AULA MECÂNICA 11/11 (1)

- LAGRANGIANOS E MECÂNICA LAGRANGIANA

- ~~SE~~ MECÂNICA CONSERVATIVO

$$m_i \ddot{X}_i = -\frac{\partial U}{X_i} \qquad (*)$$

$$\boxed{T(\dot{X}) - U(X)}$$

$$L(X_1, \dots, X_N, \dot{X}_1, \dots, \dot{X}_N) = \sum_{i=1}^{N} \frac{m_i \dot{x}_i^2}{2} - U(X)$$

T ENERGIA CINÉTICA

U ENERGIA POTÊNCIAL.

FUNCIONAL AÇÃO $\qquad \gamma : [a,b] \rightarrow R^{3N}$

$$\Phi_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma}, t)\, dt; \qquad (**)$$

ENTÃO γ É UM PONTO EXTREMAL DE Φ_L.

EM $C^2([a,b], R^{3N}, \gamma(a), \gamma(b))$ SE

É SÓ SE γ É SOLUÇÃO DE $(*)$.

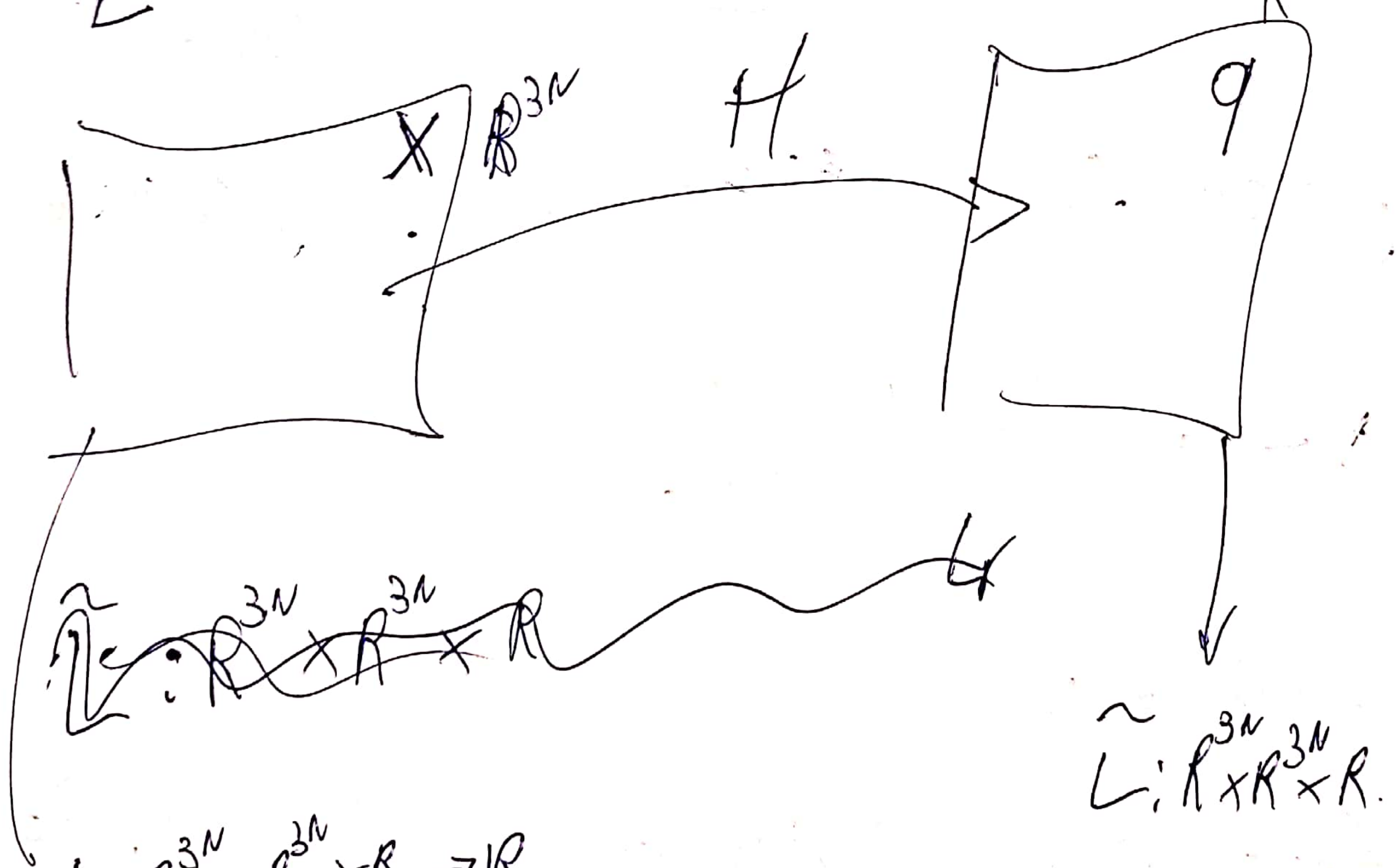
EQUAÇÕES DE EULER-LAGRANGE
P/ (~~**~~) SÃO (*).

