

PQI5776 - Fenômenos de Transporte I - 2020

Difusão em um filme líquido descendente – Absorção gasosa com reação química. Aplicação desenvolvida a partir do exemplo apresentado no §18.5 do Bird (2002). Incluir no exemplo a reação de consumo de A na fase líquida por uma reação de primeira ordem, expressa por: $-r_A = k''\rho_A$.

- a) Obter uma expressão para a taxa de absorção do gás A no solvente B em função dos adimensionais Pe e Da .
- b) Analisar o caso de reação lenta. Obter uma expressão simplificada para a taxa de absorção.
- c) Analisar o caso de reação rápida. Obter uma expressão simplificada para a taxa de absorção.

§ 18.5

$$\frac{\partial p_{\omega A}}{\partial t} + \text{div}(\rho_A \vec{v}) = -\text{div} \vec{j}_D + \dot{r}_A$$

$$\frac{\partial p_{\omega A}}{\partial t} + \text{div}(\underbrace{\rho_A \vec{v} + \vec{j}_D}_{\vec{n}_A}) = \dot{r}_A$$

$$\vec{n}_A = \vec{j}_D + \omega_A (\vec{n}_A + \vec{n}_B)$$

No caso da absorçao no ligado.

$$n_{Ax} = j_{Dx} + \omega_A (n_{Ax} + n_{Bx})$$

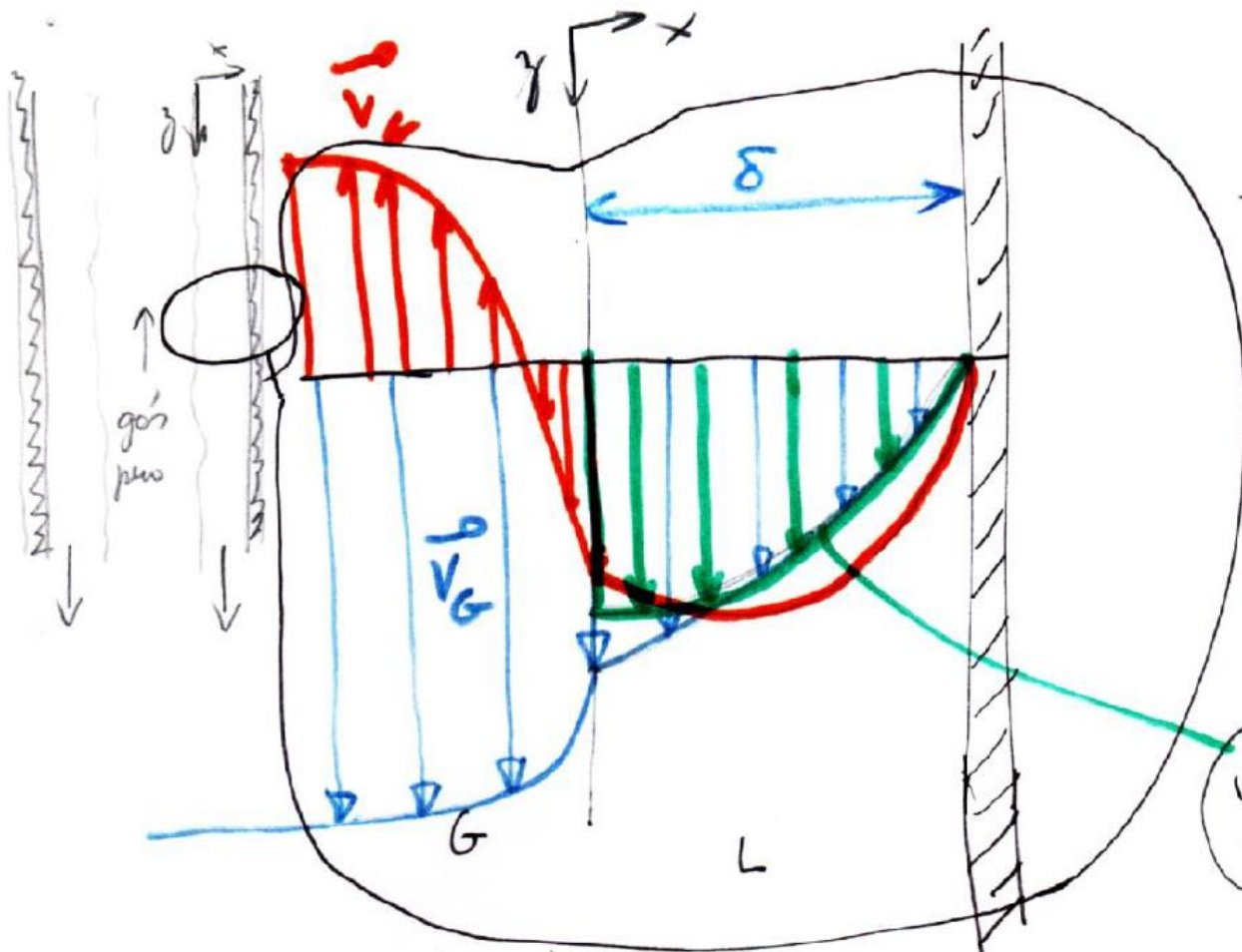
$$n_{Ax} = \frac{j_{Dx}}{(1 - \omega_A)}$$

o diluido!

