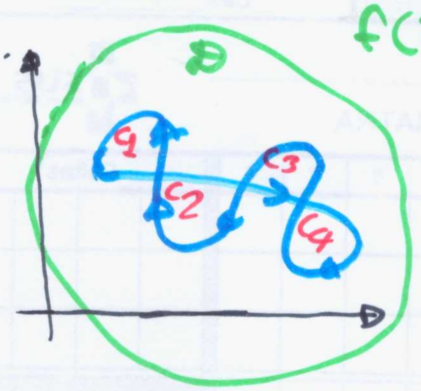
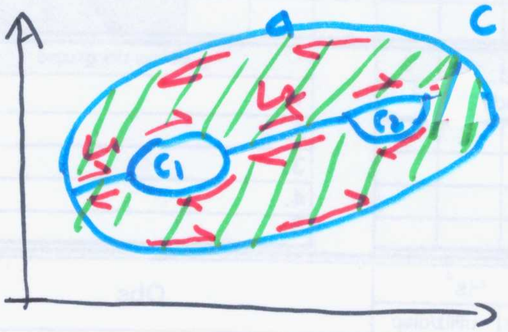
22/10/2020 ①



$f(z)$ is analytic within $\mathcal{D}$

$$\oint_C f(z)\,dz = 0$$

Theorem:



a) C is a simple closed contour → counterclockwise

b) C_k $(k = 1, 2 \ldots n)$ finite number of simple closed contours within C

$C_k \subset C$

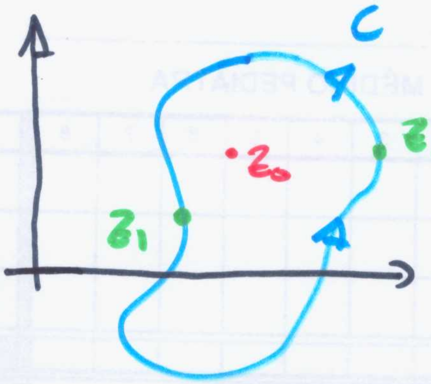
c) $f(z)$ is analytic within the boundaries delimited by C and C_k

Then

$$\oint_C f(z)\,dz + \sum_{J=1}^{n} \oint_{C_J} f(z)\,dz = 0$$

Corollary: a $f(z)$ that is analytic throughout a simply connected domain $\mathcal{D}$ must have an antiderivative in $\mathcal{D}$

Problem 3 from Page 126 (Example)



$$\oint_{C_0} (z-z_0)^{n-1}\,dz = 0$$

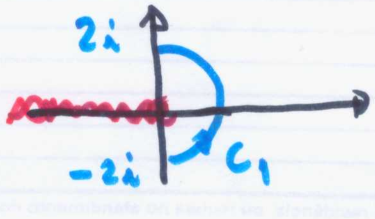
$n = \quad , \pm 2 \ldots \; \forall n \in \mathbb{Z}$

$$\int_{z_1}^{z_2} (z-z_0)^{n-1}\,dz = \left[\frac{(z-z_0)^n}{n}\right]_{z_1}^{z_2}$$

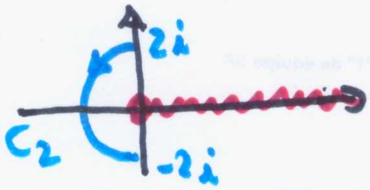
then, whenever $z_1 \equiv z_2$ and C_0 does not go through z_0, we have:

$$\oint_{C_0} (z-z_0)^{n-1}\,dz \stackrel{!}{=} 0$$

Problem 7. Page 117, another example - in fact it had already been solved. So, just a summary of the result:

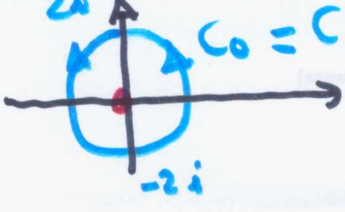


$$\oint_{C_0} \frac{dz}{z} = 2\pi i$$



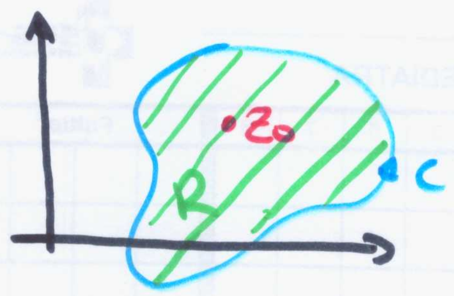
Based on the antiderivative

$$[\log z]' = \frac{1}{z}$$



$C_0 = C_1 + C_2$

Cauchy Integral formula:



$f(z)$ is analytic throughout $R \cup C$ and $z_0 \in R$

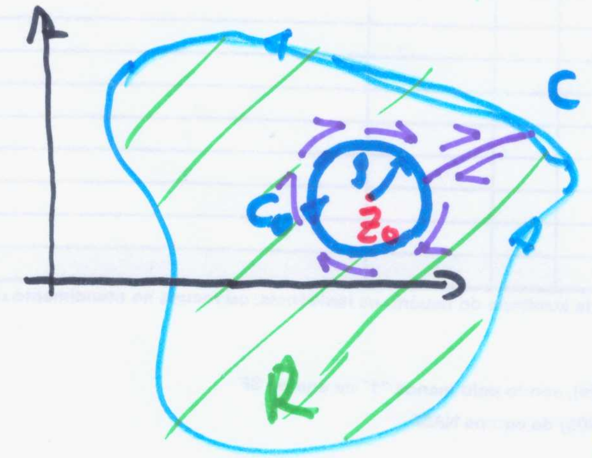
then $f(z_0) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z)dz}{(z-z_0)}$

We assume that $f(z)$ is continuous throughout $R \cup C$ — since it is analytic and, hence, differentiable there. Hence we have:

$\forall \varepsilon > 0 \ \exists \delta \mid |f(z) - f(z_0)| < \varepsilon \Leftarrow |z - z_0| < \delta$

then we define a circle of radius $\rho \mid \rho < \delta$

And get: $|f(z) - f(z_0)| < \varepsilon$ over $|z - z_0| = \rho$



$\dfrac{f(z)}{(z-z_0)}$ is analytic within the region delimited by C and C_0

{where C_0 is now run counterclockwise !!

