^2|\,\big| \leq |z^2 + 1/n^2| \leq |z^2| + |1/n^2|$$

$$|R^2 - \frac{1}{n^2}| \leq |z^2+\frac{1}{n^2}| \leq R^2 + \frac{1}{n^2}$$

$$\frac{1}{|R^2 - 1/n^2|} \geqslant \frac{1}{|z^2 + \frac{1}{n^2}|} \geqslant \frac{1}{R^2 + 1/n^2}$$

Under the above conditions:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx = \lim_{\substack{R\to\infty \\ n\to\infty}} \int_C g(z)\, dz = \lim_{n\to\infty} f(i/n) =$$

$$= \underline{\underline{f(0)}}$$

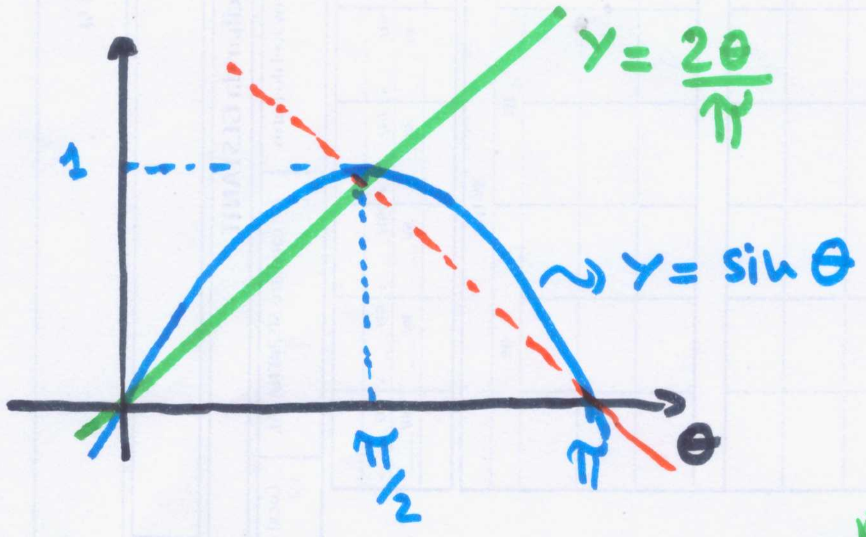
Improper integrals involving trigonometric functions.

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad ; \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

in terms of Cauchy principal values:

$$\int_{-R}^{R} f(x)e^{iax}\,dx = \int_{-R}^{R} f(x)\cos(ax)\,dx + i\int_{-R}^{R} f(x)\sin(ax)\,dx$$

Jordan's inequality: $(R>0)$ $\quad \int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$



for $0 \leq \theta \leq \frac{\pi}{2}$

$$\sin(\theta) \geq \frac{2\theta}{\pi}$$

$$-\sin(\theta) \leq -\frac{2\theta}{\pi}$$

then, for $R>0 \Rightarrow e^{-R\sin\theta} \leq e^{-2R\theta/\pi}$

$0 \leq \theta \leq \frac{\pi}{2}$

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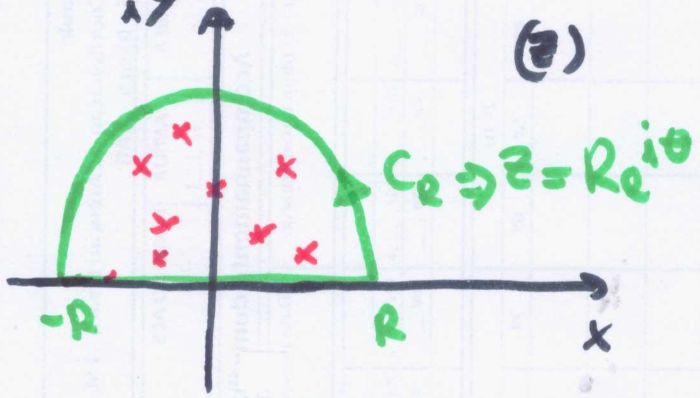
$$\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \le \int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

$$\therefore \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta < \frac{\pi}{2R}$$

the sine function is symmetric in the interval $0 \le \theta \le \pi$, with respect to $\theta = \frac{\pi}{2}$