n_{Ax} e absorvido
ou desordenado

$$n_{Ax} \Rightarrow j_{Dx} = K (\rho_{A0} - \rho_{A\infty})$$

↑
geometria, esvanecimento, difusao



Hipóteses

- filme laminar (s/ ondulações)
- cisalhamentos do gás desprezível
- diluído
- equilíbrio na interface

$$v_z = v_{máx} \left[1 - \frac{x^2}{\delta^2} \right]$$

$$\mu_G \frac{dv_G}{dx} \Big|_{x=0} = \mu_L \frac{dv_L}{dx} \Big|_{x=0}$$

- Solução diluída \Rightarrow perfil de velocidades não afetado pelo T. de Massa.

Conservação de A - filme líquido

$$\rho \frac{dW_A}{dt} = \rho \frac{\partial W_A}{\partial t} + \text{div } \rho W_A \vec{v} = -\text{div } \vec{j}_A^0 + \dot{r}_A^0$$

regime permanente, incompressível, bidimensional, ρ cte e D_{AB} cte.

$$\vec{v} \cdot \text{grad } \rho_A = + D_{AB} \Delta \rho_A + \dot{r}_A^0$$

malha homogênea de 1ª ordem e univariável:

$$\dot{r}_A^0 = -K_1'' \rho_A$$

$$v_x \frac{\partial \rho_A}{\partial x} + v_z \frac{\partial \rho_A}{\partial z} = + D_{AB} \left(\frac{\partial^2 \rho_A}{\partial x^2} + \frac{\partial^2 \rho_A}{\partial z^2} \right) - K_1'' \rho_A$$

0 - escoamento desenvolvido

$$\rightarrow v_{\max} \left[1 - \frac{x^2}{\delta^2} \right] \frac{\partial p_A}{\partial z} = +D_{AB} \left(\frac{\partial^2 p_A}{\partial x^2} + \frac{\partial^2 p_A}{\partial z^2} \right) - K_1'' p_A$$

$$\rightarrow \check{p}_A = \frac{p_A}{p_{A0}} ; \quad \check{x} = \frac{x}{\delta} ; \quad \check{z} = \frac{z}{L} ; \quad \check{v}_3 = \frac{v_3}{v_{\max}}$$

$$\frac{p_{A0} v_{\max}}{L} \left[1 - \check{x}^2 \right] \frac{\partial \check{p}_A}{\partial \check{z}} = \frac{D_{AB}}{L^2} \left[\frac{\partial^2 \check{p}_A}{\partial \check{x}^2} + \frac{\partial^2 \check{p}_A}{\partial \check{z}^2} \right] - K_1'' \frac{p_{A0}}{p_{A0}} \check{p}_A$$

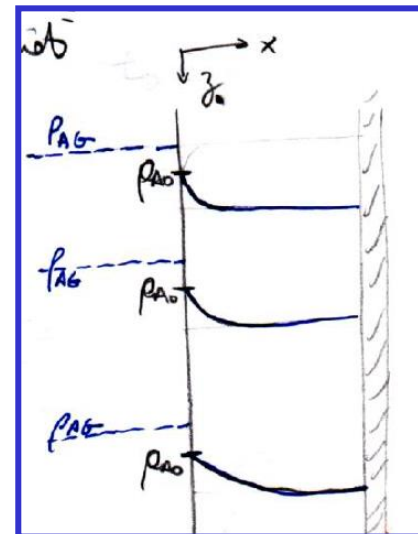
$$\rightarrow \frac{v_{\max} \delta^2}{D_{AB} L} \left[1 - \check{x}^2 \right] \frac{\partial \check{p}_A}{\partial \check{z}} = \frac{\partial^2 \check{p}_A}{\partial \check{x}^2} + \frac{\delta^2}{L^2} \frac{\partial^2 \check{p}_A}{\partial \check{z}^2} - \frac{K_1'' \delta^2}{D_{AB}} \check{p}_A$$

$$\boxed{\frac{\delta}{L} Pe (1 - \check{x}^2) \frac{\partial \check{p}_A}{\partial \check{z}} = \frac{\partial^2 \check{p}_A}{\partial \check{x}^2} - DA \check{p}_A} \quad \text{+ comparison with a 18.5.7}$$

$$Pe = \frac{v_{\max} \delta}{D_{AB}} \quad DA = \frac{K_1'' \delta^2}{D_{AB}} = \frac{t_D}{t_R}$$

$$v_{\max} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] \frac{\partial c_A}{\partial z} = D_{AB} \frac{\partial^2 c_A}{\partial x^2} \quad (18.5-7)$$

$$v_{\max} \frac{\partial c_A}{\partial z} = D_{AB} \frac{\partial^2 c_A}{\partial x^2} \quad (18.5-11)$$