Then, from C-G Theorem:

$$\oint_C \frac{f(z)}{(z-z_0)} dz - \oint_{C_0} \frac{f(z)}{(z-z_0)} = 0$$

$$\oint_C \frac{f(z)\,dz}{(z-z_0)} = \oint_{C_0} \frac{f(z)\,dz}{(z-z_0)}$$

$$\oint_C \frac{f(z)\,dz}{(z-z_0)} - f(z_0)\int_{C_0} \frac{dz}{(z-z_0)} = \int_{C_0} \frac{f(z)-f(z_0)}{(z-z_0)}\,dz$$

Now on the basis of the previous example, we can write:

$$\int_{C_0} \frac{dz}{(z-z_0)} = 2\pi i$$

$$\oint_C \frac{f(z)\,dz}{(z-z_0)} - 2\pi i\, f(z_0) = \oint_{C_0} \frac{f(z)-f(z_0)}{(z-z_0)}\,dz$$

on making use of the fact that:

$|f(z)-f(z_0)| < \varepsilon$, $|z-z_0| = \rho$ and the length of C_0 is $2\pi\rho$, we get:

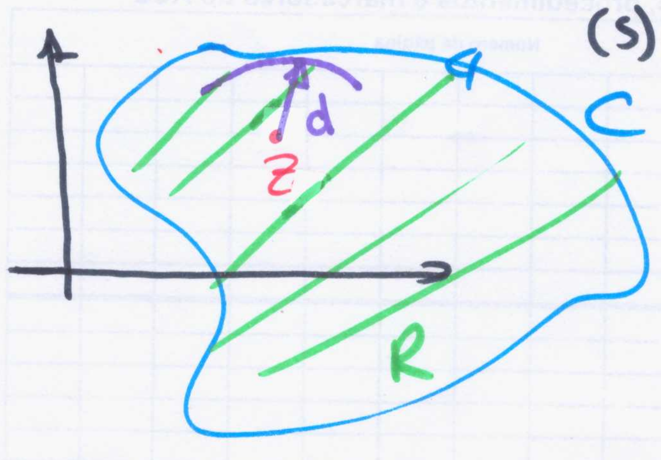
$$\left|\oint_{C_0} \frac{f(z)-f(z_0)}{(z-z_0)}\right| < \frac{\varepsilon}{\rho}\, 2\pi\rho = 2\pi\,\varepsilon$$

Hence we have:

$$\left|\int \frac{f(z)\,dz}{(z-z_0)} - 2\pi i\, f(z_0)\right| < \varepsilon$$

since $\varepsilon > 0$ is arbitrary, the LHS $= 0$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)} \qquad f(s) \text{ is analytic in } R \cup C$$



Now we shall prove that, under the same conditions
Namely, $f(s)$ is analitic in $R \cup C$, we have:

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)\,ds}{(s-z)^2}$$

In order to prove it, we make:

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i}\oint_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(z)}{\Delta z}\,ds$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)}$$

for $0 < |\Delta z| < d$, where d is the shortest distance from z to the contour C

⑥

We want to show that the above integral approaches the limit as $\Delta z \to 0$:

$$\oint_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} \right] f(s)\,ds$$

As a side note, we have:

$$\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2} = \frac{s-z-s+z+\Delta z}{(s-z-\Delta z)(s-z)^2} =$$

$$= \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}$$

So, the above integral becomes:

$$\Delta z \oint_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)^2}$$

Let M denote: $M = \max_C \{|f(s)|\}$ and L be the length of C. Also for $s \in C \Rightarrow |s-z| \geq d$ (since d is the minimum distance between z and any point on C)

$$|s-z-\Delta z| \geq \big| |s-z| - |\Delta z| \big| \geq d - |\Delta z|$$

$$\frac{1}{|s-z-\Delta z|} \leq \frac{1}{d-|\Delta z|} \quad \Bigg| \quad \frac{1}{|s-z|} \leq \frac{1}{d}$$

$$\left| \Delta z \oint_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{|\Delta z| M L}{(d-|\Delta z|)\,d^2}$$

Now we have: $\displaystyle\lim_{\Delta z \to 0} \frac{|\Delta z| M L}{(d-|\Delta z|)d^2} = 0$

Hence we get:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z)^2}$$

and Therefore, one obtains the final proof.

