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A MATHEMATICAL MODEL OF THE HUMAN THERMAL SYSTEM*

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This paper describes a mathematical model developed to simulate the physical characteristics of the human thermal system in the transient state. Physiological parameters, such as local metabolic heat generation rates, local blood flow rates, and rates of sweating, must be specified as input data. Automatic computation of these parameters will be built into the model at a later date when it is used to study thermal regulation in the human.

Finite-difference techniques have been used to solve the heat conduction equation on a Control Data Corporation 1604 computer. Since numerical techniques were used, it was possible to include many more factors in this model than in previous ones. The body was divided into 15 geometric regions, which were the head, the thorax, the abdomen, and the proximal, medial, and distal segments of the arms and legs. Axial gradients in a given segment were neglected. In each segment, the large arteries and veins were approximated by an arterial pool and a venous pool which were distributed radially throughout the segment. Accumulation of heat in the blood of the large arteries and veins, and heat transfer from the large arteries and veins to the surrounding tissue were taken into account. The venous streams were collected together at the heart before flowing into the capillaries of the lungs. Each of the segments was subdivided into 15 radial sections, thereby allowing considerable freedom in the assignment of physical properties such as thermal conductivity and rate of blood flow to the capillaries.

The program has been carefully checked for errors, and it is now being used to analyze some problems of current interest.

The synthesis of an adequate mathematical model for the human thermal system must include the following factors: (1) the manner in which heat generated by metabolic reactions is distributed throughout the body, (2) conduction of

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heat due to thermal gradients, (3) convection of heat by circulating blood, (4) the geometry of the body, (5) the relatively low thermal conductivity of the superficial layer of fat and skin, (6) countercurrent heat exchange between large arteries and veins, (7) heat loss through the respiratory tract, (8) sweating, (9) shivering, (10) the storage of heat, and (11) the condition of the environment, including its temperature, motion relative to the body, and relative humidity. Some of these factors, such as the last one, can be measured with relative ease. On the other hand, such factors as the local rate of heat generation can only be measured *in vivo* with great difficulty, and their values must be deduced from indirect measurements. Indeed, one of the principal uses of a mathematical model is to assign reasonable values to those parameters which cannot be measured directly in an experiment.

Early mathematical models, such as those developed by L. W. Eichna, W. F. Ashe, W. B. Bean, and W. B. Shelley (1945) and by W. Machle and T. F. Hatch (1947) were based on the "core and shell" concept in which the rectal temperature and the mean skin temperature were used as measures of the deep and superficial temperatures, respectively. Since the amount of information built into these models is relatively small, the formulas are simple and easy to use, but they fail in many cases. For instance, D. McK. Kerslake and J. L Waddell (1958) have observed that the relative volumes assigned to the core and shell depend on the peripheral circulation, but these models do not consider this explicitly.

Recent attempts to build more information into the models have involved the use of modern computers of both the analog and digital types. In either case, the basic problem has been to solve the transient-state heat conduction equation with internal heat generation. C. H. Wyndham and A. R. Atkins (1960) have approximated the human by a series of concentric cylinders. Assuming that the rate of heat transfer between adjacent cylinders is proportional to the difference between the temperatures of the cylinders leads to a set of first-order differential equations which are easily solved on an analog computer. The effect of peripheral circulation is implicitly included in the model by allowing the effective thermal conductivity to vary as a function of temperature. R. J. Crosbie, J. D. Hardy, and E. Fessenden (1961) have adopted a very similar approach using an infinite slab rather than a cylinder. They have built in some of the more important physiological responses to thermal stress by allowing the effective thermal conductivity, metabolic rate, and rate of vaporization to vary as the mean temperature of the body varies. Although these models do include, in a not clearly defined mean manner, some of the factors mentioned in the first paragraph, they do not include the effect of regional variations in heat generation rates and blood flow rates. Wyndham and Atkins are currently adapting their model to include regional variations by using a physical system similar to the one discussed below.