Simplificação: tempo de contato baixo $t_c = \frac{L}{v_{max}}$

"ou difusão lenta no líquido"

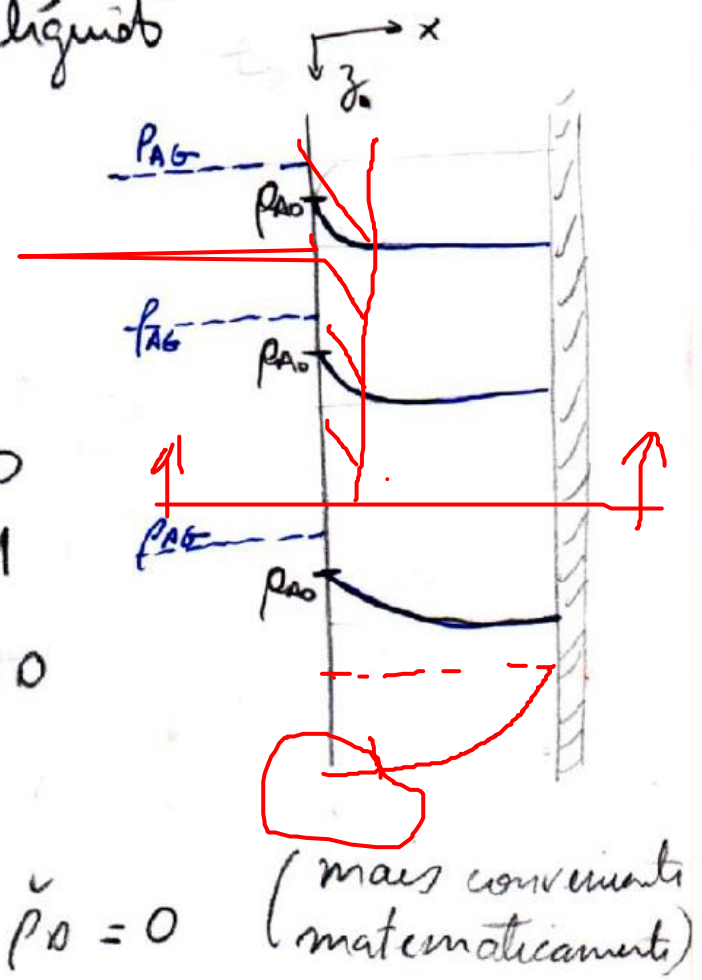
$$\tilde{x} \ll 1$$

$$\frac{\delta}{L} Pe \frac{\partial \tilde{p}_A}{\partial \tilde{z}} = \frac{\partial^2 \tilde{p}_A}{\partial \tilde{x}^2} - Da \tilde{p}_A$$

- $z=0, p_A=0 \rightarrow \tilde{z}=0, \tilde{p}_A=0$
- $x=0, p_A=p_{Ao} \rightarrow \tilde{x}=0, \tilde{p}_A=1$
- $x=\delta, \frac{\partial p_A}{\partial x} = 0 \rightarrow \tilde{x}=1, \frac{\partial \tilde{p}_A}{\partial \tilde{x}} = 0$

substituir por:

$$x \rightarrow \infty, p_A=0 \rightarrow \tilde{x} \rightarrow \infty, \tilde{p}_A=0$$



$$n_{Ax}|_{x=0} = \left. \frac{dA}{dx} \right|_{x=0} + w_A \underbrace{\frac{n_x}{\rho v_x}} \Rightarrow -D_{AB} \left. \frac{dA}{dx} \right|_{x=0}$$

$$\parallel$$

$$-D_{AB} \left. \frac{dA}{dx} \right|_{x=0}$$

$$\parallel$$

$$-\rho_{A0} \frac{D_{AB}}{\delta} \left. \frac{dA}{dx} \right|_{x=0}$$

"Sherwood" pg 315

$$W_A = \int_0^w \int_0^L n_{Ax}|_{x=0} dz dy \quad \left(\begin{array}{l} \text{taxa de transporte} \\ \text{(kg/s)} \end{array} \right)$$

$$W_A = wL \rho_{A0} D_{AB} \frac{Pe}{\sqrt{Da}} \left[\left(\frac{1}{2} + u \right) \operatorname{erf} \sqrt{u} + \sqrt{\frac{u}{\pi}} e^{-u} \right]$$

sendo $\left[u = \frac{K_1^m L}{v_{max}} = \frac{Da}{Pe} \frac{L}{\delta} \right] = \frac{t_c}{t_R}$

LEMBRAR $\rho_{A00} = 0$!