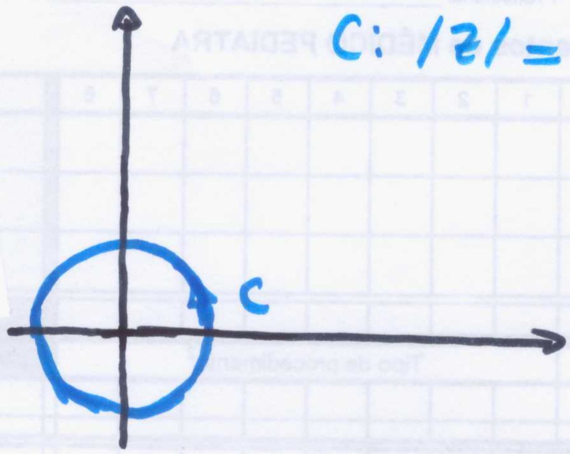
$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z)^2}$$

Corollary: if a function $f(z) = u(x,y) + i v(x,y)$ is analytic at a point $z = x+iy$, then the component functions "u" and "v" have continuous partial derivatives of ALL orders at that point. In the sense that $f^{(0)}(z) = f(z)$ and $[C]^{(n)} = 0$, we can prove by induction that:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)\,ds}{(s-z)^{n+1}} \qquad n = 0, 1, 2 \ldots$$

$f(s)$ analytic in RUC

Page 125, problem 9



$C: |z| = 1$, $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$

show that:

$$\oint_C \frac{e^{az}}{z}\,dz = 2\pi i$$

$\forall a \in \mathbb{R}$

$$z \in C:\ z = e^{i\theta} \Rightarrow dz = i e^{i\theta}\,d\theta \quad \frac{dz}{z} = i\,d\theta$$

$$\int_{-\pi}^{\pi} \exp\{a e^{i\theta}\}\, i\,d\theta = \int_{-\pi}^{\pi} \exp\{a\cos\theta + i a\sin\theta\}\, i\,d\theta =$$

$$= i\int_{-\pi}^{\pi} \underbrace{e^{a\cos\theta}}_{\text{even}} \Big\{\underbrace{\cos[a\sin\theta]}_{\text{even}} + i\underbrace{\sin[a\sin\theta]}_{\text{odd}}\Big\}\, i\,d\theta =$$

$$= i2\int_{0}^{\pi} e^{a\cos\theta} \cos[a\sin\theta]\,d\theta = ?$$

Now, on making use of the Cauchy integral

$$\oint_C \frac{e^{az}}{z}\,dz = 2\pi i\, e^{az}\Big|_{z=0} = 2\pi i$$

Morera's Theorem

if $f(z)$ is continuous throughout a domain D and if

$$\oint_C f(z)\,dz = 0$$

for every closed contour C lying in D, then $f(z)$ is analytic throughout D.

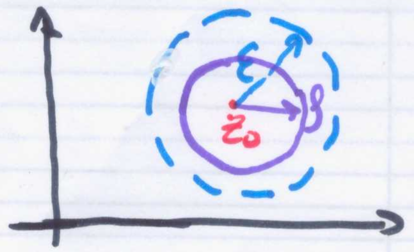
Maximum Moduli of functions

Lemma:

$f(z)$ is analytic in $|z-z_0| < \varepsilon$

if $|f(z)| \leq |f(z_0)|$ $\forall z$ in this neighborhood,

then $f(z)$ is constant there.



Let us define a circle

$$C: |z-z_0| = \rho < \varepsilon$$

$$z = z_0 + \rho e^{i\theta} \quad (0 \leq \theta \leq 2\pi)$$

$$dz = \rho i e^{i\theta} d\theta = i(z-z_0)\,d\theta$$

from Cauchy's integral, we get:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$

by hypothesis, we have:

$$|f(z_0+\rho e^{i\theta})| \leq f(z_0)$$

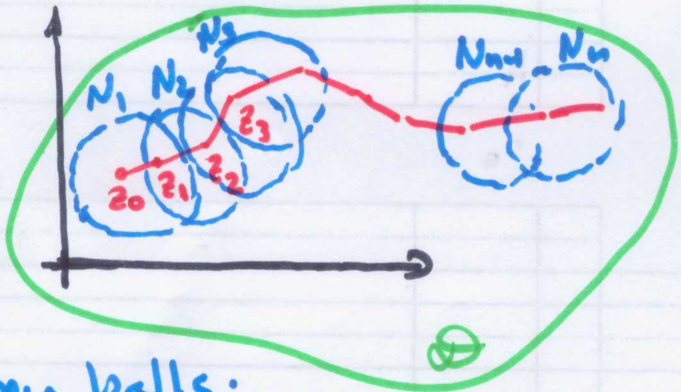
Hence, we can make:

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta = |f(z_0)|\, \frac{2\pi}{2\pi}$$

since $|f(z_0)| = |f(z_0)|$

the Lemma is proved. $|f(z_0)| = |f(z_0+\rho e^{i\theta})|$



Theorem: $f(z)$ is analytic in $\mathcal{D}$ and it is not constant. then $\max\{|f(z)|\} \in \mathcal{D}$

open balls:

open neighborhoods N_j

$|z_k - z_{k-1}| < d$

Now, if we start by assuming that there is a maximum at z_0

$|f(z_0)|$, $z_0 \in N_1$, $z_1 \in N_1$

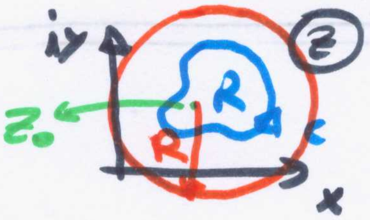
$$|f(z_0)| = |f(z_1)|$$

$$|f(z_1)| = |f(z_2)|$$

$$\vdots$$

$$|f(z_{n-1})| = |f(z_n)|$$

Liouville's theorem.



$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

$n = 0, 1, 2, \ldots$

f analytic in R∪C

we've seen that under such conditions

$\max |f(z)| \Big|_{R \cup C}$ is on C

Then let's choose C as a circle that is entirely within the Domain of analyticity of f(z), and has its center on z_0.

