Introduction to random vibrations

PEF 6000 - Special topics on dynamics of structures

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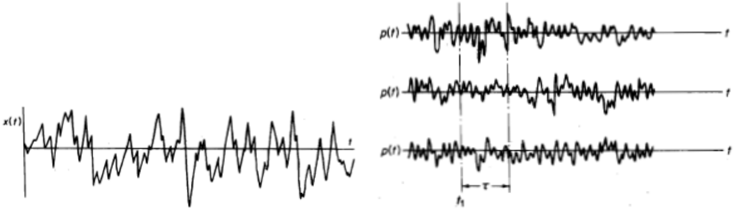
Outline

1. Objectives and references
2. Introduction
3. Fourier Series
4. Fourier Transform
5. Signal analysis with MATLAB®: Practical aspects
6. Examples of application
7. Power spectrum density (PSD)
8. Response of a 1-dof system to random excitation

- To introduce to basic aspects related to random vibrations;
- Focuses of the classes: Statistics, Fourier Series/Transform, frequency domain analysis, relation between statistics and frequency domain representation;
- Examples of references
  1. Thomson, W.T. & Dahleh, M.D., 2005. *Theory of vibration with application*. Pearson education.
  2. Meirovitch, L., 2000. *Fundamentals of vibrations*. McGraw-Hill.
  3. Aguirre, L.A., 2015. *Introdução à identificação de sistemas técnicas lineares e não lineares: Teoria e Aplicação*. Editora UFMG.

- Up to this point, focus has been placed on the deterministic dynamics, i.e., both the structural properties and the external excitation are fully known;
- In technological applications, a number of dynamic phenomena are non-deterministic (or, in other words, exhibit a random behavior). Examples: seismic excitation, loads due to surface waves in offshore structures, aerodynamic loads associated with turbulence...
- The focus of this class is on problems characterized by random excitation, but with deterministic structural properties. Despite interesting, random vibrations in which the structural properties are defined by random variables are out of the scope of this class.

- Sample: In this class, is a time-history of a certain quantity. For example, the time-history of acceleration of a certain point of a floating unit during one day. It is a random variable;
- Ensemble: A collection of samples. It is a random (or stochastic) process.



(a) Sample.   (b) Ensemble.

Figure: Extracted from Thomson & Daleh (2005).

- Time-averaging:

$$\bar{x}(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt \tag{1}$$

- This number coincides with the expected value of $x(t)$

$$E[x] = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)dt \tag{2}$$

- In practical applications, the time-histories are given in the form of a vector of finite size, containing the values of $x(t)$ at some time instants (usually employing a constant time-step). In this case

$$E[x] = \frac{1}{N} \sum_{k=1}^{N} x_k \tag{3}$$

N being the size of the time-history (sample).

- Mean-square value: Average value of $x^2(t)$.

$$E[x^2] = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt \tag{4}$$

- Variance: Average value of $(x(t) - E[x])^2$

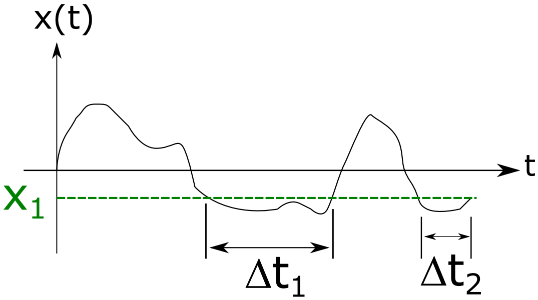
$$\sigma^2 = E[(x(t) - E[x])^2] = E[x^2 - 2xE[x] + E[x]^2] = E[x^2] - (E[x])^2 \tag{5}$$

- Standard deviation: positive root of σ^2;
- Root-mean square (r.m.s): Square-root of the mean-square value. If the signal has zero average value, the r.m.s matches the standard deviation.
- Correlation between two signals: $E[x_1(t)x_2(t)]$.
- Auto-correlation function of a real signal $R(t,\tau) = E[x(t)x(t+\tau)]$.

Definitions shown in Aguirre (2015)

- “Um processo estocástico é estacionário no sentido estrito se sua densidade de probabilidade se mantém inalterada após a mudança na origem no tempo”;
- “Um processo estocástico é estacionário no sentido amplo (ou estacionaridade fraca) se sua média é constante';'
- “Se um processo for estacionário no sentido amplo, então sua função de correlação somente dependerá da diferença temporal considerada $|t_1 - t_2| = \tau$”;
- If the process is stationary and if the average and the autocorrelation do not depend on the sample, the process is ergodic. In this case $R(t, \tau) = R(\tau)$.

- For random variables, we are interested in defining the probability of this variable assuming values larger (or smaller) than a certain value.