$0 \le \theta \le \pi$

Assume that $f(z) = \frac{P(z)}{q(z)}$ has all singularities in the upper half of the complex plane, within the semi-circle $|z| = R$

then, if $f(z)$ is such that $\exists\, M_R \geq 0 \,|\, \forall z \in C_R \Rightarrow |f(z)|\Big|_{C_R} \le M_R$ and $\lim_{R\to\infty} M_R = 0$

under such conditions, we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{iaz} = 0 \quad (a>0)$$

we start the proof with:

$$\int_{C_R} f(z)\, e^{iaz}\, dz = \int_0^{\pi} f(Re^{i\theta}) \exp(iaRe^{i\theta})\, iRe^{i\theta}\, d\theta$$

$(0 \leq \theta \leq \pi)$

$$\exp[iaRe^{i\theta}] = \exp[iaR(\cos\theta + i\sin\theta)]$$
$$= \exp[iaR\cos\theta - aR\sin\theta]$$

$$|\exp[iaRe^{i\theta}]| = |\exp[iaR\cos\theta - aR\sin\theta]|$$
$$= |e^{-aR\sin\theta}|\,|e^{iaR\cos\theta}|$$
$$= e^{-aR\sin\theta} \underbrace{|\cos(aR\cos\theta) + i\sin(aR\cos\theta)|}_{1}$$

Hence we have:

$$|f(Re^{i\theta})| \leq M_R \quad \text{and} \quad |\exp(iaRe^{i\theta})| \leq$$
$$\leq e^{-aR\sin\theta}$$

under the above conditions, we get

$$\left| \int_{C_R} f(z) e^{iaz} dz \right| \leq M_R R \int_0^{\pi} e^{-aR\sin\theta} d\theta < \frac{M_R \pi}{a}$$

But $\lim_{R \to \infty} M_R = 0$, so we get that

$$\lim_{R \to \infty} \int_{C_R} f(z) e^{iaz} dz = 0 \quad , \quad a > 0$$

And the Jordan's inequality is proved.

26/11/2020

Book: A. Sveshnikov and A. Tikhonov "The theory of functions of a Complex Variable" Mir publishers. 1978. Page 241

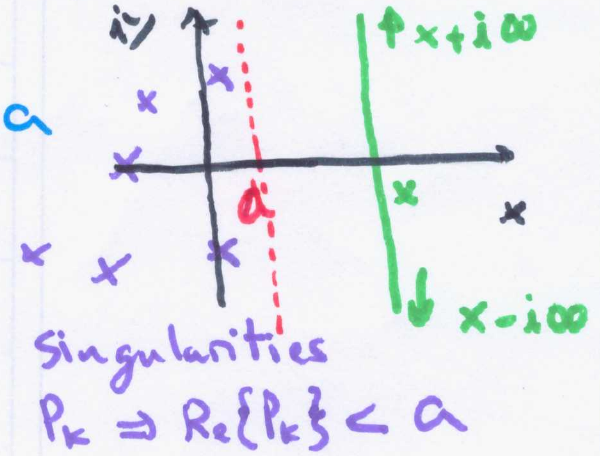
Theorem: Let $F(p) \in \mathbb{C}$, $p \in \mathbb{C}$, $p = x + iy$ meet:

a) $F(p)$ is analytic in $\mathcal{D}$: $Re\{p\} > a > 0$

b) in $\mathcal{D}$: $F(p) \xrightarrow[\substack{|p|\to\infty \\ \forall \arg\{p\}}]{} 0$ (uniformly)

c) $\forall$ $Re\{p\} = x > a$, the integral converges:

$$\int_{x-i\infty}^{x+i\infty} |F(p)| dy < M, \forall x > a$$



Then, under the above conditions,

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} F(p) dp \qquad \forall x > a$$