In two previous papers, the author has obtained both steady-state (1961a) and transient-state solutions (1961b) for a model based on a representation of the human using six cylindrical elements. Two of the elements represent the arms; two represent the legs; one represents the trunk; and the sixth represents the head. The elements are connected by the vascular system. Each element is a two-region composite cylinder, with the inner region composed of tissue, bone, and viscera and the outer region composed of fat and skin. All of the factors mentioned in the opening paragraph were explicitly included in the analysis, but such variables as local heat generation rates and local blood flow rates were assigned as parameters to be specified in the input data. The solution obtained was an analytical one expressed in terms of an infinite series of orthogonal functions, and a high-speed digital computer was used to evaluate the temperatures for a particular case. Much of the computation time was spent evaluating eigenvalues; and since this had to be repeated whenever a physiological variable changed, the program was not a very efficient one for studying thermal regulation problems in which physiological parameters were varying rapidly. Therefore, it was decided to investigate the possibility of obtaining a more versatile solution by using finite difference techniques. The purpose of this paper is to describe the result of this investigation.

Theory. The physical system on which the equations are based is shown in Figure 1. It consists of a number of cylindrical elements representing longi-

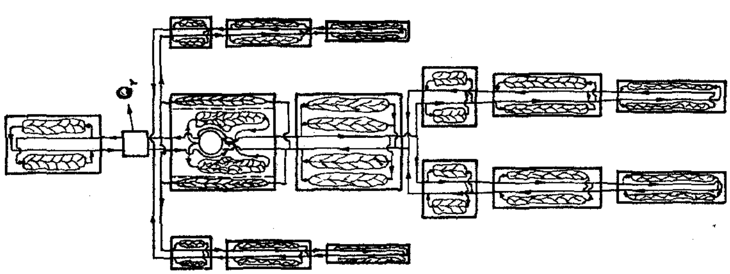

Figure 1. A schematic diagram showing the geometric arrangement of the elements and the circulatory system

tudinal segments of the arms, legs, trunk, and head. Each element, consisting of a conglomeration of tissue, bone, fat, and skin, has a vascular system which can be divided into three subsystems representing the arteries, the veins, and the capillaries. The heat which is generated in the elements by metabolic reactions is either stored in the element, carried away by circulating blood, or conducted to the surface where it is transferred to the environment. This is simply a statement of the first law of thermodynamics, which can be formulated mathematically as the heat conduction equation given below:

$$(\rho C_i)\frac{\delta T_i}{\delta t} = \frac{1}{r}\frac{\delta}{\delta r}\left(k_i r \frac{\delta T_i}{\delta r}\right) + h_{mi} + Q_{ci}(T_{ai} - T_i) + H_{ai}(T_{ai} - T_i) + H_{vi}(T_{vi} - T_i), \quad (1)$$

in which

$T_i(t, r)$ = instantaneous temperature of the tissue, bone, or viscera at a distance r from the axis of the ith element,

$\rho_i(r)$ = density of tissue,

$C_i(r)$ = specific heat of tissue,

$k_i(r)$ = effective thermal conductivity of tissue,

$h_{mi}(t, r)$ = metabolic heat generation per unit volume,

$Q_{ci}(t, r)$ = product of the mass flow-rate and specific heat of blood entering the capillary beds per unit volume,

$H_{ai}(t, r)$ = heat transfer coefficient between the arteries and tissue per unit volume,

$H_{vi}(t, r)$ = heat transfer coefficient between the veins and tissue per unit volume,

$T_{ai}(t)$ = temperature of the arterial blood,

$T_{vi}(t)$ = temperature of the venous blood.

The term on the left-hand side of equation (1) is the rate of accumulation of thermal energy per unit volume due to the changing temperature of the tissue and capillary blood in the volume. This equals the sum of the five terms on the right which represent in order the net rate of conduction of heat into a unit volume, the rate of heat generation by metabolic reactions, the net rate at which heat is carried into the volume by capillary blood, the rate at which heat is transferred from arterial blood to the tissue, and the rate at which heat is transferred from venous blood to the tissue. It should be observed that this form of the heat conduction equation is applicable only to an axially symmetrical system in which the longitudinal conduction of heat is negligible. This means that the analysis does not apply to situations in which the subject is curled up in a ball in order to conserve heat. If the subject is moving so that there is a uniform flow of air around each of the elements, the analysis should apply. H. H. Pennes (1948) has shown that longitudinal conduction in the arms is relatively unimportant. This should be true also in the legs, but probably is not true in the head. It has been assumed that there is perfect heat transfer between the blood in the capillaries and the neighboring tissue, i.e., the temperature of blood leaving the capillary beds is equal to the temperature of the neighboring tissue. Because of the small diameter of the capillaries this is probably a good approximation, but such a simple condition does not prevail in the larger vessels. As a first approximation it has been assumed in this paper that the rate of heat transfer from the blood in the large vessels to the neighboring tissue is proportional to the difference between the blood and tissue temperatures. The proportionality factor has been called H_a for the arteries and H_v for the veins.