- Assinção de reação (ou reação muito lenta)

$$t_c \ll t_R \rightarrow u \rightarrow 0$$

$(t_c \ll t_R)$

$$u \rightarrow 0 \begin{cases} \operatorname{erf} u \cong \frac{2}{\sqrt{\pi}} \left(u - \frac{u^3}{6} \dots \right) \cong \frac{2}{\sqrt{\pi}} u \\ \operatorname{erf} \sqrt{u} \cong \frac{2}{\sqrt{\pi}} \left(\sqrt{u} - \frac{u\sqrt{u}}{6} \dots \right) \cong \frac{2}{\sqrt{\pi}} \sqrt{u} \\ e^{-u} = 1 - u + \frac{u^2}{2!} \dots \cong 1 - u \end{cases}$$

$$\left(\frac{1}{2} + \underset{0}{u} \right) \cdot \frac{2}{\sqrt{\pi}} \sqrt{u} + \frac{\sqrt{u}}{\sqrt{\pi}} (1 - \underset{0}{u}) \cong \frac{2\sqrt{u}}{\sqrt{\pi}}$$

$$W_A = w_{A0} D_{AB} \frac{P_e}{\sqrt{D_a}} \times \frac{2\sqrt{u}}{\sqrt{\pi}} = w_{A0} D_{AB} \frac{P_e}{\sqrt{D_a}} \times \frac{\sqrt{D_a}}{\sqrt{P_e}} \sqrt{\frac{L}{\delta}} \times \frac{2}{\sqrt{\pi}}$$

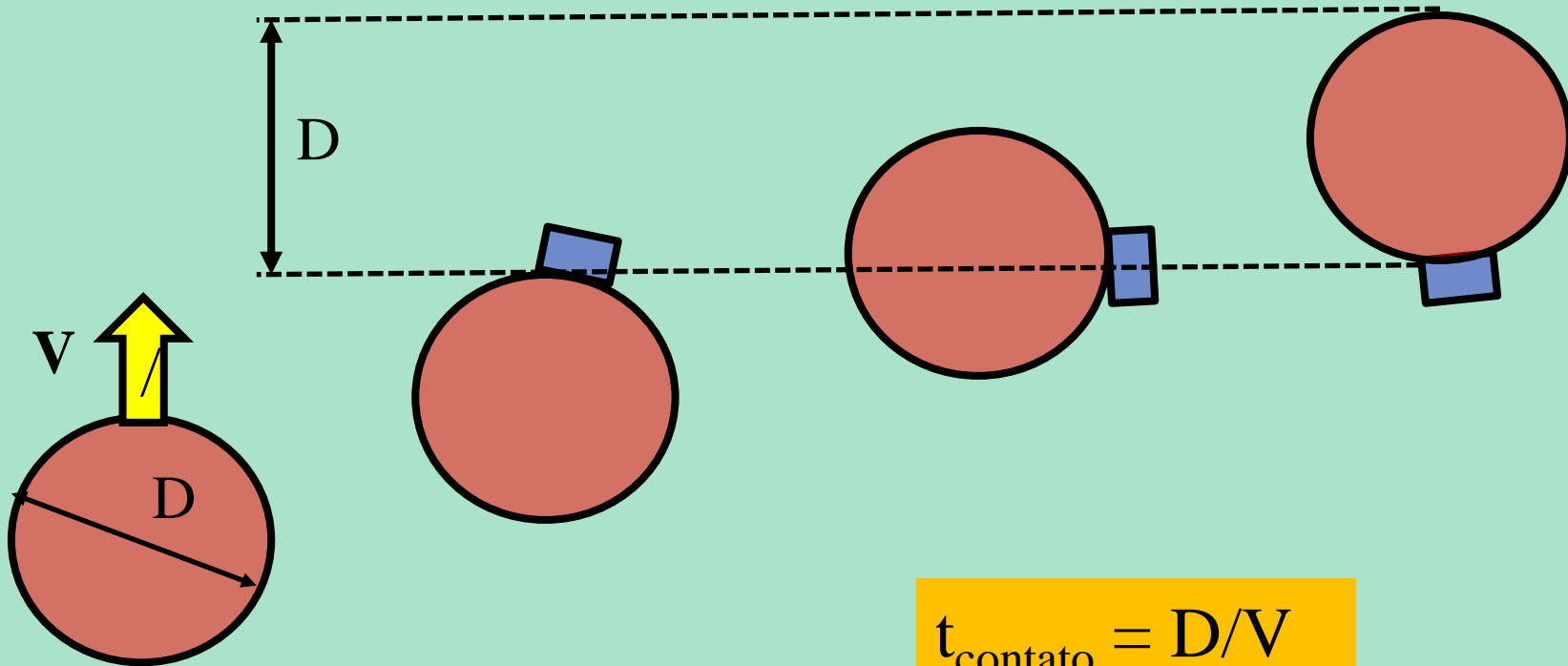
$$W_A = w_{A0} D_{AB} \frac{2}{\sqrt{\pi}} \sqrt{P_e} \sqrt{\frac{L}{\delta}} = w_{A0} L \sqrt{\frac{4 D_{AB} v_{\max}}{\pi L}}$$