— MUDANÇA DE COORDENADAS:

$H: \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ DIFEOMORFISMO.

$DH(X)$ É INVERSÍVEL EM QUALQUER X.

L — LAGRANGIANA COORDENADAS X.



$L: \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R} \to \mathbb{R}$

$\tilde{L}(q_0, \dot{q}_0, t)$

$q_0 = H(X_0)$. $\exists!\, X_0$

$\dot{q}_0 = DH(X_0)\dot{X}_0$ · $\exists!\, \dot{X}_0$

$\tilde{L}(q_0, \dot{q}_0, t) := L(X_0, \dot{X}_0, t)$

$\tilde{L}(q_0, \dot{q}_0, t) =$

$$\boxed{L\left(H^{-1}(q_0),\ DH\left(H^{-1}(q_0)\right)^{-1} \dot{q}_0,\ t\right)}$$

$x_0 = H^{-1}(q_0)$.

SE $\gamma : [a,b] \to \mathbb{R}^{3N}$ UMA CURVA C^2.

$$\Phi_L(\gamma) = \Phi_{\tilde{L}}(H \circ \gamma).$$

γ É CURVA EXTREMAL P/ Φ_L

$\Updownarrow$

$H \circ \gamma$ É CURVA EXTREMAL P/ $\Phi_{\tilde{L}}$.

$x, \dot{x} =$

$$L(x, \dot{x}) = \sum \frac{m_i \dot{x}_i^2}{2} - U(x).$$

$\tilde{U}(q) = U(H^{-1}(q))$. $\quad \tilde{T}(q, \dot{q}) =$

$\cancel{\tilde{T}(q,\dot{q})=}$ (4)

$T(\dot{X}) = \frac{1}{2}\langle M\dot{X}, \dot{X}\rangle$ ONDE:

M É A MATRIZ

$$\begin{bmatrix} m_1 & & & & & & & & & \\ & m_1 & & & & & & 0 & & \\ & & m_1 & & & & & & & \\ & & & m_2 & & & & & & \\ & & & & m_2 & & & & & \\ & & & & & m_2 & & & & \\ & 0 & & & & & \ddots & & & \\ & & & & & & & m_N & & \\ & & & & & & & & m_N & \\ & & & & & & & & & m_N \end{bmatrix}$$

$M = \mathrm{DIAG}\,(m_1, m_1, m_1, m_2, m_2, m_2, m_3, \ldots, m_N, m_N, m_N)$.

$\dot{X} = DH^{-1}(H^{-1}(q))\cdot\dot{q}$.

$T(\dot{X}) =$

$\tilde{T}(q,\dot{q}) = \frac{1}{2}\langle M\,DH^{-1}(H^{-1}(q))\dot{q}, DH^{-1}(H^{-1}(q))\dot{q}\rangle =$

$= \frac{1}{2}\langle B(q)\dot{q}, \dot{q}\rangle$. $B(q) =$

ONDE

$$B(q) = \underbrace{DH^{-1}(CH^{-1}(q))}^{T} \underline{M}\, \underline{DH^{-1}(CH^{-1}(q))}$$

$$B(q) = B(q)^{T}$$

$$\tilde{T}(q,\dot{q}) = \frac{1}{2}\langle B(q)\dot{q},\dot{q}\rangle$$

$B(q)$ É DEFINIDA POSITIVA. $T(\dot{x}) = \sum \frac{1}{2} m_i \dot{x}_i^2$

SE TÍNHAMOS $L(x,\dot{x}) = T(\dot{x}) - U(x)$

$$\boxed{\tilde{L}(q,\dot{q}) = \frac{1}{2}\langle B(q)\dot{q},\dot{q}\rangle - \tilde{U}(q)}$$

- q CHAMADO DE COORDENADA GENERALIZADA
- $\dot{q}$ " " VELOCIDADE "
- $\frac{\partial L}{\partial \dot{q}_i} = p_i$ " " MOMENTO GENERALIZADO.

$$\frac{d}{dt} p_i = + \frac{\partial L}{\partial q_i}$$

TRANSFORMADA DE LEGENDRE.

