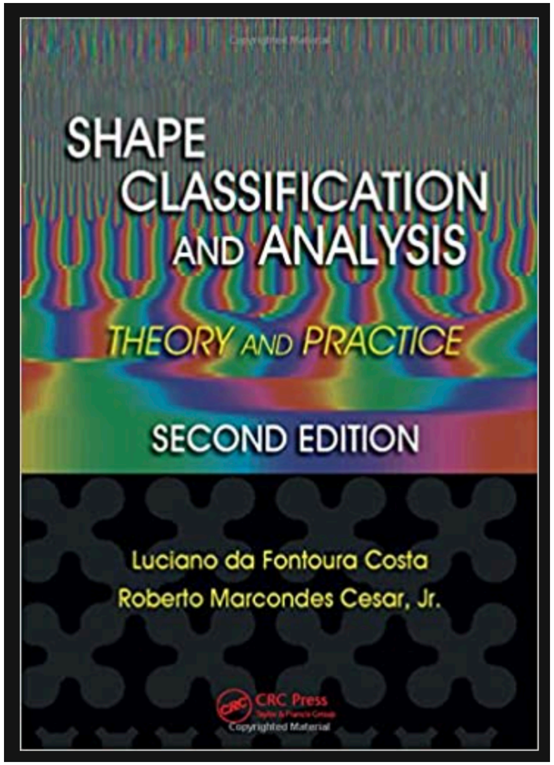
Fourier analysis



Roberto M. Cesar-Jr
IME-USP
cesar@ime.usp.br

Fourier analysis

- 3 transforms and 1 algorithm:
  - Fourier Series
  - Continuous Fourier Transform
  - Discrete Fourier Transform
  - Fast Fourier Transform

Fourier Series

- The Fourier series of a periodic function $g(t)$, with period $2L$:

$$g(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

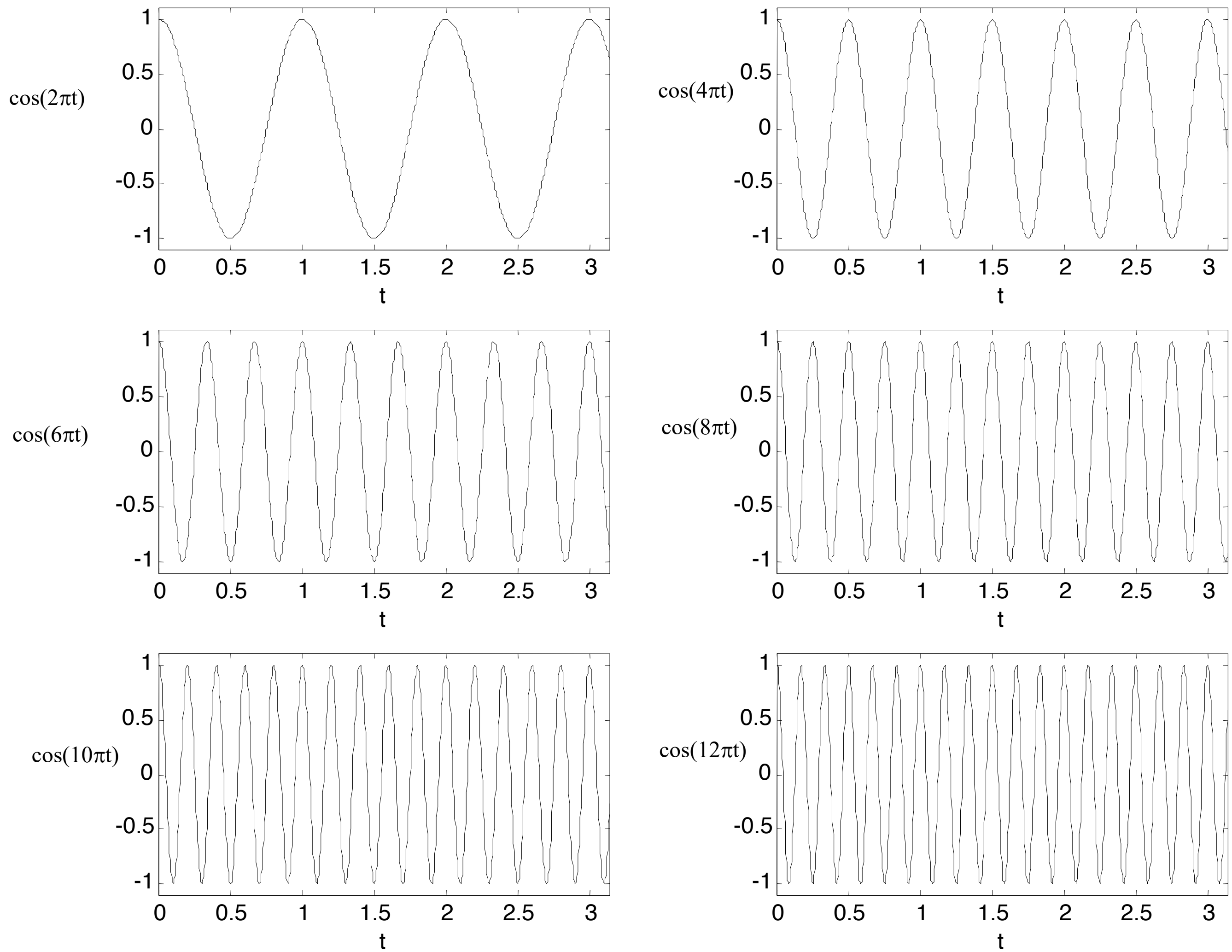
$$b_n = \frac{1}{L}\int_{-L}^{L} g(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Fourier Series