Absorção "física"

$$\frac{W_A}{WL} = \text{fluxo médio} = k_L (p_{A0} - 0) \Rightarrow$$

$$k_L = \sqrt{\frac{4 D_{AB} v_{\max}}{\mu L}}$$

Teoria de Higbie $\Delta 1/t_{\text{contato}}$



- Reação rápida

$$t_R \ll t_c \rightarrow u \gg 1$$

$$u \gg 1 \left\{ \begin{array}{l} \operatorname{erf} w \approx 1 - \frac{e^{-u^2}}{\sqrt{\pi}} \left(\frac{1}{u} - \frac{1}{2u^3} \right) \approx 1,0 \\ \operatorname{erf} \sqrt{u} \approx 1 - \frac{e^{-u}}{\sqrt{\pi}} \left(\frac{1}{\sqrt{u}} - \frac{1}{2\sqrt{u} \cdot u} \right) \approx 1,0 \\ e^{-u} \approx 0 \end{array} \right. \left. \begin{array}{l} \text{vide} \\ \text{tabela} \end{array} \right.$$

$$\left(\frac{1}{2} + u \right) \times 1 + \sqrt{\frac{4}{\pi}} \times 0 = \frac{1}{2} + u \approx u$$

$$W_A = w_{A0} D_{AB} \frac{Pe}{\sqrt{Da}} \times u = w_{A0} D_{AB} \frac{Pe}{\sqrt{Da}} \times \frac{Da}{Pe} \frac{L}{\delta}$$

$$W_A = w_{A0} D_{AB} \frac{L}{\delta} \sqrt{Da} = w_{A0} L \sqrt{2D_{AB} k_1''}$$

$$\frac{W_A}{wL} = \rho_{A0} \sqrt{2D_{AB} k_1''} = k_L (\rho_{A0} - 0) \Rightarrow$$

$$k_L = \sqrt{D_{AB} k_1''}$$

e $p/k_1'' \rightarrow \infty$?

→ não depende do escoamento!

$$W_A = w_{PAO} D_{AB} \frac{Pe}{\sqrt{Da}} \times w = w_{PAO} D_{AB} \frac{Pe}{\sqrt{Da}} \times \frac{Da L}{Pe \delta}$$

$$W_A = w_{PAO} D_{AB} \frac{L}{\delta} \sqrt{Da} = w_{PAO} L \sqrt{D_{AB} K_1''}$$

$$\frac{W_A}{wL} = \rho_{AO} \sqrt{D_{AB} K_1''} = k_L (\rho_{AO} - 0) \Rightarrow$$

$$k_L = \sqrt{D_{AB} K_1''}$$

e $p/K_1'' \rightarrow \infty$?

não depende do escoamento!



critérios para $\ddot{x} \ll 1 \rightarrow x \ll \delta$

$$t_c \ll t_D$$

$$\frac{L}{v_{max}} \ll \frac{\delta^2}{D_{AB}}$$

$$\frac{L}{\frac{1}{2} \frac{g}{v} \delta^2} \ll \frac{\delta^2}{D_{AB}}$$

$$\langle v \rangle = \frac{2}{3} v_{max} = \frac{1}{3} \frac{g}{v} \delta^2$$

$$v_{max} = \frac{1}{2} \frac{g}{v} \delta^2$$

$$\delta \gg \left(\frac{2 D_{AB} v L}{g} \right)^{1/4}$$

Exercício 12.A.4 - 2 - lista 4

$$\delta \gg 0,1 \text{ mm} \checkmark$$

GENERAÇÃO DE TURBULENTO - FILME

§18.5 DIFFUSION INTO A FALLING LIQUID FILM (GAS ABSORPTION)¹

In this section we present an illustration of *forced-convection* mass transfer, in which viscous flow and diffusion occur under such conditions that the velocity field can be considered virtually unaffected by the diffusion. Specifically, we consider the absorption of gas *A* by a laminar falling film of liquid *B*. The material *A* is only slightly soluble in *B*, so that the viscosity of the liquid is unaffected. We shall make the further restriction that the diffusion takes place so slowly in the liquid film that *A* will not "penetrate" very far into the film—that is, that the penetration distance will be small in comparison with the film thickness. The system is sketched in Fig. 18.5-1. An example of this kind of system occurs in the absorption of O_2 in H_2O .

Let us now set up the differential equations describing the diffusion process. First, we have to solve the momentum transfer problem to obtain the velocity profile $v_z(x)$ for the film; this has already been worked out in §2.2 in the absence of mass transfer at the fluid surface, and we know that the result is

$$v_z(x) = v_{\max} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] \quad (18.5-1)$$

provided that "end effects" are ignored.