1_ TRANSFORMADA DE LEGENDRE UNIDIMENSIONAL.

DEF: DADA $f: I \to \mathbb{R}$ $I \subset \mathbb{R}$ INTERVALO

CONVEXA.

VAMOS DEFINIR.

$J \subset \mathbb{R}$ INTERVALO

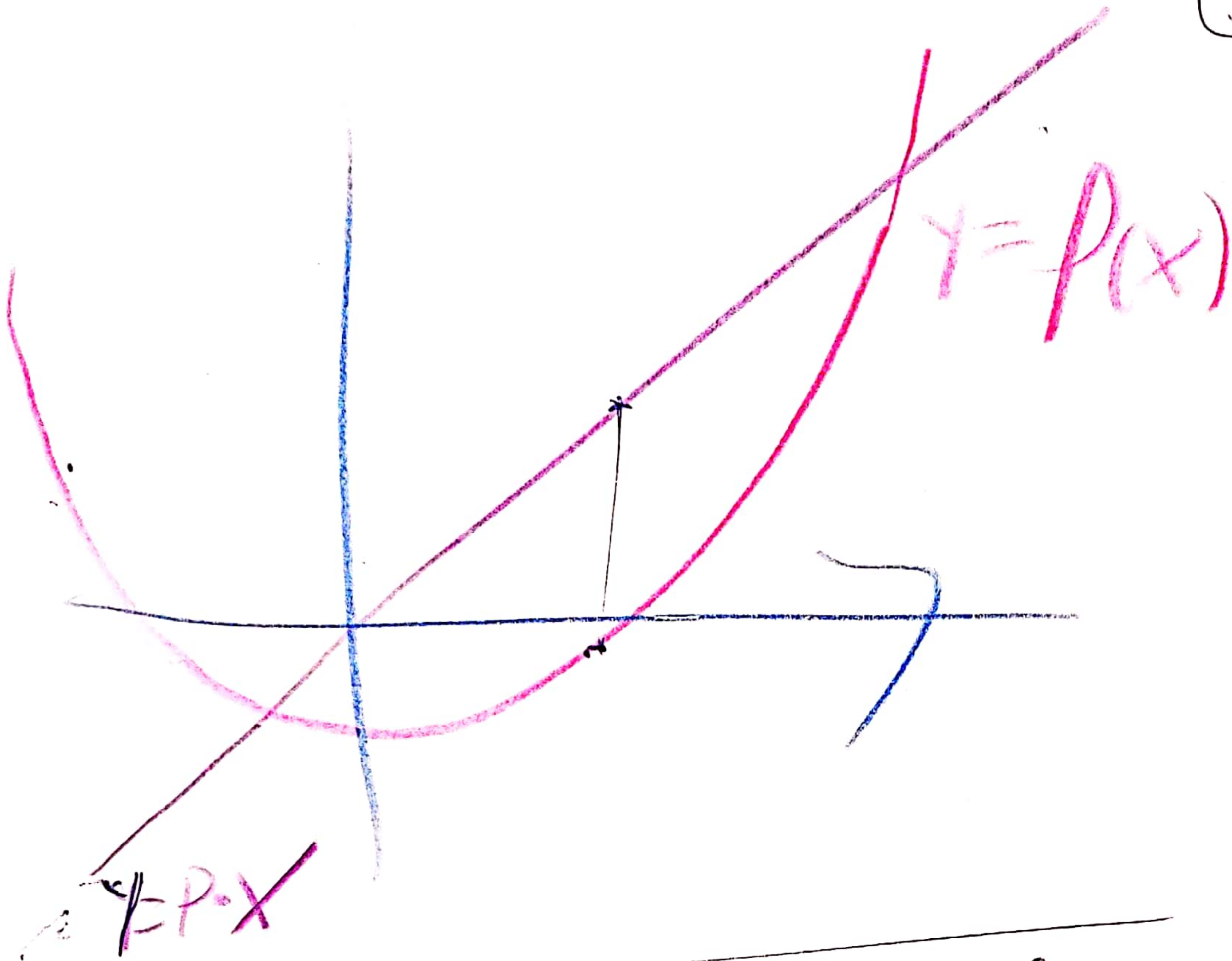
$f^*: J \to \mathbb{R}$

POR.