Since the temperature of blood in the large vessels changes with time, it is necessary to write two more thermal energy balances. In formulating the equation for the arteries it has been assumed that the arteries in the ith element form a pool having a uniform temperature, T_{ai}. The rate of accumulation of the thermal energy in this reservoir is equal to the sum of the net rate at which heat is carried into the pool by flowing blood, the rate at which heat is transferred from neighboring tissue to the blood in the pool, and the rate at which heat is transferred directly from the venous pool to the arterial pool due to the proximity of certain arteries and veins. This equality is expressed mathematically in the following equation.

$$(MC)_{ai}\frac{\delta T_{ai}}{\delta t} = Q_{ai}(T_{am} - T_{ai}) + 2\pi L_i \int_0^{a_i} H_{ai}(T_i - T_{ai})r\,dr + H_{avi}(T_{vi} - T_{ai}), \quad (2)$$

in which

$T_{am}(t)$ = temperature of the blood entering the arterial pool,
M_{ai} = mass of the blood contained in the arterial pool of the ith element,
C_{ai} = specific heat of blood,
$Q_{ai}(t)$ = product of the mass flow rate and specific heat for blood entering the arterial pool,
L_i = length of the ith element,
H_{avi} = heat transfer coefficient for direct transfer between large arteries and veins.

The integral is necessary in equation (2) because the tissue temperature is a function of r.

The corresponding equation for the venous pool is

$$(MC)_{vi}\frac{\delta T_{vi}}{\delta t} = Q_{vi}(T_{vn} - T_{vi}) + 2\pi L_i \int_0^{a_i} (Q_{ci} + H_{vi})(T_i - T_{vi})r\,dr + H_{avi}(T_{ai} - T_{vi}), \quad (3)$$

in which

$Q_{vi}(t)$ = product of mass and specific heat for venous blood flowing into the venous pool of the ith element from the nth element.

It will be assumed throughout this analysis that the M_{ai}'s and M_{vi}'s are constants so that

$$Q_{ai}(t) = Q_{vi}(t) + 2\pi L_i \int_0^{a_i} Q_{ci}(t, r) r dr. \tag{4}$$

The equation for the venous temperature in the abdominal section is slightly different than equation (3) because two veins, one from each leg, flow into this section. It was also necessary to modify the equations for the thoracic section since all of the venous streams terminate and the arterial streams originate in this section. It was assumed that the temperature of the blood entering the pulmonary capillaries is equal to the "cup mixing" mean temperature of the venous streams entering the right ventricle. This necessitated a change in equation (1) because the temperature of the venous blood entering the pulmonary capillaries is different than the temperature of the arterial blood entering the more superficial capillaries of the thorax:

$$(\rho C)_1 \frac{\delta T_1}{\delta t} = \frac{1}{r} \frac{\delta}{\delta r}\left(k_1 r \frac{\delta T_1}{\delta r}\right) + h_{m1} + Q_{ca}(T_{a1} - T_1) + Q_{cv}(T_{v1} - T_1) + H_{a1}(T_{a1} - T_1) + H_{v1}(T_{v1} - T_1), \tag{5}$$

in which

$Q_{ca}(t, r)$ = product of the mass flow rate and specific heat for arterial blood flowing into the capillaries,

$Q_{cv}(t, r)$ = product of the mass flow rate and specific heat for venous blood flowing into the pulmonary capillaries.