- Based on the above figure, calculate $P[x < x_1]$.
- Answer:

$$P[x < x_1] = P[\bar{x}] = \lim_{t \to \infty} \frac{1}{t} \sum \Delta t_k \tag{6}$$

- Notice that

$$P[x < x_1] = P[x_1] = \begin{cases} 1, & x_1 \to \infty \\ 0, & x_1 \to -\infty \end{cases} \tag{7}$$

- $P[x_1]$ is the cummulative probability distribution function.
- Now, we want to evaluate the probability of $x(t)$ lying between x_1 and $x_1 + \Delta x$. This is done by computing:

$$P[x_1 + \Delta x] - P[x_1] \tag{8}$$

- We define the density probability function as:

$$p(x) = \lim_{\Delta x \to 0} \frac{P[x_1 + \Delta x] - P[x_1]}{\Delta x} = \frac{dP}{dx} \tag{9}$$

- Hence, we can write $P[x_1]$ as function of $p(x)$ by computing the integral

$$P[x_1] = \int_{-\infty}^{x_1} p(x)dx \tag{10}$$

- Notice that $P[\infty] = 1$ (area below the curve defined by $p(x)$)
- Mean value (Centroid of the area below the curve):

$$\bar{x} = \int_{-\infty}^{\infty} x p(x) dx \tag{11}$$

- Variance (Moment of inertia around the axis defined by $\bar{x}$)

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x) dx \tag{12}$$

- Quadratic mean value (Moment of inertia around the axis defined by the ordinate axis)

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 p(x) dx \tag{13}$$

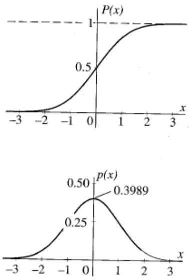
- Gaussian or normal for a random variable of null mean:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-0.5(x/\sigma)^2) \tag{14}$$

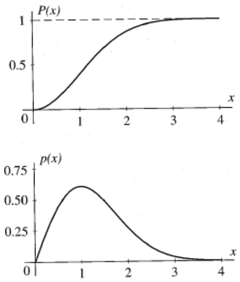
- If we have a random variable with positive values (for example, the amplitude A), Rayleigh's distribution is commonly found:

$$p(A) = \frac{A}{\sigma^2} exp(-0.5(A/\sigma)^2) \tag{15}$$

Two important probability density functions



(a) Normal distribution.



(b) Rayleigh distribution

Figure: Extracted from Meirovitch (2000).

- $x(t)$ is said to be periodic of period T if $x(t) = x(t+T), \forall t$;
- Let $x(t)$ be a periodic signal of period $T = 2\pi/\omega$. The Fourier series of this signal is the projection of $x(t)$ onto the set of harmonic function $[\cos n\omega t, \sin n\omega t], n = 1, 2, 3 \ldots, \infty$;
- Considering the usual inner product between two functions $f(t)$ and $g(t)$ $< f, g >= \int_0^T f(t)g(t)dt$ it is easy to notice that the trigonometric functions form a orthogonal basis;

- Hence

$$x(t) = \sum_{n=0}^{\infty} [a_n \sin(n\omega t) + b_n \cos(n\omega t)] =$$

$$= b_0 + \sum_{n=1}^{\infty} [a_n \sin(n\omega t) + b_n \cos(n\omega t)] \tag{16}$$

with

$$b_n = \frac{\int_0^T x(t) \cos(n\omega t) dt}{\int_0^T \cos^2(n\omega t) dt} = \frac{2}{T} \int_0^T x(t) \cos(n\omega t) dt, n = 1, 2, \ldots \tag{17}$$

$$a_n = \frac{\int_0^T x(t) \sin(n\omega t) dt}{\int_0^T \sin^2(n\omega t) dt} = \frac{2}{T} \int_0^T x(t) \sin(n\omega t) dt, n = 1, 2, \ldots \tag{18}$$

$$b_0 = \frac{\int_0^T x(t) \cos(0\omega t) dt}{\int_0^T \cos^2(0\omega t) dt} = \frac{1}{T} \int_0^T x(t) dt \tag{19}$$

- **Notice that b_0 is the mean value of the signal.**

- The Fourier series can be written either in the trigonometric or in the exponential form. In the latter case, we recall the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ and the following identities:

$$\cos(n\omega t) = \frac{e^{in\omega t} + e^{-in\omega t}}{2} \tag{20}$$

$$\sin(n\omega t) = \frac{e^{in\omega t} - e^{-in\omega t}}{2i} = \frac{i(e^{-in\omega t} - e^{in\omega t})}{2} \tag{21}$$

- Substituting the above identities in Eq. 16, we have

$$\begin{aligned} x(t) = b_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{i(e^{-in\omega t} - e^{in\omega t})}{2} \right) + b_n \left(\frac{e^{in\omega t} + e^{-in\omega t}}{2} \right) \right] = \\ = b_0 + \sum_{n=1}^{\infty} \left[\left(\frac{b_n - ia_n}{2} \right) e^{in\omega t} + \left(\frac{b_n + ia_n}{2} \right) e^{-in\omega t} \right] = \\ = b_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \underbrace{\left(\frac{b_n - ia_n}{2} \right)}_{c_n} e^{in\omega t} \end{aligned} \tag{22}$$

- Using the definitions of a_n and b_n in Eq. 22, we obtain

$$c_n = \left(\frac{b_n - ia_n}{2}\right) = \frac{1}{2}\left(\frac{2}{T}\int_0^T x(t)\cos(n\omega t)dt - i\frac{2}{T}\int_0^T x(t)\sin(n\omega t)dt\right) =$$
$$= \frac{1}{T}\int_0^T x(t)(\cos(n\omega t) - i\sin(n\omega t))dt = \frac{1}{T}\int_0^T x(t)e^{-in\omega t}dt \quad (23)$$