$J: \{ p \,/\, \sup\limits_{x \in I} \cancel{\max}\ p \cdot x - f(x) < +\infty \}$.

$$f^*(p) = \sup_{x \in I} p \cdot x - f(x).$$

⑦



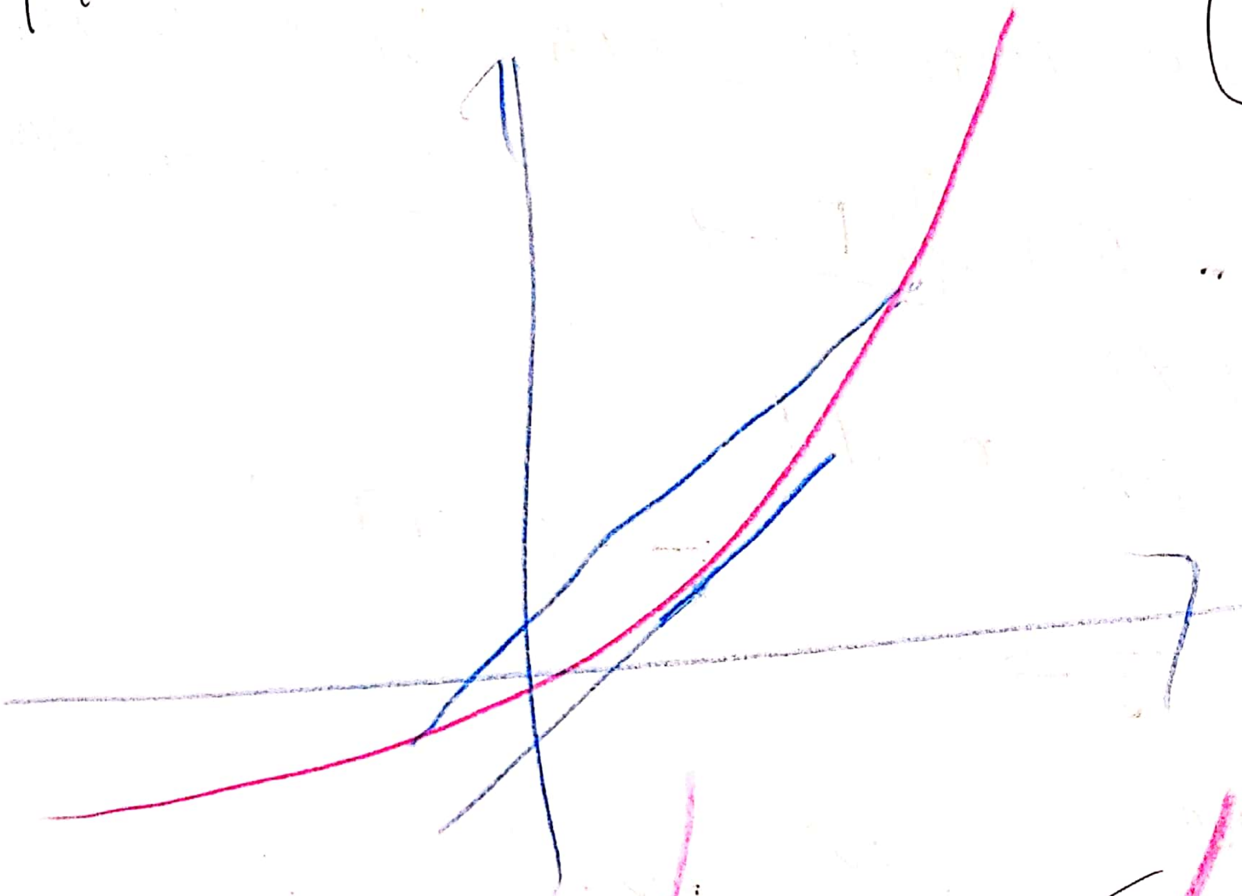
~~SE f É CONTÍNUA, E lim f(x) ... x→±∞~~