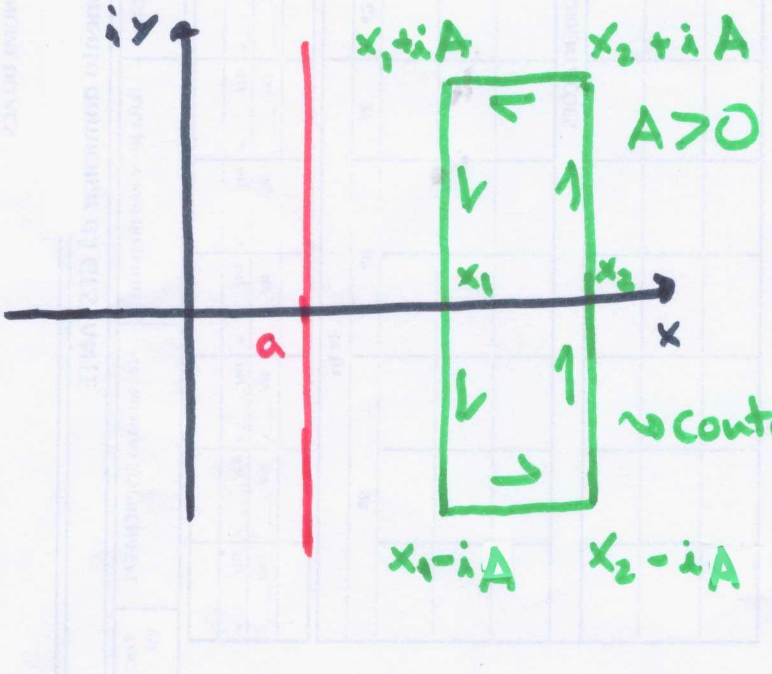
$$\left|\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt}F(p)\,dp\right| \leq \frac{1}{2\pi}\int_{x-i\infty}^{x+i\infty} |e^{pt}F(p)|.\,|dp| =$$

on this path, $|dp| = dy \Leftarrow dP = i\,dy$, $|e^{pt}| = e^{xt}$

$$= \frac{e^{xt}}{2\pi}\int_{x-i\infty}^{x+i\infty} |F(p)|\,dy \leq \frac{M}{2\pi}e^{xt} \qquad \forall \; x > a$$

1) We must prove that the integral is independent of x. So that it leads to $f = f(t)$. Also, we must show that $|f(t)| < M_e^{at}$ where $t \in \mathbb{R}$

$\mathcal{D}: R_e\{P\} > a$



$$\oint_\Gamma e^{pt}F(P)\,dP = 0 \Rightarrow \text{Cauchy Goursat theorem}$$

$$\int_{x_1-iA}^{x_1+iA} e^{pt}F(P)\,dP = \int_{x_2-iA}^{x_2+iA} e^{pt}F(p)\,dP + \int_{x_2+iA}^{x_1+iA} e^{pt}F(p)\,dp + \int_{x_1-iA}^{x_2-iA} e^{pt}F(p)$$

Now, in the limit as $A \to \infty$, the integrals over the horizontal lines should vanish, since $\lim\limits_{p\to\infty} F(p) \to 0$, and we are left

