EX. 138. $\Phi(x) = \dfrac{|x|^{2-n}}{(n-2)n\,|B(0,1)|}$, $n \geq 3$

PARA $n = 2$, TEMOS TERMO LOGARITMO (FÓRMULA NÃO VALE!)

PARA $n = 1$, TEMOS $\Phi(x) = \dfrac{|x|^{2-1}}{\underbrace{(1-2)\cdot 1\cdot 2}_{-1\times 2}} = \dfrac{-|x|}{2}$ $\quad (2-1 \to -1)$

$\underbrace{-1 \quad\quad 0 \quad\quad 1}_{2}$

a) PROVE QUE $-\dfrac{d^2}{dx^2}\Phi = \delta_0$

RESOLUÇÃO: $\Phi \longleftrightarrow T_\Phi$.

$$-\frac{d^2}{dx^2}T_\Phi(\varphi) = \delta_0(\varphi),\ \forall \varphi \in C_c^\infty(\mathbb{R})$$

$$-\frac{d}{dx}\frac{d}{dx}T_\Phi(\varphi) = \frac{d}{dx}T_\Phi\left(\frac{d\varphi}{dx}\right) = -T_\Phi\left(\frac{d^2\varphi}{dx^2}\right) = -\int_{\mathbb{R}}\Phi(x)\frac{d^2\varphi}{dx^2}(x)\,dx$$

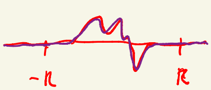
$$\delta_0(\varphi) = \varphi(0) \qquad \int f'g = fg - \int fg'$$

PERGUNTA: $-\displaystyle\int_{\mathbb{R}}\Phi(x)\frac{d^2\varphi}{dx^2}(x)\,dx = \varphi(0),\ \forall \varphi \in C_c^\infty(\mathbb{R})$? VAMOS PROVAR QUE VALE ATÉ MESMO PARA $C_c^2(\mathbb{R})$.

$$-\int_{-\infty}^{\infty}\Phi(x)\frac{d^2\varphi}{dx^2}(x) = \int_{-\infty}^{\infty}\frac{|x|}{2}\frac{d^2\varphi}{dx^2}(x)\,dx = \int_0^{\infty}\frac{x}{2}\frac{d^2\varphi}{dx^2}(x)\,dx - \int_{-\infty}^{0}\frac{x}{2}\frac{d^2\varphi}{dx^2}(x)\,dx$$

$$= \frac{1}{2}\left(\lim_{R\to\infty}\int_0^R x\,\varphi''(x)\,dx - \lim_{R\to\infty}\int_{-R}^0 x\,\varphi''(x)\,dx\right)$$

SE R É GRANDE, ENTÃO $\varphi(R), \varphi'(R) = 0$.



$$= \frac{1}{2}\lim_{R\to\infty}\left(\cancel{x\varphi'(x)}\Big|_0^R - \int_0^R \varphi'(x)\,dx - \cancel{x\varphi'(x)}\Big|_{-R}^0 + \int_{-R}^0 \varphi'(x)\,dx\right)$$

$$= \frac{1}{2}\lim_{R\to\infty}\left(-\underbrace{\varphi(R)}_{=0} + \varphi(0) + \varphi(0) - \underbrace{\varphi(-R)}_{=0}\right) = \frac{1}{2}\,2\varphi(0) = \varphi(0)$$

b) $$\mu(x) = \int_{-\infty}^{\infty} \Phi(x-y) f(y)\, dy \underset{\substack{\tilde{y} = x-y \\ d\tilde{y} = dy}}{=} \int_{-\infty}^{\infty} \Phi(\tilde{y}) f(x-\tilde{y})\, d\tilde{y}.$$

$$\mu(x) = \int_{-\infty}^{\infty} \Phi(y) f(x-y) dy.$$

USAMOS APENAS PARA JUSTIFICAR $-\frac{d}{dx^2}\int \Phi(y) f(x-y)dy = -\int \Phi(y) \frac{d^2 f}{dx^2}(x-y)dy.$

SE $f \in C_c^2(\mathbb{R})$, ENTÃO $\exists\, R > 0$ T.Q. $\operatorname{supp} f \subset [-R, R]$.

VAMOS SUPOR QUE $x \in]-r, r[$. LOGO $x - y \in]-R-r, R+r[$.