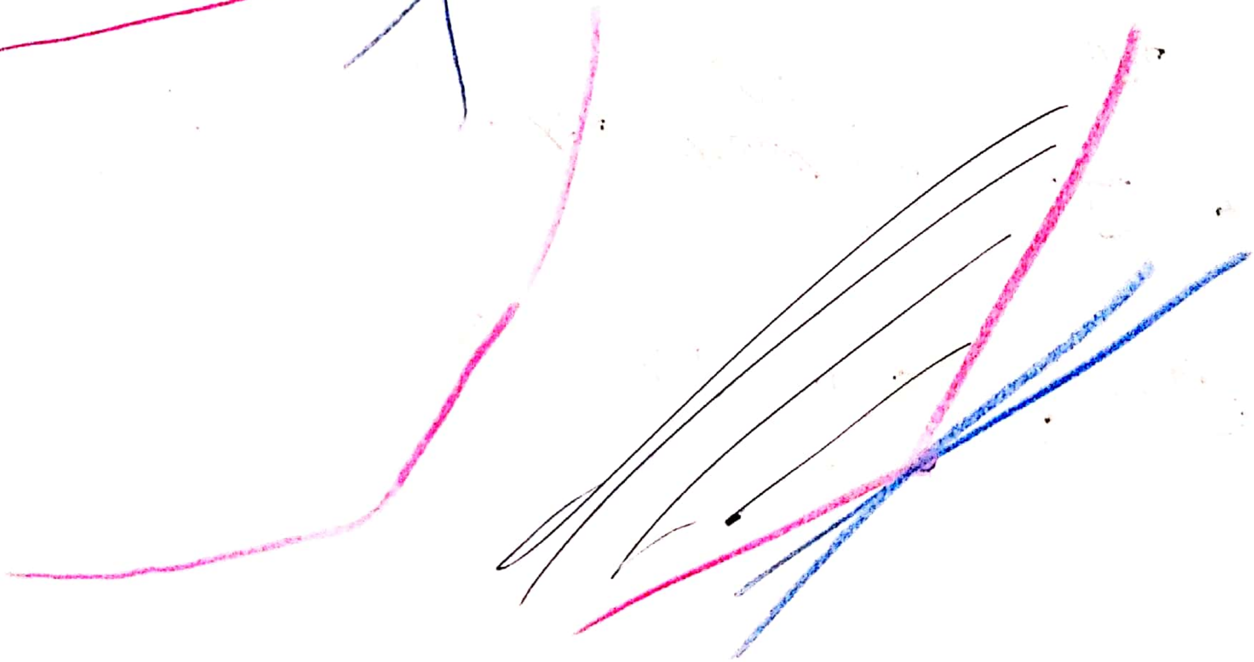
~~f: R → R.~~

~~ENTÃO~~

~~f* ESTÁ DEFINIDA P/ TODO R EXERCÍCIO.~~

$f: \mathbb{R} \to \mathbb{R}$.



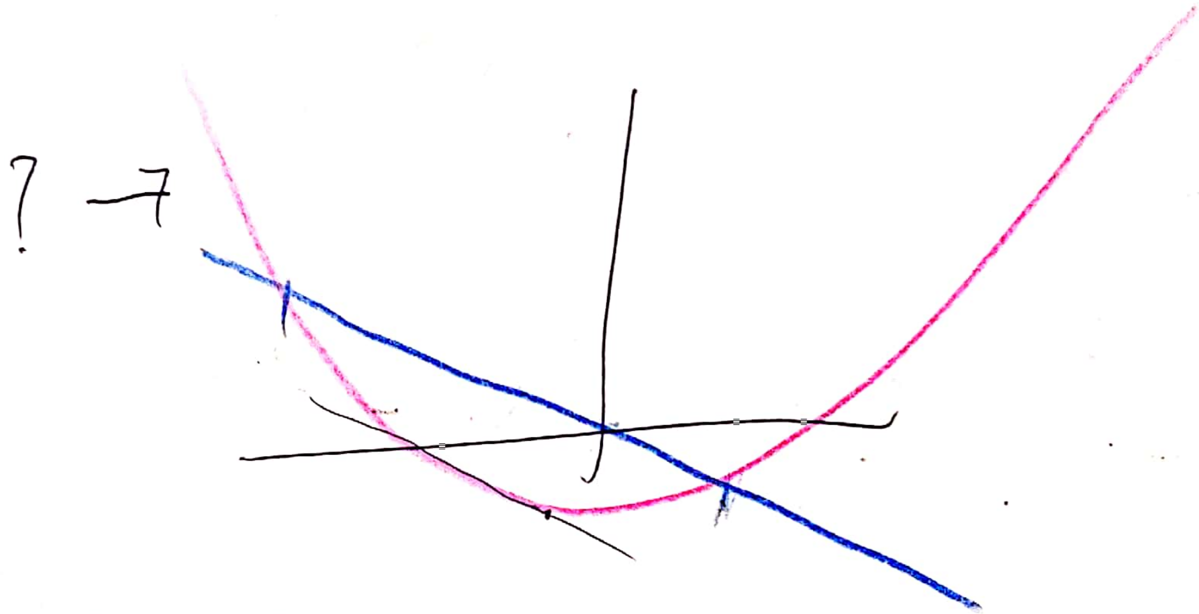


CASO ONDE f É DIFERENCIÁVEL E

$$\lim_{x \to -\infty} f'(x) = -\infty. \quad \lim_{x \to +\infty} f'(x) = +\infty,$$