Next we have to establish a mass balance on component *A*. We note that c_A will be changing with both x and z . Hence, as the element of volume for the mass balance, we select the volume formed by the intersection of a slab of thickness Δz with a slab of thickness Δx . Then the mass balance on *A* over this segment of a film of width W becomes

$$N_{Az}|_z W \Delta x - N_{Az}|_{z+\Delta z} W \Delta x + N_{Ax}|_x W \Delta z - N_{Ax}|_{x+\Delta x} W \Delta z = 0 \quad (18.5-2)$$

Dividing by $W \Delta x \Delta z$ and performing the usual limiting process as the volume element becomes infinitesimally small, we get

$$\frac{\partial N_{Az}}{\partial z} + \frac{\partial N_{Ax}}{\partial x} = 0 \quad (18.5-3)$$

¹ S. Lynn, J. R. Straatemeier, and H. Kramers, *Chem. Engr. Sci.*, 4, 49-67 (1955).

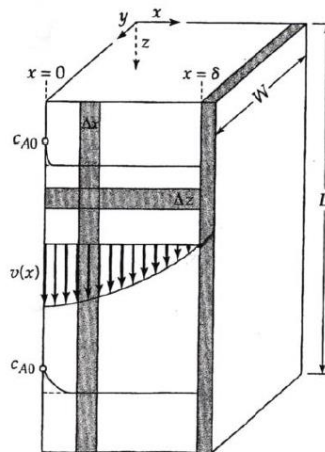


Fig. 18.5-1. Absorption of *A* into a falling film of liquid *B*.

Into this equation we now insert the expression for N_{Az} and N_{Ax} , making appropriate simplifications of Eq. 18.0-1. For the molar flux in the z direction, we write, assuming constant c ,

$$N_{Az} = -\mathcal{D}_{AB} \frac{\partial c_A}{\partial z} + \underline{x_A(N_{Az} + N_{Bz})} \approx c_A v_z(x) \quad (18.5-4)$$

We discard the dashed-underlined term, since the transport of A in the z direction will be primarily by convection. We have made use of Eq. (M) in Table 17.8-1 and the fact that \mathbf{v} is almost the same as \mathbf{v}^* in dilute solutions. The molar flux in the x direction is

$$N_{Ax} = -\mathcal{D}_{AB} \frac{\partial c_A}{\partial x} + \underline{x_A(N_{Ax} + N_{Bx})} \approx \mathcal{D}_{AB} \frac{\partial c_A}{\partial x} \quad (18.5-5)$$

Here we neglect the dashed-underlined term because in the x direction A moves predominantly by diffusion, there being almost no convective transport normal to the wall on account of the very slight solubility of A in B . Combining the last three equations, we then get for constant \mathcal{D}_{AB}

$$v_z \frac{\partial c_A}{\partial z} = \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial x^2} \quad (18.5-6)$$

Finally, insertion of Eq. 18.5-1 for the velocity distribution gives

$$v_{\max} \left[1 - \left(\frac{x}{\delta} \right)^2 \right] \frac{\partial c_A}{\partial z} = \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial x^2} \quad (18.5-7)$$

as the differential equation for $c_A(x, z)$.

Equation 18.5-7 is to be solved with the following boundary conditions:

$$\text{B.C. 1:} \quad \text{at } z = 0, \quad c_A = 0 \quad (18.5-8)$$

$$\text{B.C. 2:} \quad \text{at } x = 0, \quad c_A = c_{A0} \quad (18.5-9)$$

$$\text{B.C. 3:} \quad \text{at } x = \delta, \quad \frac{\partial c_A}{\partial x} = 0 \quad (18.5-10)$$

The first boundary condition corresponds to the fact that the film consists of pure B at the top ($z = 0$), and the second indicates that at the liquid-gas interface the concentration of A is determined by the solubility of A in B (that is, c_{A0}). The third boundary condition states that A cannot diffuse through the solid wall. This problem has been solved analytically in the form of an infinite series,² but we do not give that solution here. Instead, we seek only a limiting expression valid for "short contact times," that is, for small values of L/v_{\max} .

If, as indicated in Fig. 18.5-1, the substance A has penetrated only a short distance into the film, then the species A "has the impression" that the film is moving throughout with a velocity equal to v_{\max} . Furthermore if A does not penetrate very far, it does not "sense" the presence of the solid wall at $x = \delta$. Hence, if the film were of infinite thickness moving with the velocity v_{\max} , the diffusing material "would not know the difference." This physical argument suggests (correctly) that we will get a very good result if we replace Eq. 18.5-7 and its boundary conditions by

$$v_{\max} \frac{\partial c_A}{\partial z} = \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial x^2} \quad (18.5-11)$$

$$\text{B.C. 1:} \quad \text{at } z = 0, \quad c_A = 0 \quad (18.5-12)$$

$$\text{B.C. 2:} \quad \text{at } x = 0, \quad c_A = c_{A0} \quad (18.5-13)$$

$$\text{B.C. 3:} \quad \text{at } x = \infty, \quad c_A = 0 \quad (18.5-14)$$

² R. L. Pigford, PhD thesis, University of Illinois (1941).