ASSIM, PARA $x \in]-r, r[$, TEMOS $\mu(x) = \int_{-R-r}^{R+r} \Phi(y) f(x-y)\, dy.$

PORTANTO $$-\frac{d^2}{dx^2}\mu(x) = -\frac{d^2}{dx^2}\int_{-R-r}^{R+r} \Phi(y) f(x-y) dy = -\int_{-R-r}^{R+r} \Phi(y) \frac{d^2 f}{dx^2}(x-y)\, dy$$

$$= -\int_{-\infty}^{\infty} \Phi(y) \frac{d^2 f}{dx^2}(x-y) dy = -\int_{-\infty}^{\infty} \Phi(y) \frac{d^2}{dy^2}\big(f(x-y)\big) dy.$$

NOTE QUE $$\frac{d}{dy}\big(f(x-y)\big) = \frac{df}{dx}(x-y)\frac{d}{dy}(x-y) = -1\frac{df}{dx}(x-y)$$

$$\frac{d^2}{dy^2}\big(f(x-y)\big) = -\frac{d}{dy}\left(\frac{df}{dx}(x-y)\right) = -\frac{d^2 f}{dx^2}(x-y)\frac{d}{dy}(x-y) = \frac{d^2 f}{dx^2}(x-y).$$

SABEMOS QUE $\forall\, \varphi \in C_c^2(\mathbb{R})$, TEMOS $-\int \Phi(y) \frac{d^2\varphi}{dy^2}\, dy = \varphi(0)$ (ITEM a))

ASSIM, SE $\varphi(y) = f(x-y)$, TEMOS $$-\int \Phi(y)\frac{d^2\varphi}{dy^2}\, dy = \varphi(0) = f(x-0) = f(x)$$

$$\overset{\shortparallel}{-\int \Phi(y) \frac{d^2}{dy^2}\big(f(x-y)\big)dy.}$$

CONCLUSÃO

$$-\frac{d^2}{dx^2}u(x) = -\int \Phi(y)\frac{d^2}{dy^2}\left(f(x-y)\right)dy = f(x) \quad // \quad \boxed{D}$$

SUPONHA QUE $\exists$ $v(x)$ T.Q $-\frac{d^2 v}{dx^2}(x) = f(x)$

LOGO $\underbrace{\frac{d^2}{dx^2}\left(u(x) - v(x)\right)} = -f(x) + f(x) = 0.$

$$u(x) - v(x) = \underline{a + bx.}$$

SE u E v SÃO LIMITADAS $\Rightarrow b = 0.$

CONCLUÍMOS QUE $u(x) = v(x) + a$, OU SEJA, u É ÚNICO

MÓDULO FUNÇÕES CONSTANTES. $\boxed{D}$

EX. 239. $n=1$, $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$.

a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \iff v'' + \frac{z}{2} v' = 0$.

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial t}\left(v\left(\frac{x}{\sqrt{t}}\right)\right) = v'\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{x}{2t^{3/2}}\right)$$

$$\frac{\partial}{\partial x} u(x,t) = \frac{\partial}{\partial x}\left(v\left(\frac{x}{\sqrt{t}}\right)\right) = v'\left(\frac{x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}$$

$$\frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial}{\partial x}\left(v'\left(\frac{x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}\right) = v''\left(\frac{x}{\sqrt{t}}\right)\frac{1}{t}.$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\Updownarrow$$

$$v'\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{x}{2t^{3/2}}\right) = v''\left(\frac{x}{\sqrt{t}}\right)\frac{1}{t}$$

$$v''\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{2t^{1/2}} v'\left(\frac{x}{\sqrt{t}}\right) = 0$$

$$v''(z) - \frac{z}{2} v'(z) = 0$$

MOSTRE QUE $v(z) = c\int_0^z e^{-\frac{z^2}{4}} + d$.

$$v''(z) = \frac{z}{2} v'(z) \underset{\substack{\text{CONTA}\\ \text{FORMAL}\\ \text{(SEM RIGOR)}}}{\Longrightarrow} \frac{v''(z)}{v'(z)} = \frac{z}{2} \Rightarrow \frac{d}{dz}\ln(v'(z)) = \frac{z}{2}$$

$$\Rightarrow \int_0^z \frac{d}{dw}\ln(v'(w))\,dw = \int_0^z \frac{w}{2}\,dw$$

$$\ln(v'(z)) - \ln(v'(0)) = \frac{w^2}{4}\Big|_0^z = \frac{z^2}{4}$$

$$\ln(v'(z)) = \ln(v'(0)) + \frac{z^2}{4}$$

$$\boxed{v'(z) = v'(0)e^{z^2/4}.}$$

$$\int_0^z v'(w)\,dw = v'(0)\int_0^z e^{\frac{w^2}{4}}\,dw$$

$$v(z) - v(0) = v'(0)\int_0^z e^{\frac{w^2}{4}}\,dw$$

$$\boxed{v(z) = \underset{\substack{\|\\ d}}{v(0)} + \underset{\substack{\|\\ c}}{v'(0)}\int_0^z e^{-\frac{s^2}{4}}\,ds}$$