$C: |z - z_0| = R \qquad \max_C \{|f(z)|\} = M_R$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\frac{M_R}{R^{n+1}} 2\pi R$$

$$|f^{(n)}(z_0)| \leq n! \frac{M_R}{R^n} \overset{(n=1)}{\Longrightarrow} |f'(z_0)| \leq \frac{M_R}{R}$$

Therefore, no entire function other than a constant can be bounded over the entire (z) plane

On assuming that f(z) is entire(*)

Consider Cauchy's integral in the form

$$f'(z) = \frac{n!}{2\pi i} \oint_{C_R} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} \Rightarrow \begin{cases} \max\{|f(z)|\} = M_R \\ \text{RUC} \\ \text{where } M_R \text{ is on } C_R \\ |f'(z)| \le \dfrac{M_R}{R} \end{cases}$$

(*) an entire function is analytic over the whole complex plane (Z)

Then we want to consider the Limit:

$$\lim_{R\to\infty} |f'(z)| \le \begin{cases} \to \infty \text{ if } M_R \xrightarrow[R\to\infty]{} \infty \\ \text{and } f(z) \text{ blows up} \\ \text{as } R \to \infty \end{cases}$$

or $\to 0$ in case M_R is finite regardless of $R \to \infty$

in which case we get $|f'(z)| \le 0 \Rightarrow$

$|f'(z)| = 0 \quad \forall\, z \in \mathbb{C} \Rightarrow f(z) = \text{constant}$

Liouville's Theorem: If a function is entire and bounded in the complex plane, then it is constant.

Fundamental theorem of Algebra:

any polynomial:

$$P(z) = a_0 + a_1 z + \dots + a_n z^n \qquad (a_n \neq 0)$$

of degree $n \geqslant 1$ has at least 1 zero.

that is $\exists\, z_0 \in \mathbb{C} \mid P(z_0) = 0$

Suppose that $P(z)$ has no zeros in $\mathbb{C}$

then we define:

$$f(z) = \frac{1}{P(z)}$$

and under these conditions, $f(z)$ is entire.

it is also bounded, for we can write:

$$w \equiv \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots \frac{a_{n-1}}{z} \Rightarrow \begin{cases} \text{for } |z| > R \\ |w| < \frac{|a_n|}{2} \end{cases}$$

$$P(z) = (a_n + w) z^n \quad \text{for } |z| > R \quad (R \gg 1)$$

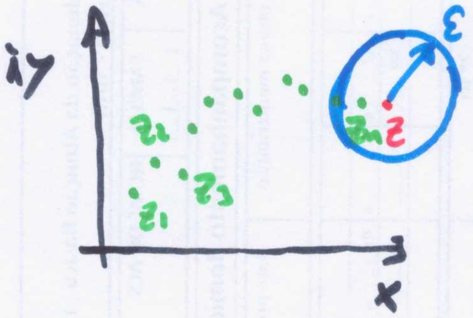
$$|a_n + w| \geqslant \big|\, |a_n| - |w| \,\big| > \frac{|a_n|}{2}$$

$$|f(z)| = \frac{1}{|P(z)|} = \frac{1}{|a_n + w||z^n|} < \frac{2}{|a_n| R^n}$$

Then we would get that $f(z)$ is entire and bounded which contradicts Liouville's theorem and is, thus, absurd.

Which, in turn, proves the fundamental theorem of Algebra.

Convergence of a sequence of complex Numbers.



the sequence is said to converge if:

$$\forall \varepsilon>0\ \exists n_0 \in \mathbb{N}\ \Big|\ |z-z_n|<\varepsilon\ (\forall n)$$

Theorem 1: $\left\{ \begin{matrix} z_n = x_n + i y_n \\ n = 1, 2, \dots \end{matrix} \right\}$ and $z = x + iy$

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Series $\Rightarrow$ summation of sequences

$$S_n \equiv \sum_{k=1}^{N} \underbrace{(x_n + i y_n)}_{z_n} = \sum_{k=1}^{N} x_n + i \sum_{k=1}^{N} y_n$$

An infinite series can be seen as an infinite sequence of partial sums

$$S_n \xrightarrow[n\to\infty]{} S \quad \begin{cases} S = X + iY = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n \\ (\Longleftrightarrow) \quad \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y \end{cases}$$

And on the basis of the above one proves that — the proof is easy and it's in the book (chapter 5 - page 133)

$$\sum_{n=0}^{\infty} z_n = S \Longleftrightarrow \begin{cases} \sum_{n=0}^{\infty} x_n = X \\ \sum_{n=0}^{\infty} y_n = Y \end{cases}$$

A necessary condition for convergence of an infinite series: $S = \sum_{n=0}^{\infty} z_n$ is $\lim_{k\to\infty} z_k = 0$

↳ proof can be read in the book.