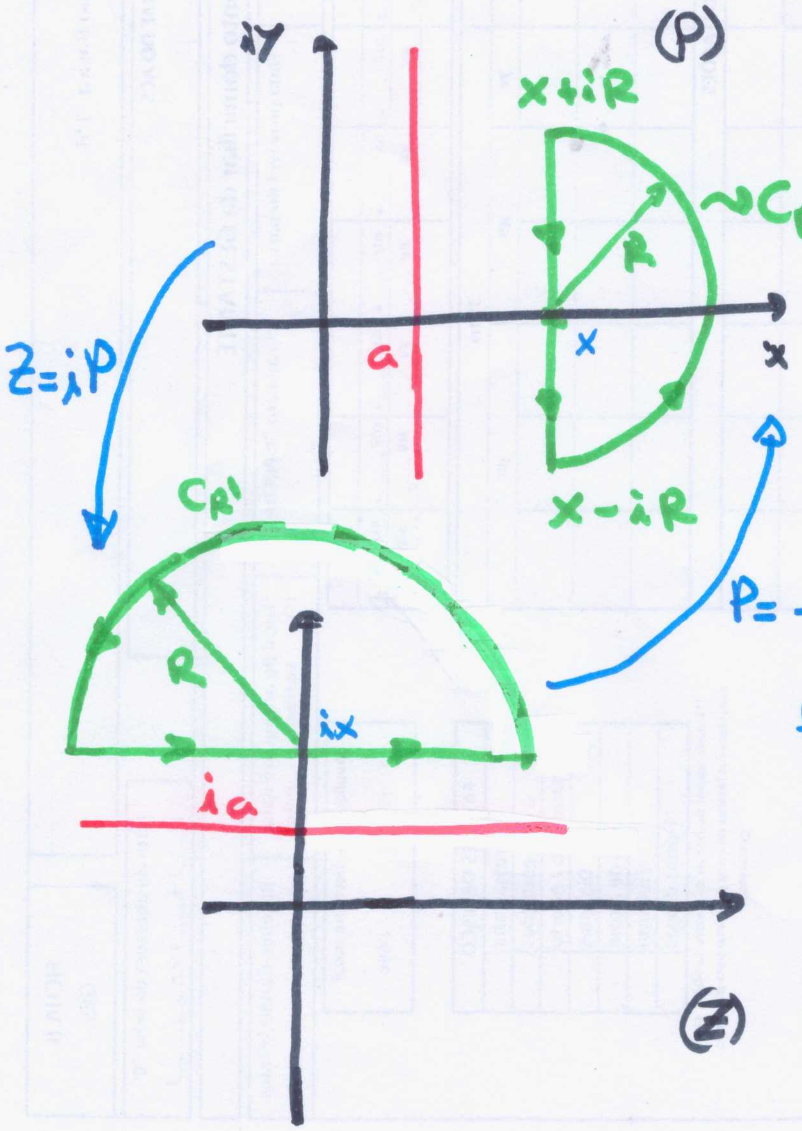
with:

$$\int_{x_1-i\infty}^{x_1+i\infty} e^{pt} F(p)\,dp = \int_{x_2-i\infty}^{x_2+i\infty} e^{pt} F(p)\,dp$$

since x_1 and x_2 are arbitrary in $\mathcal{D}: \mathrm{Re}\{p\} = x > a$ the integral does not depend on x

2) Now we must show that $f(t) \equiv 0$ for $t<0$

for $t<0$



$C_{R'}: p = x + Re^{i\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\boxed{dp = -i\,dz}$$

$$f(t) = \frac{-i}{2\pi i}\int_{\infty+ix}^{-\infty+ix} e^{-izt} F(-iz)\,dz =$$

$$= \frac{i}{2\pi i}\int_{-\infty+ix}^{\infty+ix} F(z)e^{isz}\,dz$$

$s = -t$

$t<0 \Leftrightarrow$

$\Leftrightarrow s>0$

$$C = C'_R + \left[(-R+ix),\,(R+ix)\right]$$

$$\oint_C F(z)e^{isz}\,dz = 0 \quad \text{and} \quad \int_{C'_R} F(z)e^{isz}\,dz = 0$$

$(R\to\infty)$ $(R\to\infty)$

Jordan's Lemma!