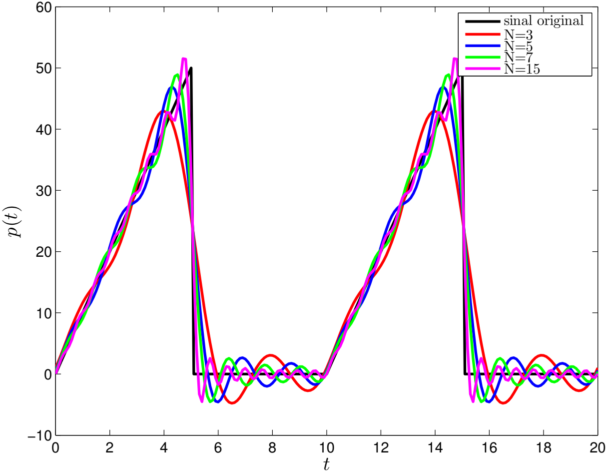
Consider the periodic function of period T_p given by:

$$p(t) = p_0 t,\ T_p k < t < T_p(2k+1)/2,\ k = 0, 1, 2 \dots \tag{24}$$

$$p(t) = 0,\ T_p(2k+1)/2 < t < T_p(k+1),\ k = 0, 1, 2, \dots \tag{25}$$

- Fundamental frequency $\bar{\omega} = \frac{2\pi}{T_p}$
- The series is truncated in N terms
- $p(t) = b_0 + \sum_{n=1}^{N} (b_n \cos(n\bar{\omega}t) + a_n \sin(n\bar{\omega}t))$
- $b_0 = \frac{1}{T_p} \int_0^{T_p} p(t)dt$
- $b_n = \frac{2}{T_p} \int_0^{T_p} p(t) \cos(n\bar{\omega}t)$
- $a_n = \frac{2}{T_p} \int_0^{T_p} p(t) \sin(n\bar{\omega}t)$

Considering $p_0 = 10$ and $T_p = 10$, we have:



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- Periodic signals can be studied using the Fourier series. What can we do if the signal is not periodic?
- We consider a non-periodic signal as a periodic one with $T \to \infty$.

- In the complex Fourier series, the frequency discretization (i.e., the interval between two identified frequencies is ω. Here, this interval is defined as $\Delta\omega = \frac{2\pi}{T}$
- Hence

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T} \left(\int_0^T x(t) e^{-in\Delta\omega t} dt \right) e^{in\Delta\omega t} \tag{26}$$

- When $T \to \infty$, the interval between frequencies goes to $d\omega$. In turn, $n\Delta\omega$ goes to ω as $T \to \infty$. Following, we obtain:

$$x(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \right) d\omega \tag{27}$$

We define

- Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{28}$$

- Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{29}$$

- Fourier transform exists if $\int_{-\infty}^{\infty} |x(t)|dt < \infty$, if the number of discontinuities and extrema in $x(t)$ are finite and if the discontinuities are also finite.

- Problem: Consider a 1-dof oscillator of mass m, linear damping constant c and stiffness k. For a known $p(t)$, what is the Fourier Transform of the displacement time-history $u(t)$?
- Equation of motion:

$$\ddot{u} + 2\zeta\omega u + \omega^2 u = \frac{p(t)}{m} \tag{30}$$

$$\omega = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{2m\omega} \tag{31}$$

- Answer: Firstly, we consider $p(t) = e^{i\bar{\omega}t}$. Steady-state responses are characterized by:

$$u(t) = He^{i\bar{\omega}t} \tag{32}$$

$$\dot{u}(t) = i\bar{\omega}He^{i\bar{\omega}t} \tag{33}$$

$$\ddot{u}(t) = -\bar{\omega}^2 He^{i\bar{\omega}t} \tag{34}$$

- $H = H(\bar{\omega})$ is the complex frequency response function.

- Substituting into the equation of motion:

$$(-\bar{\omega}^2 + \omega^2 + i2\zeta\omega\bar{\omega})H(\bar{\omega})e^{i\bar{\omega}t} = \frac{e^{i\bar{\omega}t}}{m} \tag{35}$$

$$H(\bar{\omega}) = \frac{1}{k\left[\left(1 - \left(\frac{\bar{\omega}}{\omega}\right)^2\right) + i2\zeta\frac{\bar{\omega}}{\omega}\right]} \tag{36}$$

- Now, consider

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{in\bar{\omega}t} \tag{37}$$

$$P_n = P(n\bar{\omega}) \tag{38}$$

- Owing to the linear character of the mathematical model:

$$u(t) = \sum_{n=-\infty}^{\infty} P(n\bar{\omega})H(n\bar{\omega})e^{in\bar{\omega}t} \tag{39}$$

- If the fundamental period $\bar{T} = \frac{2\pi}{\bar{\omega}} \to \infty$ and using the same approach employed for deriving the Fourier Transform, we obtain:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{H(\bar{\omega})P(\bar{\omega})}_{U(\bar{\omega})} e^{i\bar{\omega}t} d\bar{\omega} \tag{40}$$

- Hence, the Fourier Transform of $u(t)$ is $U(\bar{\omega}) = H(\bar{\omega})P(\bar{\omega})$.
- Another aspect is now discussed. Consider Duhamel's integral:

$$u(t) = \int_{-\infty}^{t} p(\tau)h(t-\tau)d\tau \tag{41}$$

- We consider $\xi = t - \tau \to \tau = t - \xi$. If $\tau = -\infty$, $\xi = \infty$. If $\tau = t$, $\xi = 0$. Hence,

$$u(t) = \int_{0}^{\infty} f(t-\xi)h(\xi)d\xi \tag{42}$$

- If $p(t) = e^{i\omega t}$

$$u(t) = \int_0^\infty e^{i\omega(t-\xi)h(\xi)d\xi} = e^{i\omega t} \underbrace{\int_0^\infty h(\xi)e^{-i\omega\xi}d\xi}_{H(\omega) \mathrm{if} h(t)=0, t<0} \tag{43}$$

- Conclusion: $H(\omega)$ is the Fourier Transform of the impulsive response.
- Note: $u(t) = f(t) * h(t)$ (convolution integral). "The Fourier Transform of the time convolution corresponds to the product in the frequency domain".