Absolute convergence:

$$\underbrace{\sum_{n=0}^{\infty} |z_n| = \sum_{n=0}^{\infty} \sqrt{x_n^2 + y_n^2}}_{\text{Real series}} \qquad \begin{cases} |x_n| \leq |z_n| \\ |y_n| \leq |z_n| \end{cases}$$

Let us assume that a given series $S = \sum_{n=0}^{\infty} z_n$ meets the criterion of absolute convergence, that is: $\sum_{n=0}^{\infty} |z_n|$ converges

where $\sum_{n=0}^{\infty} |z_n|$ is a real and positive series and, as always: $|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$

Then, by the comparison criterion for <u>real</u> positive series, we have that:

$\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ should both converge

Because:
$$\begin{cases} \sum_{n=0}^{\infty} x_n \leq \sum_{n=0}^{\infty} |x_n| \leq \sum_{n=0}^{\infty} |z_n| \\ \sum_{n=0}^{\infty} y_n \leq \sum_{n=0}^{\infty} |y_n| \leq \sum_{n=0}^{\infty} |z_n| \end{cases}$$

Then, on the basis of the above theorem for complex series, we get:

$$\sum_{n=0}^{\infty} z_n \text{ converge} \iff \left.\begin{cases} \sum_{n=0}^{\infty} x_n \\ \sum_{n=0}^{\infty} y_n \end{cases}\right\} \text{Both converge}$$

And the end result is that for complex series:

absolute convergence $\Rightarrow$ convergence

Example: Problem 3 from page 137

show that if $\lim_{n\to\infty} z_n = z \Rightarrow \lim_{n\to\infty} |z_n| = |z|$

$$\lim_{n\to\infty} z_n = z \Rightarrow \forall \varepsilon > 0 \mid |z - z_n| < \varepsilon \Leftarrow n < N_\varepsilon$$

$$||z_n| - |z|| < |z_n - z| < \varepsilon \Leftarrow n < N_\varepsilon$$

$$||z_n| - |z|| < \varepsilon \Leftarrow n < N$$

Therefore we get: $\lim_{n\to\infty} |z_n| = |z|$

Problem 4 from the same Page (137)

$$\sum_{p=0}^{N} z^p = \frac{1 - z^{p+1}}{1-z} = S_N$$

$$\sum_{n=1}^{\infty} z^n = \frac{z}{(1-z)} \quad \text{whenever } |z| < 1$$

Let's make

$$S_n = \sum_{k=0}^{n} z^k = 1 + \sum_{k=1}^{n} z^k = \frac{(1 - z^{n+1})}{1-z} - 1 = \frac{z(1-z^n)}{(1-z)}$$

Let us define an expression for the reminders of the summation as:

$$\rho_n(z) \equiv \sum_{p=1}^{\infty} z^p - \sum_{p=1}^{n} z^p = S - [S_n - 1] \quad \Big| \; |z| < 1$$

$$\rho_n(z) = \frac{z}{(1-z)} - z\frac{(1-z^n)}{(1-z)} = \frac{z^{n+1}}{(1-z)}$$

$$\lim_{n\to\infty} \rho_n(z) = \lim_{n\to\infty} \frac{z^{n+1}}{(1-z)} = \lim_{n\to\infty} \frac{r^{n+1} e^{i(n+1)\theta}}{(1-re^{i\theta})} = 0$$

where $r < 1$

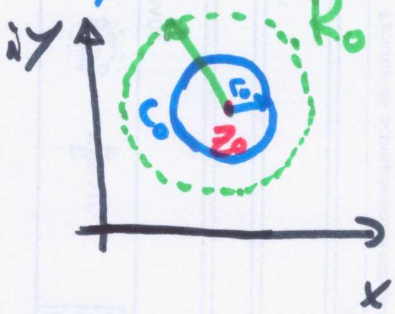
Therefore, we have proven that, indeed:

$$\boxed{\sum_{p=1}^{\infty} z^n = \frac{z}{1-z} \quad \forall\, |z| < 1}$$

$$1 + \sum_{p=1}^{\infty} z^n = 1 + \frac{z}{1-z}$$

$$\sum_{p=0}^{\infty} z^n = \frac{1-z+z}{(1-z)} = \frac{1}{(1-z)} \qquad |z| < 1$$

Taylor Series:



$|z-z_0|<R_0 \Longrightarrow f(z)$ is analytic

$C_0: |z-z_0| = r_0 < R_0$

under such conditions:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

where $a_k = \dfrac{f^{(k)}(z_0)}{k!} = \dfrac{1}{2\pi i}\oint_{C_0} \dfrac{f(z)\,dz}{(z-z_0)^{k+1}}$

In particular, we have: $f(z) = \dfrac{1}{2\pi i}\oint_{C_0} \dfrac{f(s)\,ds}{(s-z)}$

and, also, $\dfrac{1}{s-z} = \dfrac{1}{s}\left[\dfrac{1}{1-(z/s)}\right]$

for a finite sum of a G.P., we have:

$$\frac{1}{1-c} = \underbrace{1 + c + c^2 + \dots + c^{n-1}}_{\frac{1-c^n}{(1-c)}} + \frac{c^n}{(1-c)}$$

$$\boxed{\sum_{k=0}^{N-1} c^n = \frac{1}{(1-c)} - \frac{c^N}{(1-c)}}$$

Hence, we can make:

$$\frac{1}{s-z} = \frac{1}{s}\left[1 + \left(\frac{z}{s}\right) + \left(\frac{z}{s}\right)^2 + \dots + \left(\frac{z}{s}\right)^{N-1} + \underbrace{\frac{(z/s)^N}{1-(z/s)}}_{\frac{z^N}{s^{N-1}(s-z)}}\right]$$

Whence it comes that:

$$\frac{1}{(s-z)} = \frac{1}{s} + \frac{1}{s^2}z + \frac{1}{s^3}z^2 + \dots + \frac{1}{s^N}z^{N-1} + \frac{z^N}{(s-z)s^N}$$

$$\frac{1}{2\pi i}\oint_{C_0} \frac{f(s)}{(s-z)}dz = \frac{1}{2\pi i}\oint_{C_0} \frac{f(s)}{s}ds + \frac{z}{2\pi i}\oint_{C_0} \frac{f(s)}{s^2}ds + \dots$$

$$\dots + \frac{z^{N-1}}{2\pi i}\oint_{C_0} \frac{f(s)}{s^N}ds + \frac{z^N}{2\pi i}\oint_{C_0} \frac{f(s)}{(s-z)s^N}ds$$

$$f(z) = f(0) + \frac{f'(0)z}{1!} + \frac{f''(0)z^2}{2!} + \dots$$

$$+ \dots \frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1} + \rho_N(z)$$