- $g(t)$ is represented as a weighted sum of sines and cosines (ie linear combination), with frequencies defined as:

$$\frac{n\pi t}{L} = 2\pi f t \Leftrightarrow f = \frac{n}{2L}$$

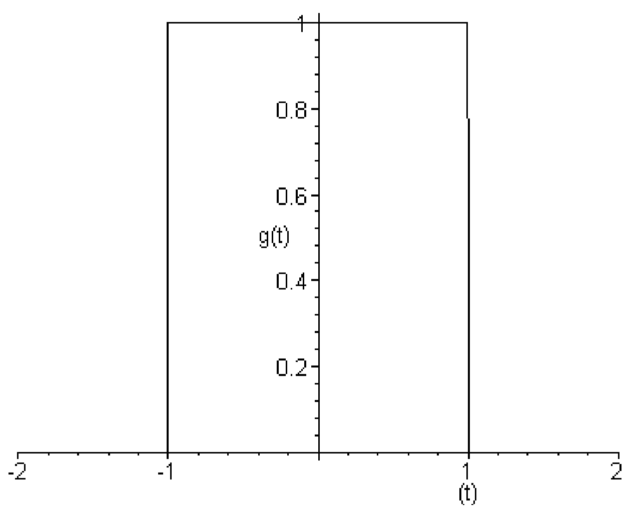
Fourier Series



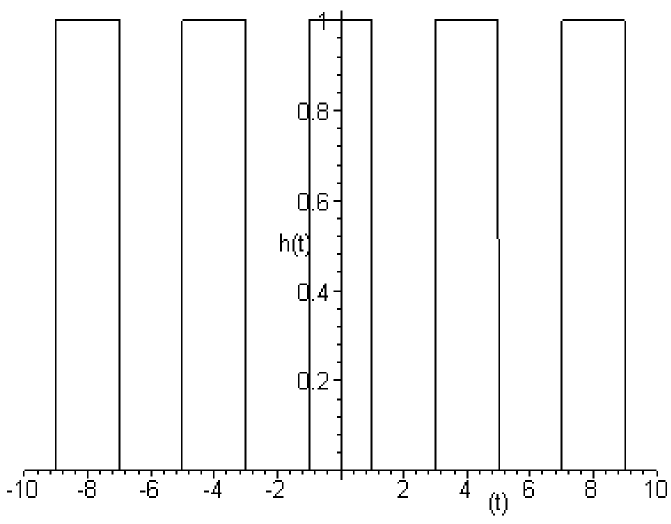
Fourier Series

Example:

$$g(t) = \begin{cases} 1 & if \ -a \le t < a \\ 0 & otherwise \end{cases}$$



Original (non-periodic) *g(t)*



Periodic version *h(t)*

Fourier Series

$$a_0 = \frac{1}{2L}\int_{-L}^{L} h(t)\,dt = \frac{1}{4a}\int_{-2a}^{2a} h(t)\,dt = \frac{1}{4a}\int_{-a}^{a} 1\,dt = \frac{1}{2}$$

Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} h(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2a}\int_{-2a}^{2a} h(t)\cos\left(\frac{n\pi t}{2a}\right)dt = \frac{1}{2a}\int_{-a}^{a} \cos\left(\frac{n\pi t}{2a}\right)dt =$$

$$= \frac{1}{2a}\left[\frac{2a}{n\pi}\sin\left(\frac{n\pi t}{2a}\right)\right]_{-a}^{a} \stackrel{a=1}{=} \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right] =$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = sinc\left(\frac{n}{2}\right)$$

Fourier Series

$$b_n = \frac{1}{L}\int_{-L}^{L} h(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2a}\int_{-2a}^{2a} h(t)\sin\left(\frac{n\pi t}{2a}\right)dt = \frac{1}{2a}\int_{-a}^{a}\sin\left(\frac{n\pi t}{2a}\right)dt =$$

$$= -\frac{1}{2a}\left[\frac{2a}{n\pi}\cos\left(\frac{n\pi t}{2a}\right)\right]_{-a}^{a} \overset{a=1}{=} -\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-\cos\left(-\frac{n\pi}{2}\right)\right] =$$

$$= -\frac{1}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-\cos\left(\frac{n\pi}{2}\right)\right] = 0$$

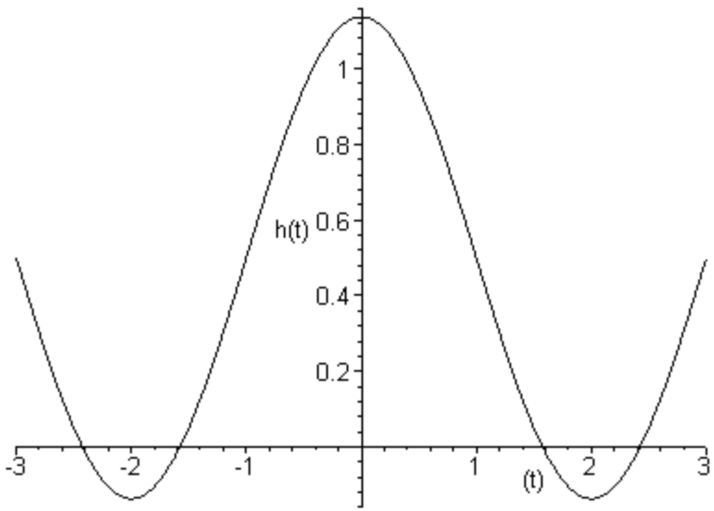
Fourier Series

$$g(t) = \begin{cases} 1 & if \; -a \le t < a \\ 0 & otherwise \end{cases}$$

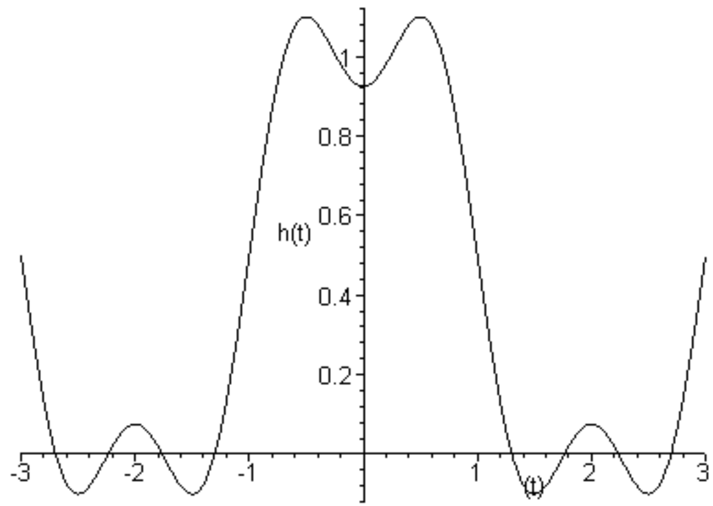
$$h(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[sinc\left(\frac{n}{2}\right) \cos\left(\frac{n\pi t}{2a}\right) \right]$$