- We rewrite the pair of Fourier Transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{44}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{45}$$

- Fourier Transform of $\dot{x}(t)$:

$$\dot{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega X(\omega)e^{i\omega t}d\omega \tag{46}$$

  Hence $FT[\dot{x}] = i\omega FT[x]$.

- If we would like to write $X(f)$ (f in Hz) instead of $X(\omega)$ (ω in rad/s), we recall that $\omega = 2\pi f$ and, then,

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \tag{47}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \tag{48}$$

- For two real signals $x_1(t)$ and $x_2(t)$

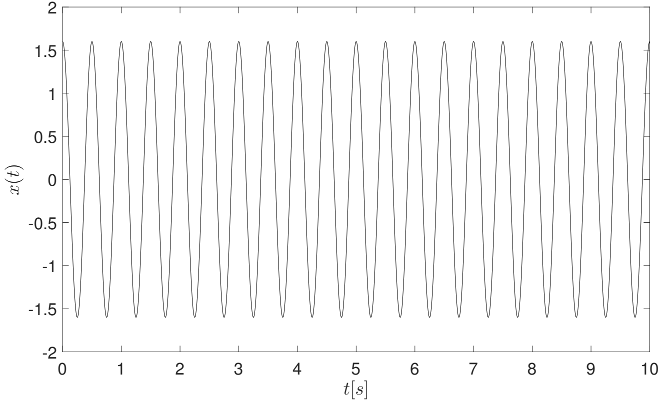
$$\int_{-\infty}^{\infty} x_1(t)x_2(t)dt = \int_{-\infty}^{\infty} X_1^*(f)X_2(f)df \tag{49}$$

- Dem:

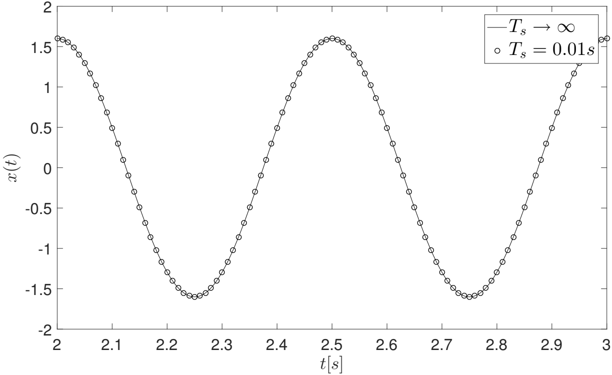
$$\int_{-\infty}^{\infty} x_1(t)x_2(t)dt = \int_{-\infty}^{\infty} x_1(t)\left[\int_{-\infty}^{\infty} X_2(f)e^{i2\pi ft}df\right]dt =$$
$$\int_{-\infty}^{\infty} X_2(f)\left[\int_{-\infty}^{\infty} x_1(t)e^{i2\pi ft}dt\right]df = \int_{-\infty}^{\infty} X_1^*(f)X_2(f)df =$$
$$= \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df \tag{50}$$

- In practical applications (experiments or numerical simulations), time is discretized (sampled) and, hence, the signals (for example, acceleration of a certain point) are given in the form of a vector of finite size;
- Focus of the class: Cases in which time is given by a vector with N points (size of the sample), sampled at constant frequency (rate) f_s (sampling frequency). The sample period is $T_s = 1/f_s$.
- We need to use the Discrete Fourier Transform (DFT). Fast Fourier Transform (FFT) is an efficient algorithm for calculating the DFT.

- Now, we discuss how to obtain the amplitude spectrum $X(f)$ from $x(t)$ using MATLAB®/Octave (and also in Julia).
- From the physical point of view, $X(f)$ illustrates the amplitude of each frequency component of $x(t)$;
- Since time is given in discrete form, the amplitude spectrum is also obtained at discrete values of frequency;
- For the sake of illustration, consider the signal given for continuum time $x(t) = A\cos(\omega t)$, with $A = 1.6$ e $\omega = 4\pi\mathrm{rad/s}$


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
$x(t)$
0 1 2 3 4 5 6 7 8 9 10
$t[s]$

- Now, we consider the discrete signal x_d, obtained from an experiment with constant sample period $T_s = 0{,}01\mathrm{s}$.



Step 1

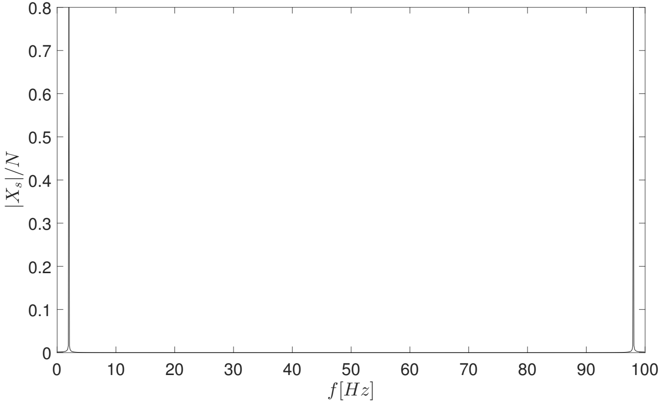
Firstly, we obtain the complex amplitude components by means of the DFT. Following, we define the vector with the frequencies in which the amplitude spectrum will be defined.