Where: $\rho_N(z) = \frac{z^N}{2\pi i}\oint_{C_0} \frac{f(s)\,ds}{s^N(s-z)}$

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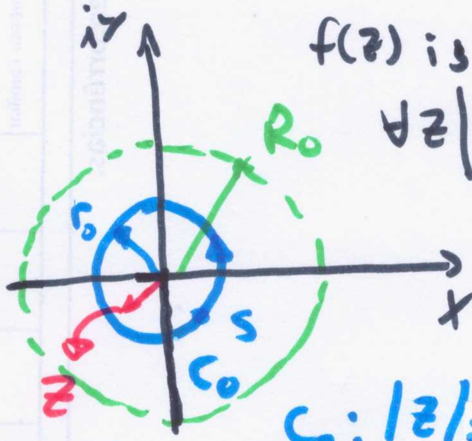
$$f(z) = f(0) + \frac{f'(0)z}{1!} + \frac{f''(0)z^2}{2!} + \cdots \frac{f^{(N-1)}(0)z^{N-1}}{(N-1)!} + \rho_N(z)$$

$$\rho_N(z) = \frac{z^N}{2\pi i}\oint_{C_0} \frac{f(s)\,ds}{s^N(s-z)}$$



$f(z)$ is analytic $\forall z \,|\, |z-0| < R_0$

$C_0 : |z| = r_0$

for a point s on C_0:

(*) $|s-z| \geq \big||s|-|z|\big| = r_0 - r$

$$|\rho_N(z)| \leq \frac{r^N}{2\pi}\frac{M\,2\pi r_0}{(r_0-r)r_0^N} = \frac{Mr_0}{r_0-r}\left(\frac{r}{r_0}\right)^N$$

$z = re^{i\theta}$, $|z| = r$

$r < r_0$

$M = \max_{C_0}\{f(z)\}$

Since $r < r_0 \Rightarrow \left(\frac{r}{r_0}\right) < 1$

Therefore: $\lim_{N\to\infty}\left(\frac{r}{r_0}\right)^N = 0 \Rightarrow \lim_{N\to\infty}\rho_N(z) = 0$

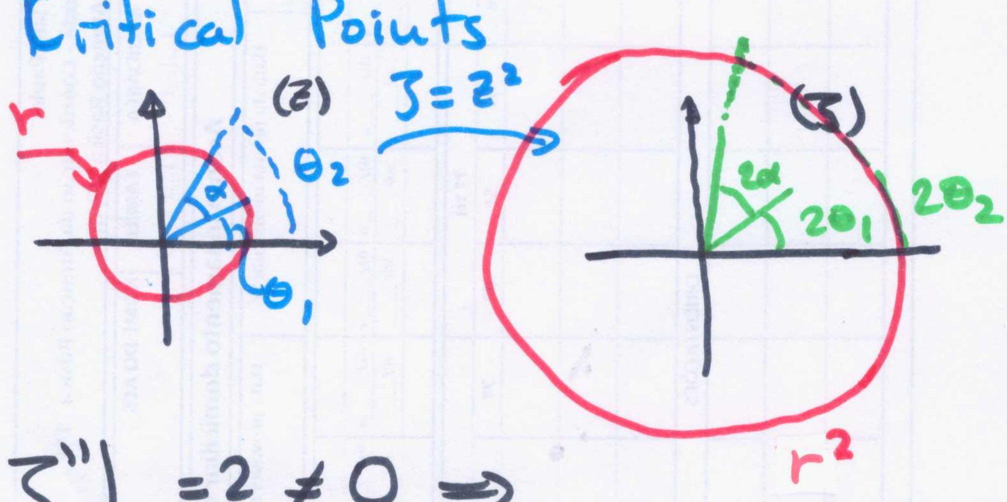
In fact, this result holds for any contour inside the open disk $|z| < R_0$

(*) $$\frac{1}{|s-z|} \leq \frac{1}{(r_0-r)}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

this (e^z) is an entire function. It's analytic over the whole complex plane, so, for the series $R_0 \to \infty$. It converges for $\forall z \mid |z| < \infty$

Critical Points



$$\zeta = z^2$$

$$\frac{d\zeta}{dz} = 2z$$

$$\zeta\Big|_{z=0} = 0, \quad \zeta'\Big|_{z=0} = 0$$

$$\zeta''\Big|_{z=0} = 2 \neq 0 \Rightarrow$$

Now, for $\zeta = z^n \Rightarrow \rho e^{i\varphi} = r^n e^{in\theta}$

$$\rho = r^n, \quad \varphi = n\theta$$

And for a Taylor Series: $f^{(k)}(z_0) = 0 \ \forall k < n$

$$\zeta = f(z) \Rightarrow \zeta = f(z_0) + a_n (z - z_0)^n \left[1 + \frac{a_{n+1}}{a_n}(z - z_0) + \dots \right]$$

$$(\zeta - \zeta_0) = (z - z_0)^n g(z), \quad \frac{dg}{dz}\Big|_{z=z_0} \neq 0$$

Laurent Series

$f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ and C is an arbitrary simple closed contour around z_0 and within $\mathcal{D}$