Fourier Series

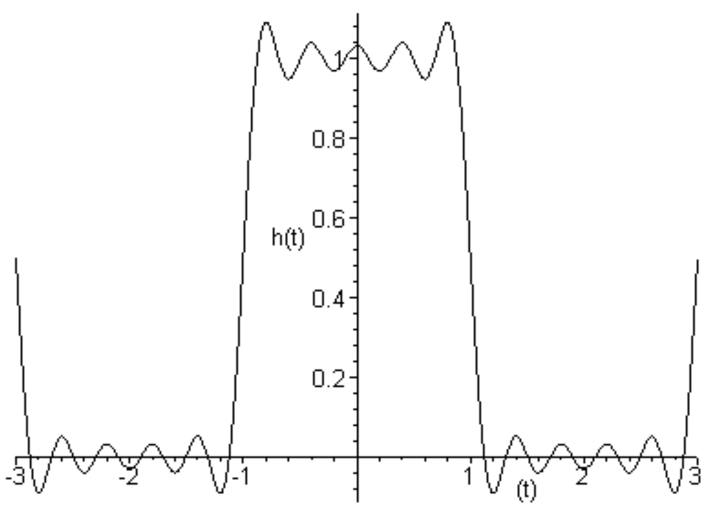
Considering a_0, b_1



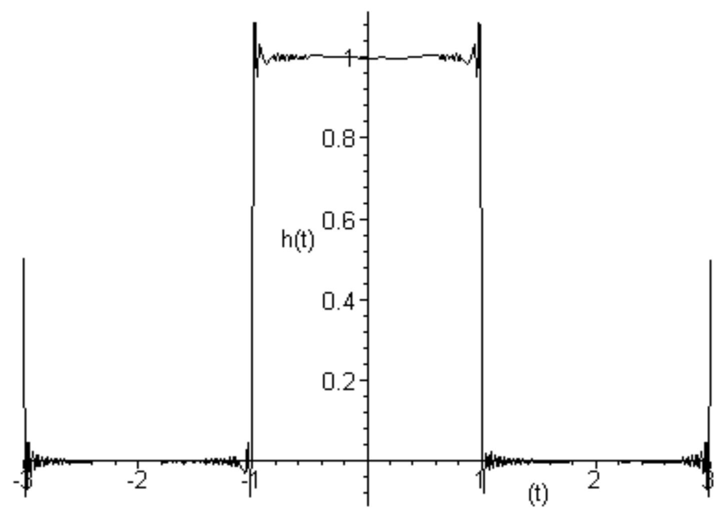
Considering a_0, b_1, b_2, b_3



Considering $a_0, b_1, b_2, \cdots, b_5$



Considering $a_0, b_1, b_2, \cdots, b_{100}$



Fourier Series

Euler formula: $\exp\{j\theta\} = \cos(\theta) + j\sin(\theta)$

$$g(t) = \sum_{n=-\infty}^{\infty}\left[c_n \exp\left\{\frac{jn\pi t}{L}\right\}\right]$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(t)\exp\left\{-\frac{jn\pi t}{L}\right\}dt$$

Continuos Fourier Transform

Continuous Fourier transform:

$$G(f) = \Im\{g(t)\} = \int_{-\infty}^{\infty} g(t)\exp\{-j2\pi ft\}dt$$

Inverse Fourier transform:

$$g(t) = \Im^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)\exp\{j2\pi ft\}df$$

Continuos Fourier Transform

2D Continuous Fourier transform:

$$G(u,v) = \Im\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\exp\{-j2\pi(ux+vy)\}dxdy$$

2D Inverse Fourier transform:

$$g(x,y) = \Im^{-1}\{G(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)\exp\{j2\pi(ux+vy)\}dudv$$