- Command 1: $Xs = fft(x);$
- Command 2: $N = size(x);$
- Command 3: $freq = [0 : 1 : N - 1] * fs/N;$

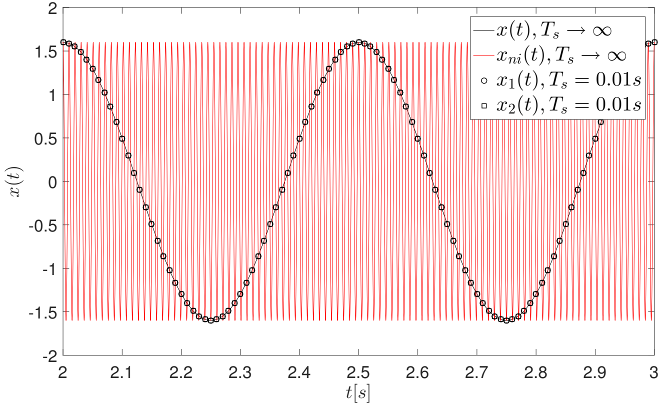
We obtain the amplitude associated with each frequency component. This can be made by taking the absolute value of the complex numbers. Factor N is due to the algorithm.

- Command 4: $absXs = abs(Xs)/N$;

Result at the end of step 2



The spectral analysis has identified two frequencies, $2\mathrm{Hz}$ and $98\mathrm{Hz}$, both with half the amplitude from the continuum time signal. Notice the symmetry with respect to $f_s/2$. We will see what happens....


$x(t), T_s \rightarrow \infty$
$x_{ni}(t), T_s \rightarrow \infty$
$x_1(t), T_s = 0.01s$
$x_2(t), T_s = 0.01s$
$x(t)$
$t[s]$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2
2.1
2.2
2.3
2.4
2.5
2.6
2.7
2.8
2.9
3

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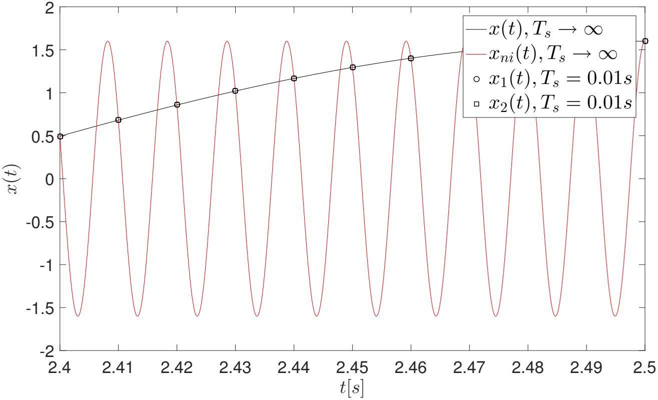
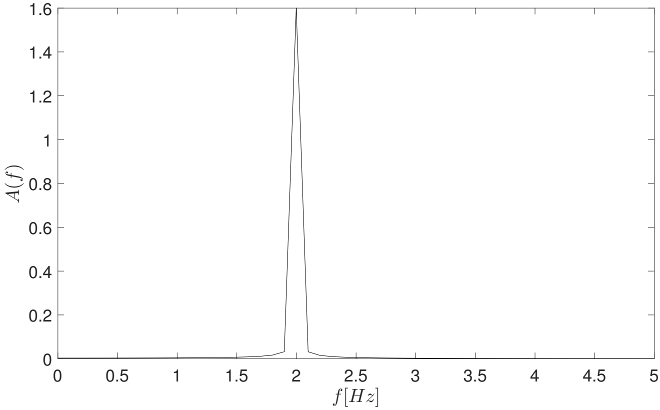


Amplitude spectrum $A(f)$

- We have two frequencies, the one of interest and another one of much larger value (*aliasing*). Each frequency is associated with half the amplitude of the original data;
- Aiming at obtaining a representative amplitude spectrum, we must consider only the frequencies below $f_s/2$. The amplitudes must be multiplied by 2;
- From this brief discussion, we can state that the largest frequency of interest of the signal sampled at f_s must be lower than $f_s/2$ (Sampling theorem or Nyquist theorem).