An exactly analogous problem occurred in Example 4.1-1, which was solved by the method of combination of variables. It is therefore possible to take over the solution to that problem just by changing the notation. The solution is³

$$\frac{c_A}{c_{A0}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4\mathcal{D}_{AB}z/v_{\max}}} \exp(-\xi^2) d\xi \quad (18.5-15)$$

or

$$\boxed{\frac{c_A}{c_{A0}} = 1 - \operatorname{erf} \frac{x}{\sqrt{4\mathcal{D}_{AB}z/v_{\max}}} = \operatorname{erfc} \frac{x}{\sqrt{4\mathcal{D}_{AB}z/v_{\max}}}} \quad (18.5-16)$$

In these expressions “erf x ” and “erfc x ” are the “error function” and the “complementary error function” of x , respectively. They are discussed in §C.6 and tabulated in standard reference works.⁴

Once the concentration profiles are known, the local mass flux at the gas-liquid interface may be found as follows:

$$N_{Ax}|_{x=0} = -\mathcal{D}_{AB} \left. \frac{\partial c_A}{\partial x} \right|_{x=0} = c_{A0} \sqrt{\frac{\mathcal{D}_{AB}v_{\max}}{\pi z}} \quad (18.5-17)$$

Then the total molar flow of A across the surface at $x = 0$ (i.e., being absorbed by a liquid film of length L and width W) is

$$\begin{aligned} W_A &= \int_0^W \int_0^L N_{Ax}|_{x=0} dz dy \\ &= Wc_{A0} \sqrt{\frac{\mathcal{D}_{AB}v_{\max}}{\pi}} \int_0^L \frac{1}{\sqrt{z}} dz \\ &= W L c_{A0} \sqrt{\frac{4\mathcal{D}_{AB}v_{\max}}{\pi L}} \end{aligned} \quad (18.5-18)$$

The same result is obtained by integrating the product $v_{\max}c_A$ over the flow cross section at $z = L$ (see Problem 18C.3).

Equation 18.5-18 shows that the mass transfer rate is directly proportional to the square root of the diffusivity and inversely proportional to the square root of the “exposure time,” $t_{\text{exp}} = L/v_{\max}$. This approach for studying gas absorption was apparently first proposed by Higbie.⁵

The problem discussed in this section illustrates the “penetration model” of mass transfer. This model is discussed further in Chapters 20 and 22.

We now turn to a falling film problem that is different from the one discussed in the previous section. Liquid *B* is flowing in laminar motion down a vertical wall as shown in Fig. 18.6-1. The film begins far enough up the wall so that v_z depends only on y for $z \geq 0$. For $0 < z < L$ the wall is made of a species *A* that is slightly soluble in *B*.

For short distances downstream, species *A* will not diffuse very far into the falling film. That is, *A* will be present only in a very thin boundary layer near the solid surface. Therefore the diffusing *A* molecules will experience a velocity distribution that is characteristic of the falling film right next to the wall, $y = 0$. The velocity distribution is given in Eq. 2.2-18. In the present situation $\cos \theta = 1$, and $x = \delta - y$, and

$$v_z = \frac{\rho g \delta^2}{2\mu} \left[1 - \left(1 - \frac{y}{\delta} \right)^2 \right] = \frac{\rho g \delta^2}{2\mu} \left[2 \left(\frac{y}{\delta} \right) - \left(\frac{y}{\delta} \right)^2 \right] \quad (18.6-1)$$

At and adjacent to the wall $(y/\delta)^2 \ll (y/\delta)$, so that for this problem the velocity is, to a very good approximation, $v_z = (\rho g \delta / \mu) y = ay$. This means that Eq. 18.5-6, which is applicable here, becomes for short distances downstream

$$ay \frac{\partial c_A}{\partial z} = \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial y^2} \quad (18.6-2)$$

where $a = \rho g \delta / \mu$. This equation is to be solved with the boundary conditions

$$\text{B.C. 1:} \quad \text{at } z = 0, \quad c_A = 0 \quad (18.6-3)$$

$$\text{B.C. 2:} \quad \text{at } y = 0, \quad c_A = c_{A0} \quad (18.6-4)$$

$$\text{B.C. 3:} \quad \text{at } y = \infty, \quad c_A = 0 \quad (18.6-5)$$

In the second boundary condition, c_{A0} is the solubility of *A* in *B*. The third boundary condition is used instead of the correct one ($\partial c_A / \partial y = 0$ at $y = \delta$), since for short contact times we feel intuitively that it will not make any difference. After all, since the mole-

¹ H. Kramers and P. J. Kreyger, *Chem. Eng. Sci.*, **6**, 42-48 (1956); see also R. L. Pigford, *Chem. Eng. Prog. Symposium Series No. 17*, Vol. 51, pp. 79-92 (1955) for the analogous heat-conduction problem.