Continuos Fourier Transform

Fourier pair

$$g(x, y) \leftrightarrow G(u, v)$$

Discrete Fourier Transform

2D Discrete Fourier transform:

$$G_{r,s} = \Im\left\{g_{p,q}\right\} = \frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} g_{p,q} \exp\left\{-j2\pi\left(\frac{pr}{M}+\frac{qs}{N}\right)\right\}$$

2D Discrete Inverse Fourier transform:

$$g_{p,q} = \Im^{-1}\left\{G_{r,s}\right\} = \sum_{r=0}^{M-1}\sum_{s=0}^{N-1} G_{r,s} \exp\left\{j2\pi\left(\frac{pr}{M}+\frac{qs}{N}\right)\right\}$$

Fourier analysis

2D Fourier transform properties

Property	Description
Separability (DFT)	The discrete Fourier transform can be computed in terms of 1D Fourier transforms of the image rows followed by 1D transforms of the columns (or vice-versa).
Spatial Translation (Shifting)	$g(x - x_0, y - y_0) \leftrightarrow \exp[-j2\pi(ux_0 + vy_0)]G(u, v)$
Frequency Translation (Shifting)	$\exp[j2\pi(xu_0 + yv_0)]g(x, y) \leftrightarrow G(u - u_0, v - v_0)$
Conjugate Symmetry	If $g(x, y)$ is real, then $G(u, v) = G^*(-u, -v)$
Rotation by θ	$g(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \leftrightarrow$ $\leftrightarrow G(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta)$
Linearity – Sum	$g_1(x, y) + g_2(x, y) \leftrightarrow G_1(u, v) + G_2(u, v)$

Fourier analysis

Linearity – Multiplication by Scalars	$ag(x,y) \leftrightarrow aG(u,v)$
Scaling	$g(ax,by) \leftrightarrow \frac{1}{\lvert ab \rvert} G\left(\frac{u}{a}, \frac{v}{b}\right)$
Average Value	The image average value is directly proportional to $G(0,0)$ (the so called *DC* component).
Convolution Theorem	$g(x,y) * h(x,y) \leftrightarrow G(u,v)H(u,v)$ and $g(x,y)h(x,y) \leftrightarrow G(u,v) * H(u,v)$
Correlation Theorem	$g(x,y) \circ h(x,y) \leftrightarrow G^*(u,v)H(u,v)$ and $g^*(x,y)h(x,y) \leftrightarrow G(u,v) \circ H(u,v)$
Differentiation	$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n g(x,y) \leftrightarrow (j2\pi u)^m (j2\pi v)^n G(u,v)$

Convolution Theorem

Convolution Theorem	$g(x, y) * h(x, y) \leftrightarrow G(u, v)H(u, v)$ and $g(x, y)h(x, y) \leftrightarrow G(u, v) * H(u, v)$

Frequency Filtering

2D Discrete Fourier transform:

Algorithm: ***Frequency Filtering***

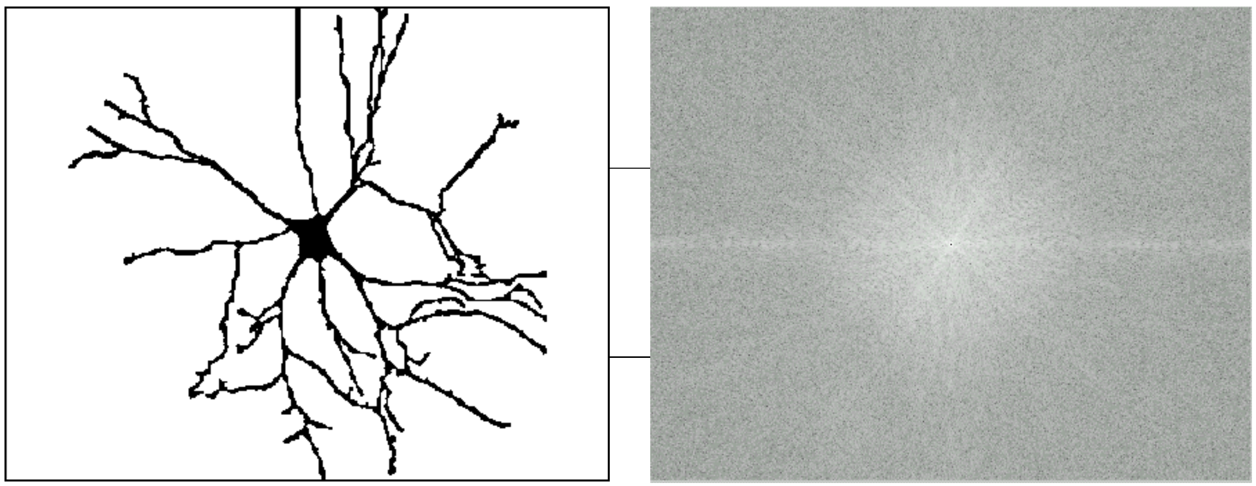
```
1.Choose G(r,s);
2.Calculate the Fourier transform F(r,s);
3.H(r,s) = F(r,s) G(r,s);
4.Calculate the inverse Fourier Transform h(p,q);
```