$\mathcal{D}: \ R_1 < |z - z_0| < R_2$

under such circumstances, we have:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad n = 0, 1, 2 \ldots$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} (z - z_0)^n\,dz$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \ ; \ c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

$$\forall n \in \mathbb{Z} \ ; \ n = 0, \pm 1, \pm 2, \ldots$$

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z} \; ; \; \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} = \frac{1}{(1-z)} - \frac{z^N}{(1-z)}$$

$f(z)$ is analytic throughout

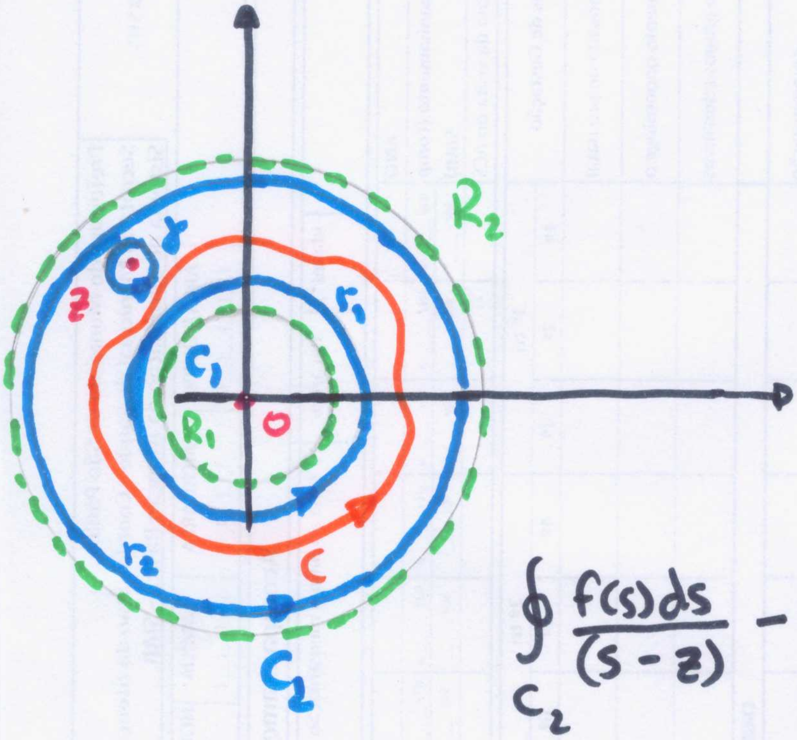
$\mathcal{D}: R_1 < |z| < R_2$

closed annular region

$r_1 \leq |z| \leq r_2$

$C_1: |z| = r_1$

$C_2: |z| = r_2$



$$\oint_{C_2} \frac{f(s)\,ds}{(s-z)} - \oint_{C_1} \frac{f(s)\,ds}{(s-z)} - \underbrace{\oint_{\gamma} \frac{f(s)\,ds}{(s-z)}}_{2\pi i\, f(z)} = 0$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)\,ds}{(s-z)} - \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)\,ds}{(s-z)}$$

$$\frac{1}{(s-z)} = \frac{1}{s} + \frac{1}{s^2} z + \frac{1}{s^3} z^2 + \ldots + \frac{1}{s^N} z^{N-1} + \frac{z^N}{(s-z)s^N}$$

Now, for the 2nd integral, we make:

$$-\frac{1}{s-z} = \frac{1}{z-s}$$

$$-\frac{1}{(s-z)} = \frac{1}{z} + \frac{1}{s^{-1}}\frac{1}{z^2} + \ldots + \frac{1}{s^{-N+1}}\frac{1}{z^N} + \frac{1}{z^N}\left(\frac{s^N}{z-s}\right)$$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{N-1} z^{N-1} + \rho_N(z) +$$

$$+ \frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_N}{z^N} + \sigma_N(z)$$

$$a_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)\,ds}{s^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)\,ds}{s^{-n+1}}$$

$$\rho_N = \frac{z^N}{2\pi i} \oint_{C_2} \frac{f(s)\,ds}{(s-z)s^N} \quad ; \quad \sigma_N = \frac{1}{2\pi i\, z^N} \oint_{C_1} \frac{s^N f(s)\,ds}{(z-s)}$$

$$M = \max_{C_1 \text{ and } C_2} \{ |f(z)| \}$$

on C_1: $|z-s| \geq r - r_1$

$$|\rho_N| \leq \frac{M r_2}{(r_2 - r)} \left(\frac{r}{r_2}\right)^N \quad , \quad (r < r_2) \text{ on } C_2$$

$$|\sigma_N| \leq \frac{M r_1}{(r - r_1)} \left(\frac{r_1}{r}\right)^N \quad ; \quad (r > r_1) \text{ on } C_1$$

Hence, $\lim_{N\to\infty} \sigma_N = 0$ and $\lim_{N\to\infty} \rho_N = 0$

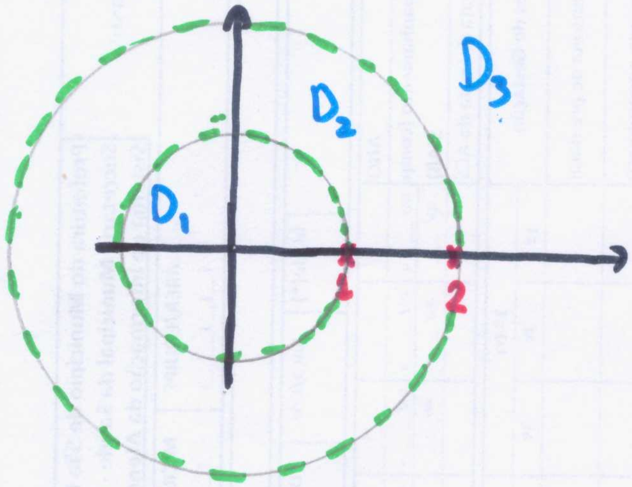
A Few examples:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\cosh(z) = \cos(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n (i)^{2n}}{(2n)!} z^{2n} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