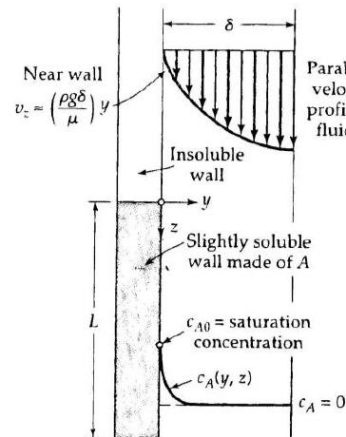


Fig. 18.6-1. Solid *A* dissolving into a falling film of liquid *B*, moving with a fully developed parabolic velocity profile.

cles of A penetrate only slightly into the film, they cannot get far enough to “see” the outer boundary of the film, and hence they cannot distinguish between the true boundary condition and the approximate boundary condition that we use. The same kind of reasoning was encountered in Example 12.2-2 and Problem 12B.4.

The form of the boundary conditions in Eqs. 18.6-3 to 5 suggests the method of combination of variables. Therefore we try $c_A/c_{A0} = f(\eta)$, where $\eta = y(a/9\mathcal{D}_{AB}z)^{1/3}$. This combination of the independent variables can be shown to be dimensionless, and the factor of “9” is included to make the solution look neater.

When this change of variable is made, the partial differential equation in Eq. 18.6-2 reduces to an ordinary differential equation

$$\frac{d^2f}{d\eta^2} + 3\eta^2 \frac{df}{d\eta} = 0 \quad (18.6-6)$$

with the boundary conditions $f(0) = 1$ and $f(\infty) = 0$.

This second-order equation, which is of the form of Eq. C.1-9, has the solution

$$f = C_1 \int_0^\eta \exp(-\bar{\eta}^3) d\bar{\eta} + C_2 \quad (18.6-7)$$

The constants of integration can then be evaluated using the boundary conditions, and one obtains finally

$$\boxed{\frac{c_A}{c_{A0}} = \frac{\int_\eta^\infty \exp(-\bar{\eta}^3) d\bar{\eta}}{\int_0^\infty \exp(-\bar{\eta}^3) d\bar{\eta}} = \frac{\int_\eta^\infty \exp(-\bar{\eta}^3) d\bar{\eta}}{\Gamma(\frac{4}{3})}} \quad (18.6-8)$$

for the concentration profiles, in which $\Gamma(\frac{4}{3}) = 0.8930 \dots$ is the gamma function of $\frac{4}{3}$. Next the local mass flux at the wall can be obtained as follows

$$\begin{aligned} N_{Ay}|_{y=0} &= -\mathcal{D}_{AB} \frac{\partial c_A}{\partial y} \Big|_{y=0} = -\mathcal{D}_{AB} c_{A0} \left[\frac{d}{d\eta} \left(\frac{c_A}{c_{A0}} \right) \frac{\partial \eta}{\partial y} \right] \Big|_{y=0} \\ &= -\mathcal{D}_{AB} c_{A0} \left[\frac{\exp(-\eta^3)}{\Gamma(\frac{4}{3})} \left(\frac{a}{9\mathcal{D}_{AB}z} \right)^{1/3} \right] \Big|_{y=0} = + \frac{\mathcal{D}_{AB} c_{A0}}{\Gamma(\frac{4}{3})} \left(\frac{a}{9\mathcal{D}_{AB}z} \right)^{1/3} \end{aligned} \quad (18.6-9)$$

Then the molar flow of A across the entire mass transfer surface at $y = 0$ is

$$W_A = \int_0^W \int_0^L N_{Ay}|_{y=0} dz dx = \frac{2\mathcal{D}_{AB} c_{A0} WL}{\Gamma(\frac{2}{3})} \left(\frac{a}{9\mathcal{D}_{AB}L} \right)^{1/3} \quad (18.6-10)$$

where $\Gamma(\frac{2}{3}) = \frac{4}{3} \Gamma(\frac{4}{3}) = 1.1907 \dots$

The problem discussed in §18.5 and the one discussed here are examples of two types of asymptotic solutions that are discussed further in §20.2 and §20.3 and again in Chapter 22. It is therefore important that these two problems be thoroughly understood. Note that in §18.5, $W_A \propto (\mathcal{D}_{AB}L)^{1/2}$, whereas in this section $W_A \propto (\mathcal{D}_{AB}L)^{2/3}$. The differences in the exponents reflect the nature of the velocity gradient at the mass transfer interface: in §18.5, the velocity gradient was zero, whereas in this section, the velocity gradient is nonzero.