Frequency Filtering

Example: Box filter

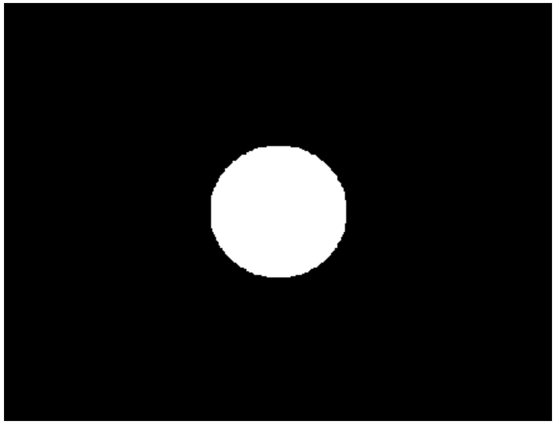
$$G_{r,s} = \begin{cases} 1, \ \text{if} \ \left(r^2 + s^2\right) \leq T \\ 0, \ \text{if} \ \left(r^2 + s^2\right) > T \end{cases}$$

Frequency Filtering

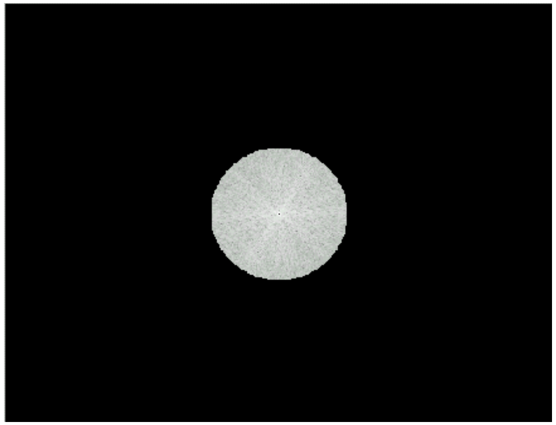


(a) (b)

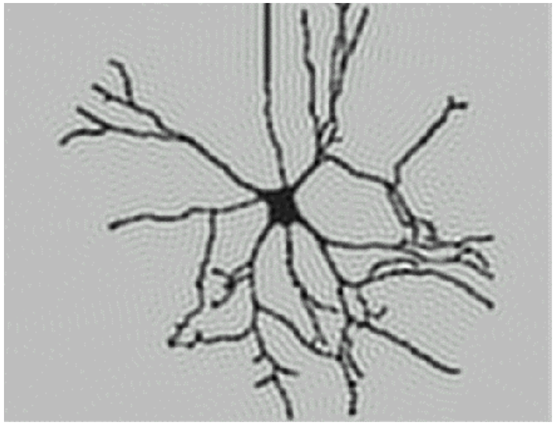
Frequency Filtering



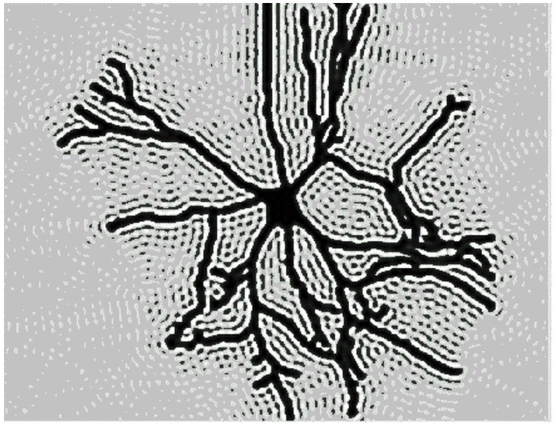
(a)