$$\frac{\partial}{\partial x}\left(u(x,t)\right) = \frac{\partial}{\partial x}\left(v\left(\frac{x}{\sqrt{t}}\right)\right) = v'\left(\frac{x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}.$$

$$v(z) = d + c\int_0^z e^{-\frac{s^2}{4}}\,ds \Rightarrow \frac{d}{dz}v(z) = \frac{d}{dz}\left(d + c\int_0^z e^{-\frac{s^2}{4}}\,ds\right)$$

$$= c\,e^{-z^2/4}.$$

LOGO $\frac{\partial u}{\partial x}(x,t) = \frac{1}{\sqrt{t}}\,c\,e^{-\frac{x^2}{4t}}.$

LEMBRAMOS QUE $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}$. ASSIM, BASTA

ESCOLHER $c = \frac{1}{\sqrt{4\pi}}$.

PQ OBTEMOS SOLUÇÃO FUNDAMENTAL?

1) SE $\varphi \in S(\mathbb{R})$, $\int \varphi\,dx = 1$, ENTÃO $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$. LOGO

$T_{\varphi_\varepsilon} \xrightarrow{\varepsilon\to 0} \delta_0$.

<u>DEMO:</u> $T_{\varphi_\varepsilon}(\psi) = \int \varphi_\varepsilon(x)\psi(x)\,dx = \int \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)\psi(x)\,dx$

$\psi \in C_c^\infty(\mathbb{R})$

$$= \int \varphi(y)\psi(\varepsilon y)\,dy$$

$y = \frac{x}{\varepsilon}$

$$\lim_{\varepsilon\to 0} T_{\varphi_\varepsilon}(\psi) = \lim_{\varepsilon\to 0}\int \varphi(y)\psi(\varepsilon y)\,dy = \int \varphi(y)\lim_{\varepsilon\to 0}\psi(\varepsilon y)\,dy = \int \varphi(y)\psi(0)\,dy$$

$$= \psi(0)\underbrace{\int \varphi(y)\,dy}_{=1} = \psi(0). \quad \square$$

No caso do exercício

$$\frac{\partial}{\partial x} u(x,t) = \frac{1}{\sqrt{t}}\, v\left(\frac{x}{\sqrt{t}}\right).$$

Logo $\varepsilon = \sqrt{t}$ $\quad \underline{\frac{1}{\sqrt{t}}\, v\left(\frac{x}{\sqrt{t}}\right) \longrightarrow c\,\delta_0}.$

Como, sabemos $\quad \dfrac{\partial \Phi}{\partial t} - \dfrac{\partial^2 \Phi}{\partial x^2} = 0, \quad t > 0,\ x \in \mathbb{R}$

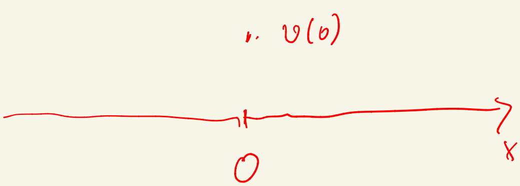
$$\lim_{t \to 0} \Phi(x,t) = \delta_0, \quad \text{em } \mathcal{D}'(\mathbb{R}).$$

Então é "razoável" esperar que $\frac{1}{\sqrt{t}}\, v\left(\frac{x}{\sqrt{t}}\right)$ esteja relacionado com Φ.

$$u(x,t) = v\left(\frac{x}{\sqrt{t}}\right) \qquad t \to 0 \qquad \sqrt{t} \to \infty$$

Se $\quad v(\pm\infty) = 0 \qquad u(x,0) = 0, \quad x \neq 0$

$$u(0,0) = v(0), \quad x = 0.$$

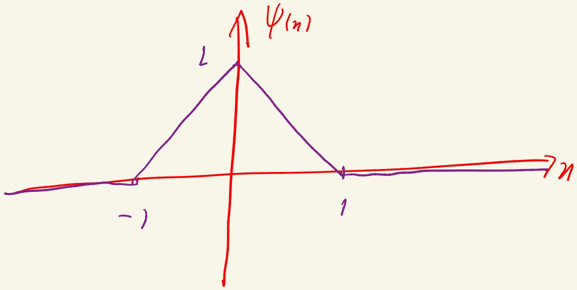


$$\frac{\partial u}{\partial x}(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$

MUITO INFORMAL.