Therefore: $\displaystyle f(t) = \int_{-\infty+ix}^{+\infty+ix} F(z)e^{isz}\,dz = 0$

$t<0$

$s>0$

3) Finally, we must prove $f(t)$ is indeed the inverse Laplace transform of $F(P)$

for an arbitrary $P_0 \mid Re\{P_0\} > a$, we make:

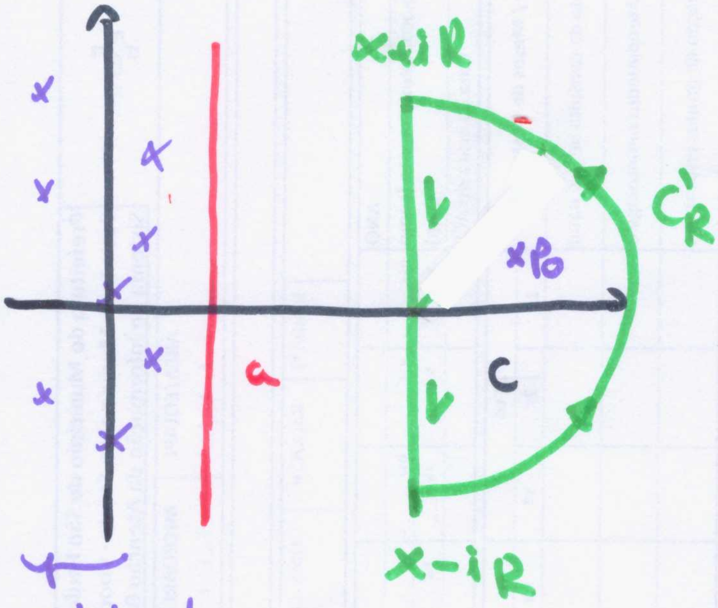
$$\int_0^\infty e^{-P_0t} f(t)\,dt = \frac{1}{2\pi i}\int_0^\infty e^{-P_0t}\left\{\int_{x-i\infty}^{x+i\infty} e^{pt}F(p)\,dp\right\}dt =$$

$$= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F(p)\left\{\int_0^\infty e^{-(P_0-P)t}\,dt\right\}dP = \underbrace{\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{F(P)\,dP}{(P_0-P)}}$$

$$\int_0^\infty e^{-(P_0-P)dt}\,dt = \left[\frac{-e^{-(P_0-P)t}}{(P_0-P)}\right]_0^\infty = \frac{1}{(P_0-P)}$$

Cauchy integral can be used on closing a contour.

for $Re\{P_0\} > x > a$



original singularities of $F(P)$

$$\frac{1}{2\pi i}\underbrace{\oint_C \frac{F(P)\,dP}{(P_0-P)}}_{} = -F(P_0)$$

$$\frac{1}{2\pi i}\cancel{\int_{C_R'} \frac{F(P)\,dP}{(P_0-P)}} + \underbrace{\int_{x+iR}^{x-iR} \frac{F(P)\,dP}{(P_0-P)}}_{}$$

0 as $R \to \infty$ (Jordan)

$$-\int_{x-iR}^{x+iR} \frac{F(P)}{(P_0-P)}\,dP = -F(P_0)$$

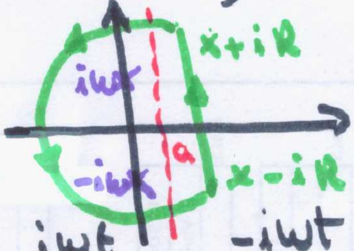
$$\Rightarrow \mathcal{L}\{f(t)\} = F(P_0)$$

$t > 0$

Example: $F(P) = \dfrac{\omega}{P^2+\omega^2}$

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{e^{Pt}\,\omega\,dP}{P^2+\omega^2} \qquad x > 0 \quad , Re\{P\} > 0 \, , \omega^2 > 0$$

on making use of the contour:



$$f(t) = \sum_{k=1}^{2} Res\left\{\frac{e^{Pt}\,\omega}{(P+i\omega)(P-i\omega)}, P_k\right\} = \frac{\omega e^{i\omega t}}{2i\omega} - \frac{\omega e^{-i\omega t}}{2i\omega} =$$