(b)



(c)



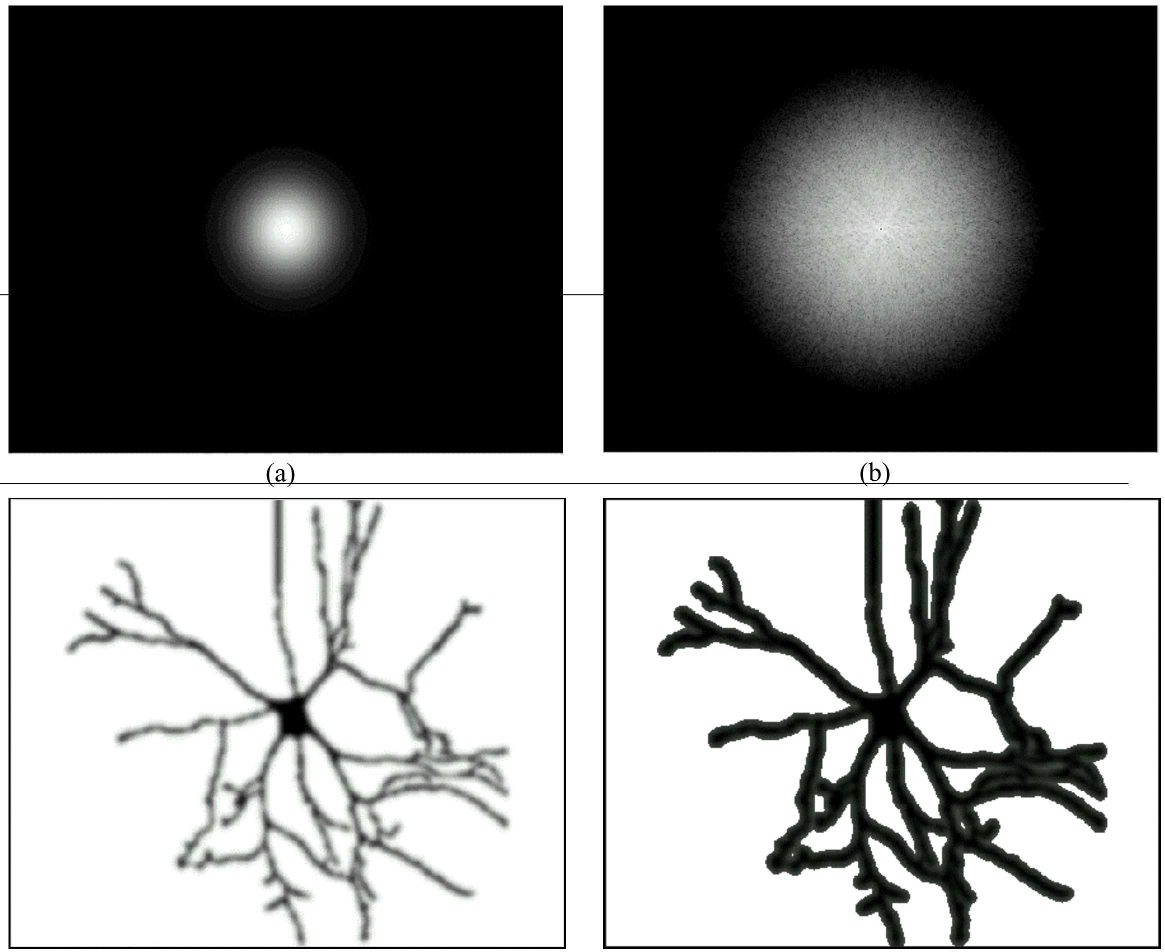
(d)

Frequency Filtering

Example: Gaussian filter

$$G_{r,s} = \exp\left(-\frac{\left(r^2 + s^2\right)}{2\sigma^2} \right)$$

Frequency Filtering



(a) (b) (c) (d)