Ex. 134 $\quad \psi : \mathbb{R} \to \mathbb{R}$

$$\psi(x) = \begin{cases} 0, & x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$



$\psi \geq 0$. $\quad \int \psi \, dx = \frac{1}{2} + \frac{1}{2} = 1$, $\psi(x) = 0$, $|x| \geq 1$.

1) $\psi_j(x) = j\,\psi(jx)$, $j \in \mathbb{N}$. Mostre que $\lim \psi_j = \delta$.

Demo: $\underbrace{\lim_{j \to \infty} T_{\psi_j}(\varphi) = \delta_0(\varphi)}$, $\forall \varphi \in C_c^\infty(\mathbb{R})$.

$$\lim_{j \to \infty} \int \psi_j(x)\varphi(x)\,dx = \varphi(0) \quad (*)$$

Note que $\underline{\int \psi_j(x)\,dx = \int j\,\psi(jx)\,dx} \underset{\substack{y = jx \\ dy = j\,dx}}{=} \underline{\int \psi(y)\,dy = 1}$. $\Rightarrow \varphi(0) = \varphi(0) \int \psi_j(x)\,dx$

Assim, $\lim_{j \to \infty} \int \psi_j(x)(\varphi(x) - \varphi(0))\,dx \overset{?}{=} 0 \quad (**)$ $\qquad (*)$ equivale a $(**)$.

$\psi(x) = 0$, $|x| > 1$, então $\psi(jx) = 0$, $|jx| > 1 \Leftrightarrow |x| > \frac{1}{j}$. $\Rightarrow \psi_j(x) = 0$, $|x| > \frac{1}{j}$.

Portanto, $\int_{-\infty}^{\infty} \psi_j(x)(\varphi(x) - \varphi(0))\,dx = \int_{-1/j}^{1/j} \psi_j(x)(\varphi(x) - \varphi(0))\,dx$

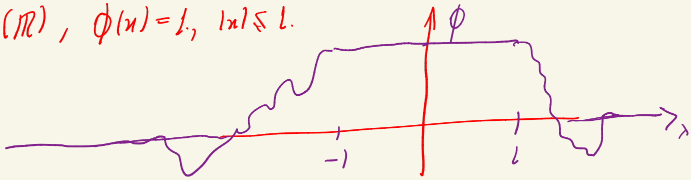
Como φ é contínua, dado $\varepsilon > 0$, $\exists\, \delta > 0$ t.q. se $|x - 0| < \delta$, então $|\varphi(x) - \varphi(0)| < \varepsilon$.

Seja J t.q. $\frac{1}{J} < \delta$. Logo, se $j > J$, temos

$$\left| \int_{-1/j}^{1/j} \psi_j(x)(\varphi(x) - \varphi(0))\,dx \right| \underset{\substack{\text{usamos} \\ \psi \geq 0}}{\leq} \int_{-1/j}^{1/j} \psi_j(x) \underbrace{|\varphi(x) - \varphi(0)|}_{< \varepsilon} dx < \varepsilon \int_{-1/j}^{1/j} \psi_j(x)\,dx = \varepsilon \underbrace{\int_{-\infty}^{\infty} \psi_j(x)\,dx}_{=1} = \varepsilon$$

$$\Rightarrow \lim_{j \to \infty} \int \psi_j(x)(\varphi(x) - \varphi(0))\,dx = 0 \quad \blacksquare$$

b) $\phi \in C_c^\infty(\mathbb{R})$, $\phi(x) = 1$, $|x| \le 1$.



Logo $\int \psi_j^2 \phi \, dx$ não converge

Demo: $\int \psi_j^2(x)\phi(x)dx = \int_{-1/j}^{1/j} \psi_j(x)^2 \underbrace{\phi(x)}_{1} dx = \int_{-1/j}^{1/j} \psi_j^2(x)dx = \int_{-\infty}^{\infty} \psi_j^2(x)dx = \int_{-\infty}^{\infty} (j\psi(jx))^2 dx \underset{\substack{y = jx \\ dy = j dx}}{=} \int_{-\infty}^{\infty} j\psi(y)^2 dy = j\int_{-\infty}^{\infty} \psi(y)^2 dy = j$

Logo o $\lim_{j\to\infty} \int \psi_j^2(x)\phi(x)dx = \lim_{j\to\infty} j = \infty$.

$$T_{\psi_j}(\phi) \longrightarrow \infty \Rightarrow T_{\psi_j} \not\longrightarrow \mu \in \mathcal{D}'(\mathbb{R}).$$

c) Conclua que $\nexists$ ext. contínua de $Q: C(\mathbb{R}) \to C(\mathbb{R})$ $Q(f) = f^2$.

para $Q: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$.