Equations (2) and (3) were also modified to take cognizance of the fact that venous blood flows into the pulmonary capillaries which in turn empty into the arterial pool:

$$(MC)_{a1} \frac{\delta T_{a1}}{\delta t} = 2\pi L_1 \int_0^{a_1} Q_{cv}(T_1 - T_{a1}) r dr + 2\pi L_1 \int_0^{a_1} H_{a1}(T_1 - T_{a1}) r dr + H_{av1}(T_{v1} - T_{a1}) \tag{6}$$

$$(MC)_{v1} \frac{\delta T_{v1}}{\delta t} = \sum_i Q_{v1i}(T_{vi} - T_{v1}) + 2\pi L_1 \int_0^{a_1} H_{vi}(T_1 - T_{v1}) r dr + H_{av1}(T_{a1} - T_{v1}) + q_{rv1}, \tag{7}$$

in which

$q_{rv1}(t)$ = rate at which heat is transferred from venous blood in the thorax to air in the respiratory tract,

$Q_{v1i}(t)$ = rate at which venous blood flows from the *ith* element into the venous pool in the thorax $= Q_{ai}(t)$ for those elements which are connected to the thoracic segment.

The total rate of heat loss through the respiratory tract depends on the respiratory rate and the temperature and humidity of the inspired air. In this analysis it was assumed that the expired air was saturated with water vapor at a mean temperature T_r:

$$T_r = 0.25T_{v-\text{head}} + 0.25T_{a-\text{head}} + 0.5T_{v-\text{chest}}. \tag{8}$$

Furthermore, it was assumed that 25 per cent of the heat loss through the respiratory tract came from the arterial pool in the head, 25 per cent from the venous pool in the head, and 50 per cent from the venous pool in the thorax.

Before these equations can be solved uniquely, certain constraining conditions must be specified. Some of these take the form of initial conditions, which specify all of the temperatures at the instant the transient begins:

$$T_i(0, r) = T_{0i}(r) \tag{9}$$

$$T_{ai}(0) = T_{a0i} \tag{10}$$

$$T_{vi}(0) = T_{v0i}. \tag{11}$$

Also needed are boundary conditions which relate the subject to his environment. In general they are based on the fact that the local rate of conduction of heat to the surface through the tissue is equal to the rate of heat transfer from the surface to the environment:

$$-\left[k_i \frac{\delta T_i}{\delta r}\right]_{r=a_i} = H_i[T_i(t, a_i) - T_{ei}], \tag{12}$$

in which

H_i = heat transfer coefficient,
T_{ei} = effective environmental temperature.

The heat transfer coefficient depends on the physical properties of the fluid surrounding the element, the velocity of the fluid, the wetness of the surface, and the relative humidity of the environment. If heat transfer by evaporation is important, the effective temperature of the environment will be lower than the dry-bulb temperature. In this paper, the heat transfer coefficient for a subject in air has been computed using the equation

$$H_i = H_{ci} + H_{ri} + \lambda_i \left(\frac{dp}{dT}\right)_i (K_i F_i + K_{Di}), \tag{13}$$

in which

H_{ci} = heat transfer coefficient for convection,
H_{ri} = heat transfer coefficient for radiation,

λ_i = latent heat of water at T_i,
$(dp/dT)_i$ = rate of change of partial pressure of water with temperature at T_i,
K_i = mass transfer coefficient for convection,
F_i = wetted fraction of the surface,
K_{Di} = mass transfer coefficient for passive diffusion of water through the epidermis.

A summary of the equations used has been published previously (Wissler, 1961a). Finally, since each element possesses axial symmetry,

$$\left(\frac{\delta T_i}{\delta r}\right)_{r=0} = 0. \qquad (14)$$

Solution of the equations. The use of numerical techniques and large, high-speed digital computers to solve the heat conduction equation has received considerable attention lately. A good description of these techniques is given in the recent book by G. E. Forsythe and R. W. Wasow (1960). The principal feature of finite-difference techniques is that they can be used even if the physical properties vary with position, and it is for this reason that they were employed to solve the equations presented in the preceding section.