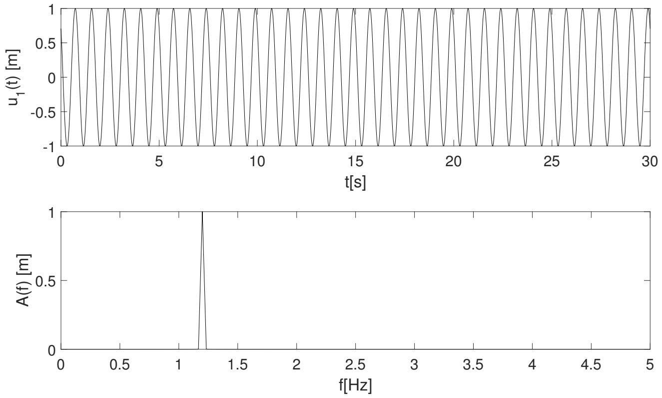
Amplitude spectrum $A(f)$



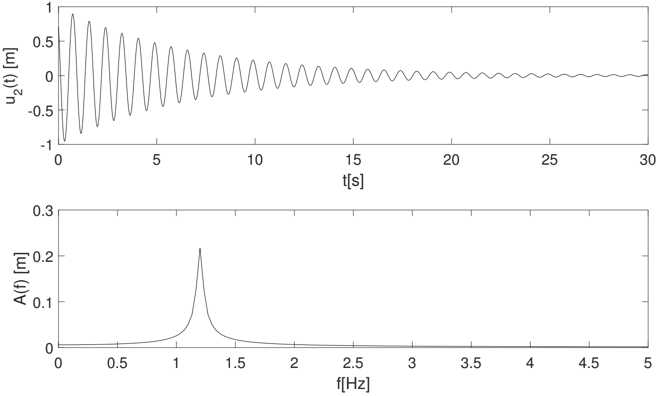
```matlab
function [freq,Amp,fd,Ad]=EspectroAmplitude(t,sinal)

%% Calcula o espectro de amplitude do sinal
%[freq,Amp,fd,Ad]=EspectroAmplitude(t,sinal)
% Entradas - t -> vetor de tempo
%            sinal
% saida    - freq -> vetor de frequencias
%          - Amp  -> vetor de amplitudes
%          - fd   -> frequencia dominante
%          - Ad   -> amplitude na frequencia dominante
%%
N=length(t);
deltat=t(2)-t(1);
deltaf=1/deltat;
freq=[0:N-1]*deltaf/N;
Xs=fft(sinal);
Amp=abs(Xs)/N;
Amp=2*Amp(1:fix(N/2));
freq=freq(1:fix(N/2));
[Ad,indice]=max(Amp);
fd=freq(indice);
```

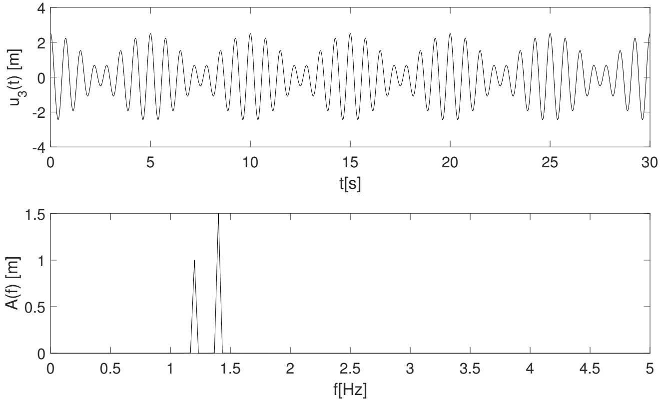
$$\rho = 1\mathrm{m}; f = 1{,}2\mathrm{Hz}; \theta = \pi/4$$



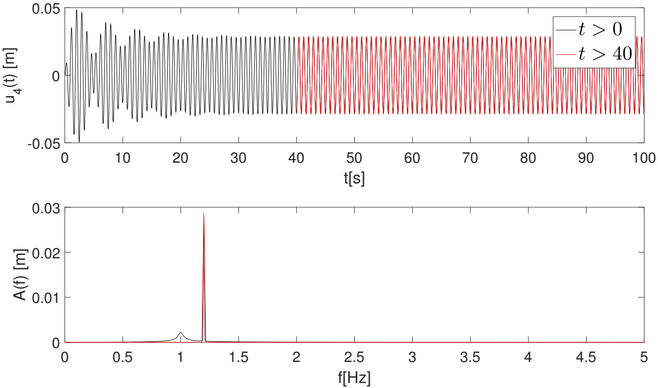
$$\rho = 1\mathrm{m}; f = 1{,}2\mathrm{Hz}; \zeta = 0.02; \theta = \pi/4$$



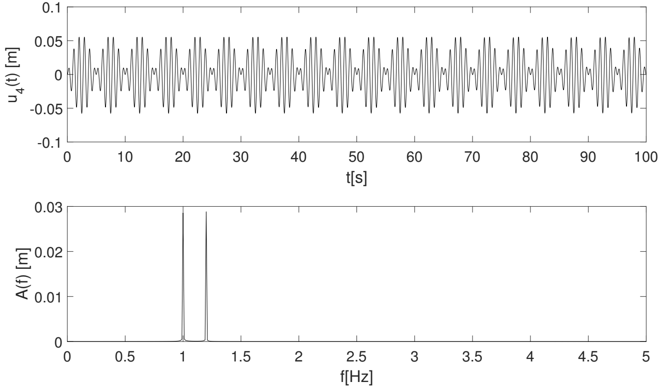
$$\rho_1 = 1\mathrm{m};\ f_1 = 1{,}2\mathrm{Hz};\ \rho_2 = 1.5\mathrm{m};\ f_2 = 1{,}4\mathrm{Hz}$$



$$\ddot{u} + 2\zeta\omega\dot{u} + \omega^2 u = p_0/m\cos(\bar{\omega}t)$$
$$u(0) = 0\mathrm{m}; \dot{u}(0) = 0\mathrm{m/s}; m = 1\mathrm{kg}; k = 4\pi^2\mathrm{N/m}; \bar{\omega} = 1.2\omega; \zeta = 2\%$$



$$\ddot{u} + 2\zeta\omega\dot{u} + \omega^2 u = p_0/m\cos(\bar{\omega}t)$$
$$u(0) = 0\mathrm{m}; \dot{u}(0) = 0\mathrm{m/s}; m = 1\mathrm{kg}; k = 4\pi^2\mathrm{N/m}; \bar{\omega} = 1.2\omega; \zeta = 0$$



- For a stationary and ergodic process, the autocorrelation function of $f(t)$ is:

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)f(t+\tau)dt \tag{51}$$

- We define the power spectrum density as the Fourier Transform of the autocorrelation function

$$S_f(\omega) = \int_{-\infty}^{\infty} R_f(\tau)e^{-i\omega\tau}d\tau \tag{52}$$

$$R_f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)e^{i\omega\tau}d\omega \tag{53}$$

- Equations 52 and 52 are known as Wiener-Khintchine relations.
- Important properties:

$$R_f(\tau=0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^2(t)dt = \overline{f^2} \tag{54}$$

$$R_f(\tau=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)d\omega = \int_{-\infty}^{\infty} S_f(f)df = \overline{f^2} \tag{55}$$

- Now, we discuss the relation between the Fourier Transform $X(f)$ and the PSD $S_x(f)$ of $x(t)$.
- We start from Parseval's theorem and from the definition of the mean square value of $x(t)$:

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} X(f)X^*(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{56}$$

$$\begin{aligned} \overline{x^2} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} x^2(t)dt = \\ &= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} X(f)X^*(f)df = \int_{-\infty}^{\infty} S_x(f)df \end{aligned} \tag{57}$$

- Hence

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} X(f)X^*(f) \tag{58}$$

- $p(t)$ is a real signal of null mean. Its Fourier series is

$$p(t) = \sum_{n=-\infty, n\neq 0}^{\infty} c_n e^{in\omega_0 t} \tag{59}$$

- This Fourier series can be given in terms of the positive frequency, as follows

$$p(t) = \sum_{n=1}^{\infty} \frac{1}{2}(c_n e^{in\omega_0 t} + c_n^* e^{-in\omega t}) \tag{60}$$

- Now, we recall the orthogonality condition to obtain

$$\overline{p^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \frac{1}{4}(c_n e^{in\omega_0 t} + c_n^* e^{-in\omega_0 t})^2 dt \tag{61}$$

- Due to the orthogonality condition, we have

$$\overline{p^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \frac{1}{4} 2 c_n c_n^* dt = \sum_{n=1}^{\infty} \frac{1}{2} c_n c_n^* \tag{62}$$

- Notice that c_n are the amplitude of the harmonic components;
- The power spectrum $G_n = G(f_n)$ is given by $\frac{1}{2} c_n c_n^*$;
- The power spectrum density $S_n = S(f)$ is given by $G_n/\Delta f$, Δf being the frequency interval;
- Units: If $p(t)$ represents the time-history of the applied force in N, $[G]$=N^2 and $[S]$=N^{2}s;
- If we would like to express the PSD considering the frequency in rad/s, we have:

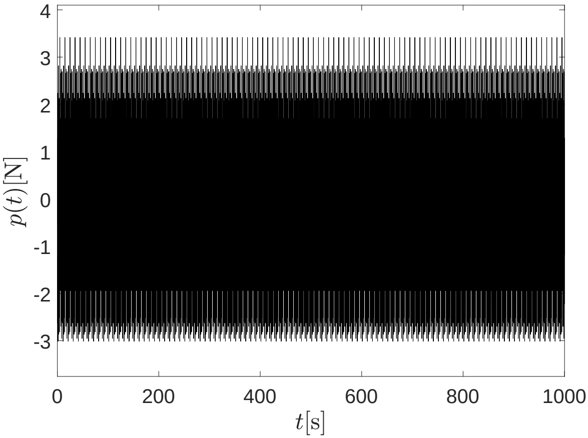
$$S(f)df = S(\omega)d\omega \leftrightarrow S(f)df = S(\omega)2\pi df \rightarrow S(f) = 2\pi S(\omega) \qquad (63)$$

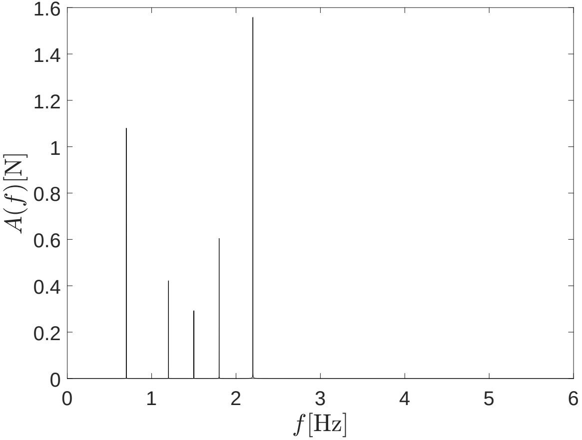
- Spectral moments of order k (m_k)

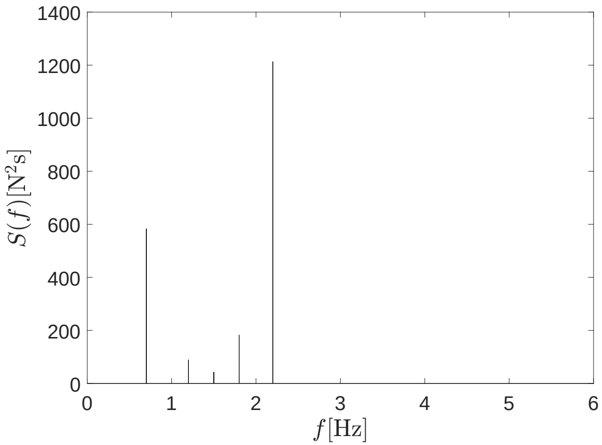
$$m_k = \int_0^\infty \omega^k S(\omega) d\omega \qquad (64)$$

- Notice that the standard-deviation of the time-history is $\sqrt{m_0}$.