Demo: Suponha que $\exists$ extensão contínua.

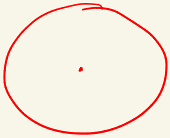
Como $\lim_{j\to\infty} T_{\psi_j} = \delta_0$, então $\lim_{j\to\infty} Q(T_{\psi_j}) = Q(\delta_0)$

Mas como $\psi_j \in C(\mathbb{R})$, então $Q(T_{\psi_j}) = T_{\psi_j^2}$. Assim,

deveria $\exists \lim_{j\to\infty} T_{\psi_j^2}$. Mas vimos que não existe.

Portanto, a distribuição $Q(\delta_0)$ "=" δ_0^2 não existe.

Ex. 167



$$\begin{cases} \Delta u(x) = 0, & x \in U \\ u(x) = \tilde{g}(x), & x \in \partial U \end{cases}$$

ESTE PROBLEMA EM GERAL NÃO TEM SOLUÇÃO.

$U = B(0,1) \setminus \{0\}$

$\partial U = \partial B(0,1) \cup \{0\}$

a) $$\begin{cases} \Delta u(x) = 0, & x \in U \\ u(x) = 0, & x \in \partial B(0,1) \\ u(0) = 1. \end{cases} \qquad \tilde{g}(x) = \begin{cases} 0, & x \in \partial B(0,1) \\ 1, & x = 0. \end{cases}$$

$\nexists$ $u : \overline{B(0,1)} \to \mathbb{R}$ CONTÍNUA QUE SEJA HARMÔNICA EM $B(0,1) \setminus \{0\}$ E SOLUÇÃO DO PROBLEMA.

DEMO: SUPONHA QUE $\exists$ UMA FUNÇÃO u DA FORMA ACIMA.

LOGO $\lim_{x \to 0} u(x) = 1. \Rightarrow \exists$ O LIMITE! PELO TEOREMA DE REMOÇÃO DE SINGULARIDADES, u É HARMÔNICA EM $B(0,1)$.

LOGO PELA FÓRMULA DO VALOR MÉDIO, TEMOS

$$u(0) = \frac{1}{|\partial B(0,r)|} \int_{\partial B(0,r)} u(y)\, dS(y), \quad \forall\, 0 < r < 1$$

TOMANDO O LIMITE $r \to 1^-$, TEMOS $u(0) = ⨍_{\partial B(0,1)} \cancelto{0}{u(y)}\, dS(y) = 0.$

MAS $u(0) = 1$. ABSURDO!

b) $\exists$ SOLUÇÃO $\Leftrightarrow$ $u_0 = \fint_{\partial B(0,1)} g(x)dx$.

$$\begin{cases} \Delta u(x) = 0 \\ u(x) = g(x), \ x \in \partial B(0,1) \\ u(0) = u_0 \end{cases}$$

DEMO: ($\Leftarrow$) SEJA u SOLUÇÃO DE $\begin{cases} \Delta u(x) = 0, & x \in B(0,1) \\ u(x) = g(x), & x \in \partial B(0,1) \end{cases}$

(u É DADO PELA FÓRMULA DE POISSON).

LOGO u É HARMÔNICA EM $B(0,1)$. LOGO

$$u(0) = \fint_{\partial B(0,r)} u(y)dS(y), \ \forall 0 < r < 1 \underset{r \to 1^-}{\Rightarrow} u(0) = \fint_{\partial B(0,1)} g(x)dx = u_0.$$

LOGO u É SOLUÇÃO DE $\begin{cases} \Delta u(x) = 0, & x \in B(0,1) \\ u(x) = g(x), & x \in \partial B(0,1) \\ u(0) = u_0 \end{cases}$

($\Rightarrow$) SE EXISTE SOLUÇÃO, ENTÃO $\lim_{x \to 0} u(x) = u(0)$. LOGO, PELO T.R.S., u É HARMÔNICA EM $B(0,1)$. LOGO

$$u(0) = \fint_{\partial B(0,r)} u(y)dS(y) \Rightarrow u_0 = u(0) = \fint_{\partial B(0,1)} g(x)dx$$

$\blacksquare$