$$= \sin(\omega t)$$

$P_k = \pm i\omega$

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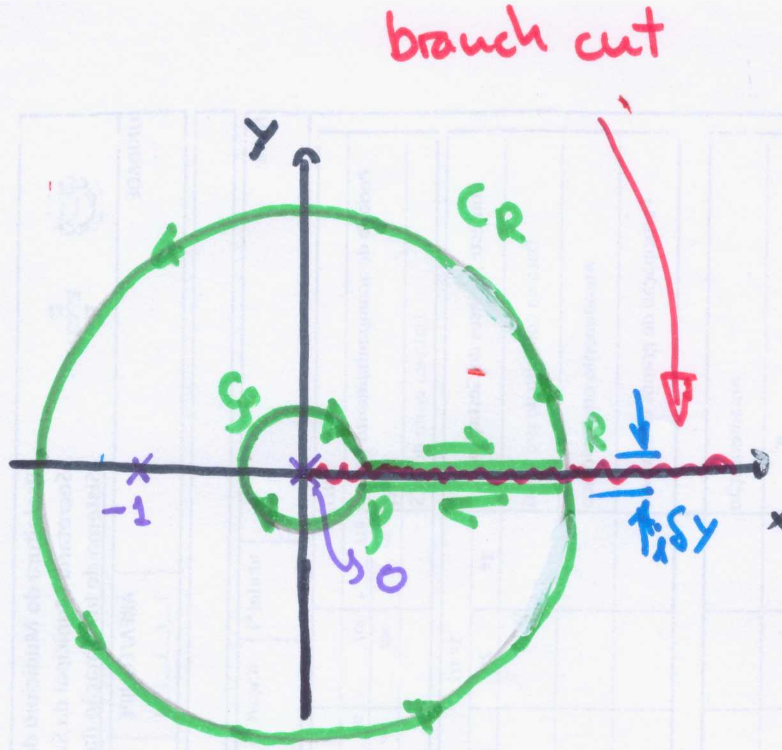


$$\int_0^\infty \frac{x^{-a}}{(x+1)}$$

$(0 < a < 1)$

$(x > 0)$

complex extension:

$$f_1(z) = \frac{z^{-a}}{(z+1)}$$

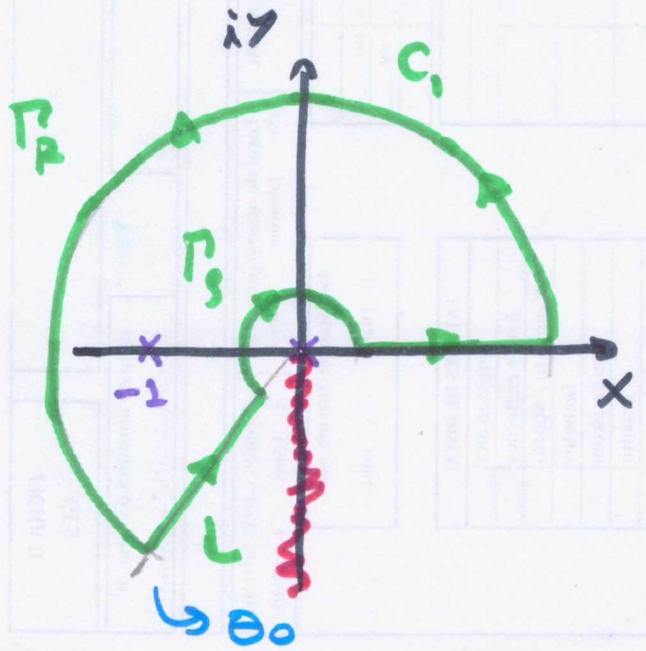
Churchill pages 193 – 195 (please read it) Symbolic Form