$$\boxed{\text{EX: } f(x) = \frac{x^2}{2}}$$

$$f^*(p) = \sup_{x \in \mathbb{R}} p \cdot x - f(x) \overset{?}{=} \boxed{\max_{x \in \mathbb{R}} p \cdot x - f(x)}$$



ALÉM DISSO,

$f^*(p) = p\bar{x} - f(\bar{x})$ ONDE $\bar{x}$ É TAL QUE $f'(\bar{x}) = p$.

SE f É C^2, E $a > 0$

$f''(x) \geq a$, $\forall x \in R$, ENTÃO:

(1) $\lim_{x \to -\infty} f'(x) = -\infty$.

– E

(2) $\lim_{x \to +\infty} f'(x) = +\infty$.

$f'(x)$ É UM DIFEOMORFISMO DE $\mathbb{R}$ EM $\mathbb{R}$.

LOGO, $h(p) = (f')^{-1}(p)$.

$$\boxed{f^*(p) = \max_{x \in \mathbb{R}} p \cdot x - f(x)}$$

$$f^*(p) = p \cdot h(p) - f(h(p)).$$

NESTE CASO.

$$\frac{d}{dp} f^*(p) = h(p) + p \cdot h'(p) - f'(h(p)) \cdot h'(p) =$$

$$= h(p) + p \cdot h'(p) - p \cdot h'(p) = h(p).$$

$$(\varphi^*)' = (\varphi')^{-1}$$

$$(\varphi^*)'' = \frac{d}{dp} h(p) = h'(p) = \frac{1}{\varphi''(h(p))} > 0.$$

$\Rightarrow (\varphi^*)'' > 0$ φ^* É CONVEXA.

NESTE CASO, QUEM É $(\varphi^*)^*$?

TENHO QUE

$$(\varphi^*)^*(y) = \max_{p \in \mathbb{R}} y \cdot p - \varphi^*(p).$$

QUE OCORRE EM $\bar{p}$ ONDE $(\varphi^*)'(\bar{p}) = y$.

$g_p(x) = p \cdot x - f(x).$ | NAS HIPÓTESE DE f SER C^2. $f''(x) \geq a > 0.$ (11)

① $\forall p, \lim_{x \to \pm\infty} g_p(x) = -\infty.$

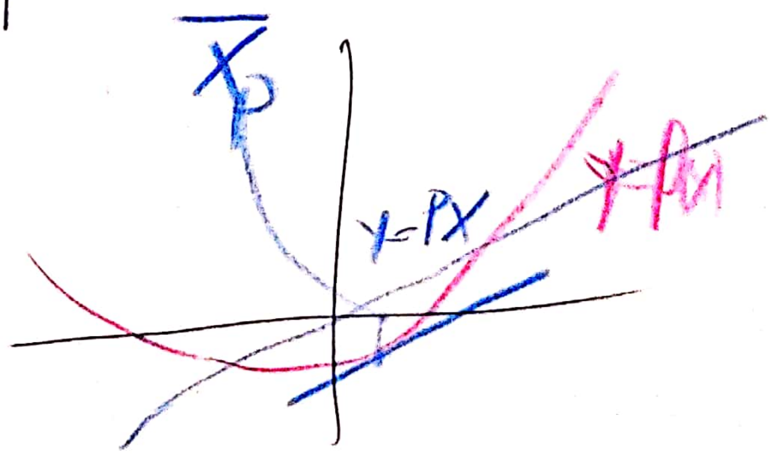
② $\forall p / g_p$ É ESTRITAMENTE CÔNCAVA E ATINGE MÁXIMO EM UM ÚNICO PONTO.