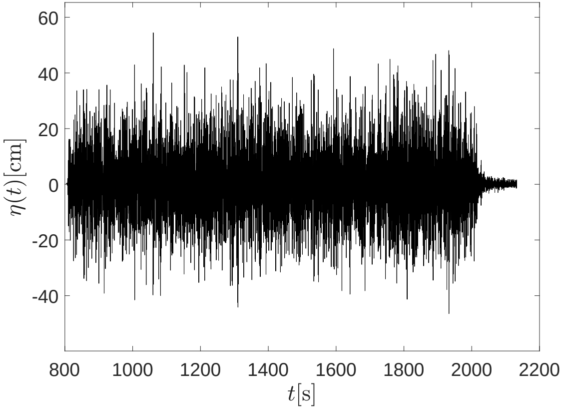
Example 1 : time-history $p(t)$

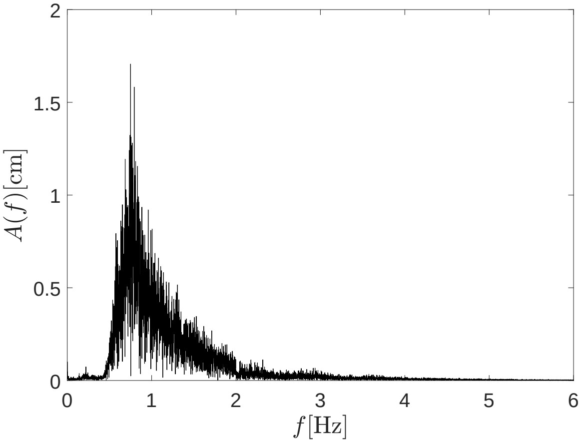



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$A(f)\,[\mathrm{N}]$
0
1
2
3
4
5
6
$f\,[\mathrm{Hz}]$

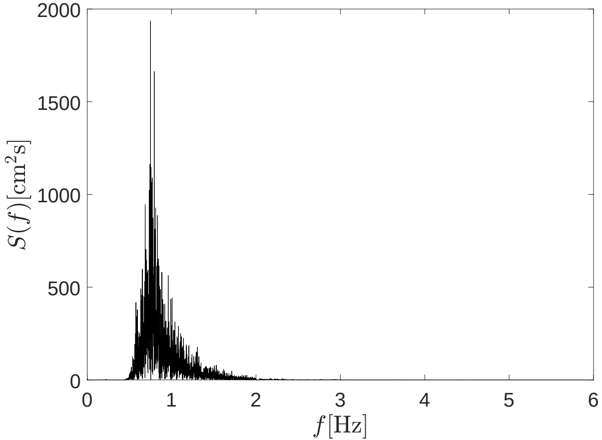

1400
1200
1000
800
600
400
200
0
$S(f)\,[\mathrm{N^2 s}]$
0
1
2
3
4
5
6
$f\,[\mathrm{Hz}]$

- $\eta(t)$ is the free-surface elevation time-history at a particular position. Data obtained from experiments carried out at TPN Wave Basin and kindly sent by PhD. Pedro Mello.
- The averaged value (offset) has been removed from data.




2
1.5
1
0.5
0
$A(f)\,[\mathrm{cm}]$
0
1
2
3
4
5
6
$f\,[\mathrm{Hz}]$

Example 2: PSD $S(f)$



- Standard deviation of $\eta(t)$ computed using std command: 12.4765
- Standard deviation of $\eta(t)$ computed $\int_0^\infty S(f)df$: 12.4764

- Consider that we have the excitation time-history $p(t)$, its Fourier Transform $P(\omega)$ and PSD $S_p(\omega)$;
- As already discussed $U(\omega) = H(\omega)P(\omega) \rightarrow U(f) = H(f)P(f)$;
- We take the complex conjugate of both sides:

$$U(f)U^*(f) = H(f)H^*(f)P(f)P^*(f) = |H(f)|^2 P(f)P^*(f) \tag{65}$$

- Now, we use the definition:

$$\lim_{T\rightarrow\infty} \frac{1}{T} U(f)U^*(f) = |H(f)|^2 P(f)P*(f) \leftrightarrow S_u(f) = |H(f)|^2 S_p(f) \tag{66}$$

- The operation given by Equation Eq. 66 is also know as spectral crossing;
- We can obtain the statistics of the response following the approach already developed;
- In some applications, the PSD of the excitation is given by empirical expressions. Hence, we can estimate statistics of the response without the numerical integration of the equations of motion in time domain.

- PSD is also obtained by $S_u(f) = \frac{A_u^2(f)}{2df}$
- Notice that we can obtain the amplitude of the harmonic components of velocity $\dot{u}$ as $2\pi f A(f)$. Then, the PSD of the velocity is written as

$$S_{\dot{u}}(f) = (2\pi f)^2 \frac{A_u^2(f)}{2df} = (2\pi f)^2 S_u(f) \tag{67}$$

- Using the same approach:

$$S_{\ddot{u}}(f) = (2\pi f)^2 S_{\dot{u}}(f) = (2\pi f)^4 S_u(f) \tag{68}$$

- If we have $S_u(f)$, we can obtain a realization of the time-history $u(t)$ by computing:

$$u(t) = \sum_{n=1}^{N} \sqrt{2S_u(f)df} \cos(2\pi f_n t + \phi_n) \tag{69}$$

ϕ_n being a random phase.