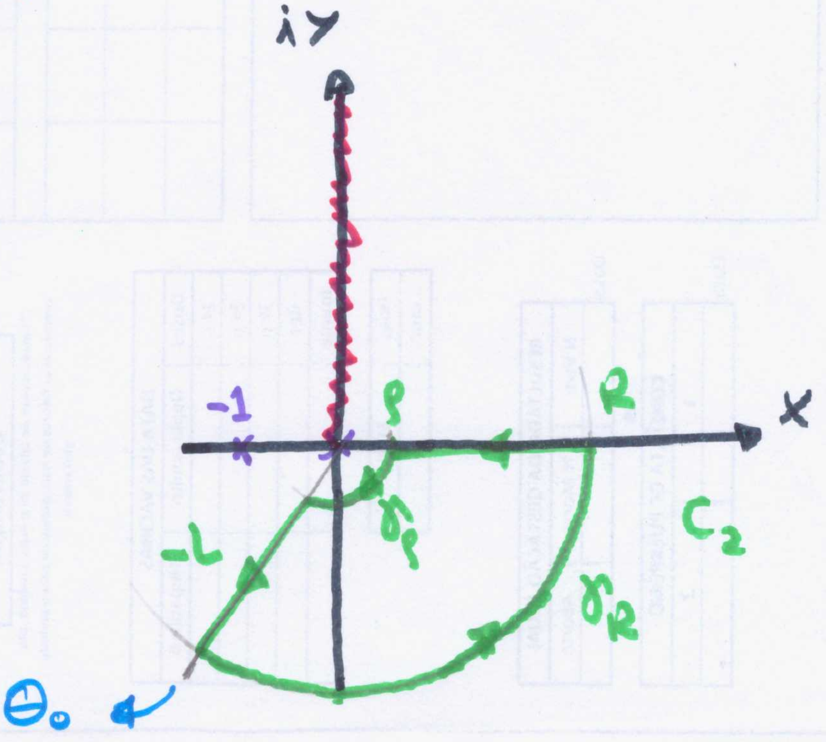
$\log(z) = \ln(r) + i\,\theta$

$z = re^{i\theta}$, $|z| = r > 0$

$\arg(z) = \theta \Rightarrow 0 < \theta < 2\pi$

The actual Proof of the result and the rationale behind it is in Problem 19 on Page 199 – which I solve below:





for C_1, we'll use the Branch:

$$|z|>0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \quad ; \quad f_1 = \frac{z^{-a}}{(z+1)}$$

$$\int_{\rho}^{R} \frac{r^{-\alpha}}{(r+1)} dr + \int_{\Gamma_R} f(z) dz + \int_L f(z) dz + \int_{\Gamma_\rho} f(z) dz =$$

$$= 2\pi i \ \operatorname{Res}\{f_1(z), -1\}$$

for C_2, we'll take the branch:

$$|z|>0,\ \frac{\pi}{2} < \theta < \frac{\pi 5}{2} \quad ; \quad f_2(z) = \frac{z^{-a}}{(z+1)}$$

$$-\int_{\rho}^{R} \frac{r^{-a} e^{-i2a\pi}}{r+1} dr + \int_{\gamma_\rho} f_2(z) dz - \int_L f_2(z) dz + \int_{\gamma_R} f_2(z) dz = 0$$

Solution

first, let's consider C_1 : $f_1(z) = \dfrac{r^{-a} e^{-ia\theta}}{re^{i\theta}+1}$

on $\Gamma_R \Rightarrow f_1(z) = \dfrac{R^{-a} e^{-ia\theta}}{Re^{i\theta}+1}$; $0 \leq \theta \leq \theta_0$, $\pi < \theta_0 < \frac{3\pi}{2}$

on $\Gamma_\rho \Rightarrow f_1(z) = \dfrac{\rho^{-a} e^{-ia\theta}}{\rho e^{i\theta}+1}$; $0 \leq \theta \leq \theta_0$

on $L \Rightarrow f_1(z) = \dfrac{r^{-a} e^{-ia\theta_0}}{re^{i\theta_0}+1}$; $\theta = \theta_0$; $\rho \leq r \leq R$

on $z = x \Rightarrow f_1(z) = \dfrac{r^{-a}}{r+1}$, $\theta = 0$, $\rho \leq r \leq R$

$$\oint_{C_1} f_1(z)\,dz = 2\pi i \,\mathrm{Res}\left[f_1(z), -1\right] = 2\pi i\, e^{-ia\pi}$$

$$\oint_{C_1} f_1(z)\,dz = \underbrace{\cancel{\int_{\Gamma_R} f_1(z)\,dz}}_{0} + \int_{L} f_1(z)\,dz + \underbrace{\cancel{\int_{\Gamma_\rho} f_1(z)\,dz}}_{0} + \int_{\rho}^{R} \frac{r^{-a}}{r+1} = 2\pi i\, e^{-ia\pi}$$

proof is on my notes $\rightarrow$ in the limits as $\rho \to 0$ and $R \to \infty$