③ SE $\bar{x}_p$ É O PONTO DE MÁXIMO DE g_p, ENTÃO $g_p'(\bar{x}_p) = 0.$

MAS

$g_p'(x) = p - f'(x) \Rightarrow$

$f'(\bar{x}_p) = p.$



logo

$$\varphi^*(p) = \cancel{p\,\overline{x}_p - \varphi}$$

$$p\cdot\overline{x}_p - \varphi(\overline{x}_p) = g_p(\overline{x}_p)$$

$h : \mathbb{R} \to \mathbb{R}$

$h(p) = \overline{x}_p$. $\quad h = (\varphi')^{-1}$

$$\varphi^*(p) = p\cdot h(p) - \varphi(h(p)).$$

$$\frac{d}{dp}\varphi^*(p) = h(p) + \cancel{p\,h'(p) - \overbrace{\boxed{\varphi'(h(p))}}^{=p}\,h'(p)} =$$
$$= h(p).$$

$$(\varphi^*)'' = h'(p) = \frac{1}{\varphi''(h(p))} > 0$$

φ^* É CONVEXA.

LOGO, POSSO DEFINIR.

$$(\varphi^*)^*$$ ~~...~~ QUE VAMOS DENOTAR POR

$$\varphi^{**}.$$

$$\varphi^{**}(Y) = \sup_{P \in \mathbb{R}} Y \cdot P - \varphi^*(P).$$

TEMOS QUE

$$\varphi^{**}(Y) = Y \cdot \bar{P}_Y - \varphi^*(\bar{P}_Y)$$

ONDE $\bar{P}_Y$ É TAL QUE $(\varphi^*)'(\bar{P}_Y) = Y.$

MAS $(\varphi^*)' = (\varphi')^{-1}$

$$\varphi^{*'}(\bar{P}_Y) = Y \iff \bar{P}_Y = \varphi'(Y)$$

~~$\bar{P}_Y = h(Y)$~~.

$Y = h(P)$

$$f^{**}(Y) = h(\bar{P}_Y) \cdot \bar{P}_Y - f^{*}(\bar{P}_Y) \quad (1)$$

$$f^{*}(\bar{P}_Y) = h(\bar{P}_Y) \cdot \bar{P}_Y - f(h(\bar{P}_Y))$$

$$\boxed{f^{*}(p) = p \cdot h(p) - f(h(p))} \quad (2).$$

SUBSTITUINDO (2) EM (1)

$$f^{**}(Y) = \cancel{h(\bar{P}_Y)\bar{P}_Y} - \left(\cancel{\bar{P}_Y \cdot h(\bar{P}_Y)} - f(h(\bar{P}_Y))\right)$$

$$f^{**}(Y) = f(\underbrace{h(\bar{P}_Y)}_{= Y}) = f(Y)$$

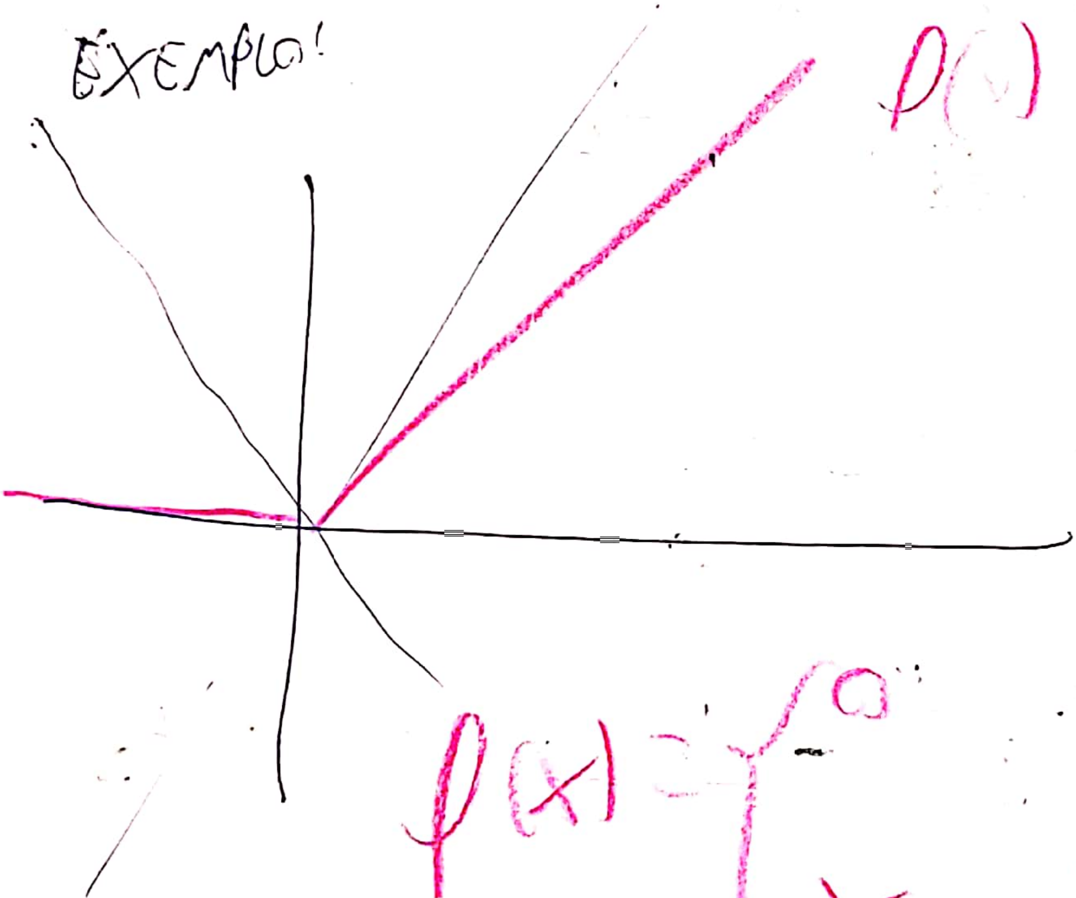
~~Coral~~ LOGO A TRANSFORMADA DE LEGENDRE ~~DE~~ É UMA INVOLUÇÃO.