Example 3

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$



$D_1 \Rightarrow |z| < 1$

$$f(z) = -\frac{1}{1-z} + \frac{1}{2}\left[\frac{1}{1-(z/2)}\right]$$

for $|z| < 1 \Rightarrow \frac{|z|}{2} < 1$

Hence, we make : $f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

D_2: $1 < |z| < 2 \Rightarrow$ change of variables

$\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z}\left[\frac{1}{1-(1/z)}\right] + \frac{1}{2}\left[\frac{1}{1-(z/2)}\right]$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

D_3: $|z| > 2 \Rightarrow |1/z| < 1$ and $|2/z| < 1$

$$f(z) = \frac{1}{z}\left[\frac{1}{1-(1/z)}\right] - \frac{1}{z}\left[\frac{1}{1-(2/z)}\right]$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$\frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^n \quad |z|<1 \quad \text{and} \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\forall z \in \mathbb{C} \,|\, |z| < \infty$$

$$\frac{1}{(1+z)} = \sum_{n=0}^{\infty} (-1)^n z^n \quad |z|<1$$

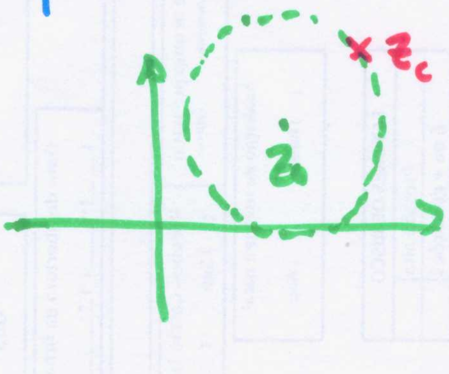
Theorems about Power Series representation of Analytic functions.

They are proved in the book for Taylor Series, but they hold just the same for negative powers, except on the singularity itself, as can be shown by the inversion transformation.

Theorem 1 $\Rightarrow$ $S = \sum_{n=0}^{\infty} a_n z^n$ if it converges

at $z = z_1 \neq 0$ $\Longrightarrow$ it converges $\forall z \in \mathbb{C} \,|\, |z| < |z$

(expansion about $z_0 = 0$)



Taylor converges in

$C: |z - z_0| < |z_c - z_0|$

negative Powers

$|z - z_0| > |z_c - z_0|$

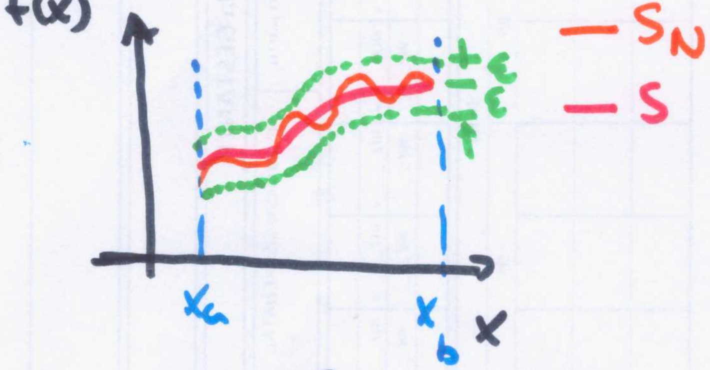
Theorem 2

if z_1 is an interior point to the convergence circle $|z|<R$, then this series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly at $|z| \le |z_1|$

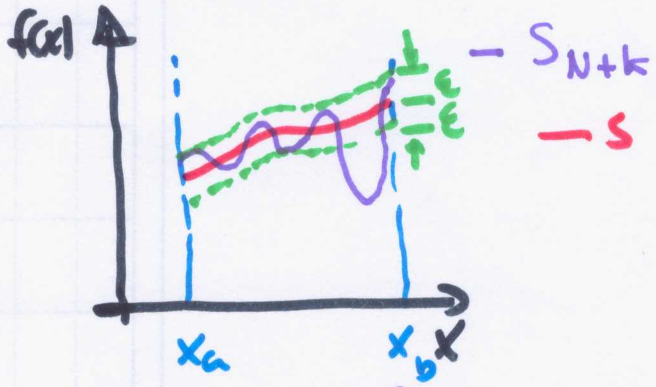
Uniform convergence

$\forall \varepsilon > 0 \; \exists N \in \mathbb{I}, N>0 \mid \forall n \geq N$

$|S - S_N| < \varepsilon$ and N is the same throughout the convergence Region



uniform convergence



non uniform convergence $\Rightarrow N = N(x)$

Uniform convergence is crucial for one to be able to differentiate or to integrate a series term by term.

Theorem 3. if $g(z)$ is continuous on C

$$\oint_C S(z)g(z)\,dz = \sum_{n=0}^{\infty} a_n \oint_C g(z) z^n \, dz$$

theorem 4 _ within the domain of convergence

$$S'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Uniqueness : if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges within the open disk $|z-z_0| < R$ then it <u>IS</u> the Taylor Series expansion of $f(z)$ there.

Theorem 5

if $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ converges to $f(z)$

within an annular region about z_0, then it is the Laurent series of $f(z)$ within that region.

Finally, Laurent series can be combined, and there is also the so-called Cauchy product of series in the book. That must also be read.