Now, on accounting for C_2, we have:

$$f_2(z) = \frac{z^{-a}}{z+1}\,; \quad z = re^{i\theta}\,; \quad |z| > 0\,; \ \frac{\pi}{2} < \theta < \frac{5\pi}{2}$$

$$L : \theta_0 \Rightarrow \pi < \theta_0 < \frac{3\pi}{2}$$

on $\gamma_R \Rightarrow f_2(z) = \dfrac{R^{-a}e^{-ia\theta}}{Re^{i\theta}+1}$ $\quad \theta_0 \leq \theta \leq 2\pi$

on $\gamma_\rho \Rightarrow f_2(z) = \dfrac{\rho^{-a}e^{-ia\theta}}{\rho e^{i\theta}+1}$; $\theta_0 \leq \theta \leq 2\pi$

on $L \Rightarrow f_2(z) = \dfrac{r^{-a}e^{-ia\theta_0}}{re^{i\theta_0}+1}$; $\theta = \theta_0$, $\rho \leq r \leq R$

on $z = x \Rightarrow f_2(z) = \dfrac{r^{-a}e^{-ia2\pi}}{r+1}$; $\theta = 2\pi$; $\rho \leq r \leq R$

the proof is on my notes

$$\oint_{C_2} f_2(z)\,dz = 0$$

$$\oint_{C_2} f_2(z)\,dz = \underbrace{\int_{\gamma_R} f_2(z)\,dz}_{0 \text{ as } R\to\infty} - \int_{\rho}^{R} \frac{r^{-a}e^{-ia2\pi}}{r+1}\,dr + \underbrace{\int_{\gamma_\rho} f_2(z)\,dz}_{0 \text{ as } \rho\to 0} +$$

$$- \int_{L} f_2(z)\,dz = 0$$

Therefore, on joining C_1 and C_2 we get; for $f(z) = \dfrac{z^{-a}}{z+1}$; $z = re^{i\theta}$, $|z|>0$

$(0 < \theta < 2\pi)$

$$\oint_{C} f(z)\,dz = \oint_{C_1} f_1(z)\,dz + \int_{C_2} f_2(z)\,dz$$

$$\int_{\rho}^{R} \frac{r^{-a}}{r+1}\,dr + \underbrace{\int_{C_R} f(z)\,dz}_{0\,(R\to\infty)} + \int_{L} f(z)\,dz + \underbrace{\int_{C_\rho} f(z)\,dz}_{0\,(\rho\to 0)} +$$

$$- \int_{\rho}^{R} \frac{r^{-a}e^{-ia2\pi}}{r+1}\,dr - \int_{L} f(z)\,dz = 2\pi i\, e^{-ia\pi}$$

$$\lim_{\substack{R\to\infty \\ \rho\to 0}} \int_{\rho}^{R} \frac{(1-e^{ia2\pi})\cdot r^{-a}}{(r+1)}\,dr = 2\pi i\, e^{-ia\pi}$$

$$\int_0^\infty \frac{r^{-a}}{(r+1)}dr = \frac{2\pi i\, e^{-ia\pi}}{(1-e^{-ia2\pi})} = \frac{2\pi i}{e^{ia\pi}(1-e^{-ia2\pi})}$$

$$\int_0^\infty \frac{r^{-a}}{(r+1)}dr = \frac{\pi}{\sin(a\pi)} \quad ; \quad 0<a<1$$

$$\sin(\alpha) = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$$

Therefore we finally get:

$$\boxed{\int_0^\infty \frac{x^{-a}}{(x+1)}dx = \frac{\pi}{\sin(a\pi)} \quad \text{for } x>0 \text{ and } (0<a<1)}$$