EXERCÍCIO:

MOSTRE QUE, ~~SE~~ EM QUALQUER CASO,

f^* É CONVEXA.

EXEMPLO:



$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$

$$f^* = \sup_{x \in \mathbb{R}} \boxed{px - f(x) = g_p(x)}$$

$$g_p(x) = p \cdot x - f(x)$$

SE $p < 0$. $\lim_{x \to -\infty} g_p(x) = +\infty$.

SE $p > 1$ $\lim_{x \to +\infty} g_p(x) = +\infty$.

é, se $0 \leq p \leq 1$,

$$\sup_{x \in \mathbb{R}} g_p(x) = g_p(0) = 0.$$

$$\varphi^*: [0,1] \to \mathbb{R}.$$

$$\varphi^*(p) = 0.$$

EXEMPLO 2:

$$\boxed{\varphi(x) = C \cdot x^2} \longrightarrow \frac{p^2}{4C}$$

$\varphi^*(p)$?

$$p = \varphi'(x) = 2Cx$$

$$x = \frac{p}{2C}.$$

$$\varphi^*(p) = p \cdot \frac{p}{2C} - C \cdot \left(\frac{p}{2C}\right)^2 = \frac{p^2}{2C} - \frac{p^2}{4C} = \frac{p^2}{4C}.$$

Em particular, se $C = \frac{1}{2}$

$$\varphi(x) = \frac{x^2}{2} \quad \varphi^*(p) = \frac{p^2}{2}$$

$$f(x) = \frac{x^{\alpha}}{\alpha} \qquad \alpha > 1$$

$$p = f'(\bar{x}_p) = \bar{x}_p^{\,\alpha-1}$$

$$\bar{X}_p = p^{\frac{1}{\alpha-1}}$$

$$f^*(p) = p \cdot p^{\frac{1}{\alpha-1}} - \frac{p^{\frac{\alpha}{\alpha-1}}}{\alpha} = \left(1-\frac{1}{\alpha}\right) \cdot p^{\frac{\alpha}{\alpha-1}} =$$

$$\frac{p^{\frac{\alpha}{\alpha-1}}}{\left(\frac{\alpha}{\alpha-1}\right)} \qquad \boxed{\frac{\alpha}{\alpha-1} = \beta} \qquad f^*(p) = \frac{p^{\beta}}{\beta}$$

ONDE $\frac{1}{\beta}+\frac{1}{\alpha} = 1$

$\varphi(x)$	$\varphi^*(p)$
$\frac{x^2}{2}$	$\frac{p^2}{2}$
$\frac{x^\alpha}{\alpha}$	$\frac{x^\beta}{\beta}$

$$\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

MAS EM GERAL.

$$\cancel{\varphi^*(p)+\varphi(x)\ \geq\ p\cdot x}.$$

$$\varphi^*(p) \geq p\cdot x-\varphi(x) \quad \forall x.$$

LOGO.

$$\varphi^*(p)+\varphi(x) \geq p\cdot x \quad \forall p,x.$$

$$\frac{p^2}{2}+\frac{x^2}{2} \geq p\cdot x$$

$$\boxed{\frac{p^2}{2}+\frac{x^\beta}{\beta} \geq p\cdot x} \quad \forall p,x.$$