Basically, the procedure used consists of subdividing each of the circular elements into a number of annular shells and assigning a single characteristic temperature to the material in each of the shells. Then the right-hand side of equation (1) at a particular value of r can be approximated by a linear algebraic equation. Furthermore, no attempt is made to compute the temperatures of the shells as continuous functions of time. Instead, one employs a marching procedure in which the initial temperatures are used to compute the temperatures a short interval of time, Δt, later. These new temperatures are then used to compute the temperatures at time, $2\Delta t$, and so on as long as necessary.

Figure 2 is helpful in visualizing this process. Normally, the temperatures in a given element are all specified at $t = 0$, the row $k = 1$, by the initial conditions; and the problem is to devise a procedure for computing temperatures in the next row ($k = 2$), and so forth until the entire table has been completed. It should be noted that the space and time steps need not all be the same size. One can use small space steps near the outside of the cylinder where the temperature gradients are the largest and large space steps near the center where the temperature gradients are small. Similarly, small time intervals can be used at the beginning of the interval when the temperature is changing rapidly, and larger time intervals can be used near the end of the transient.

In the development of the finite-difference equations, we will let $T_{i,kj}$ be the temperature existing at the jth radial point r_j, in the ith element after the $(k-1)$th time step, $t = t_k$. The difference equation used to approximate equation (1) was obtained by integrating each term in the equation over an annular region ranging from $r = r_j - (h_-/2)$ to $r = r_j + (h_+/2)$, in which h_-

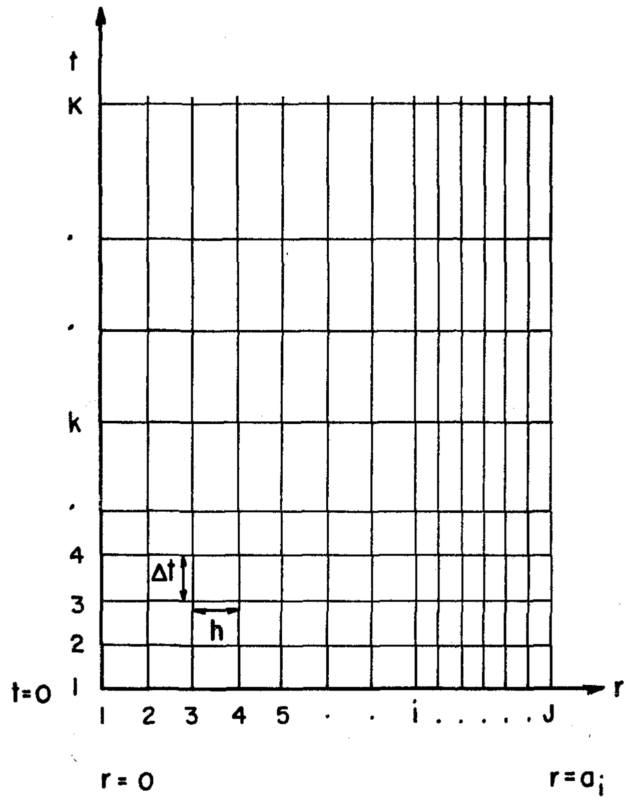

Figure 2. Diagram showing the temporal and spacial steps used in deriving the finite difference equations

is the space increment to the left of r_j and h_+ is the increment to the right of r_j. Assuming that $T_{i,kj}$ is characteristic of the temperature in this interval and allowing the physical properties to have one value (subscript $-$) to the left of r_j and another value (subscript $+$) to the right of r_j, one obtains the following equation:

$$\left[\frac{h_-(r_j - h_-/4)}{2}(\rho C)_{i-} + \frac{h_+(r_j + h_+/4)}{2}(\rho C)_{i+}\right]\frac{\delta T_{i,kj}}{\delta t} \cong$$

$$k_{i+}(r_j + h_+/2)\frac{\delta T_{i,k(j+1/2)}}{\delta r} - k_{i-}(r_j - h_-/2)\frac{\delta T_{i,k(j-1/2)}}{\delta r} +$$
$$\left\{\frac{h_-(r_j - h_-/4}{2}[(Q_{ci-} + H_{ai-})(T_{ai,k} - T_{i,kj}) + H_{vi-}(T_{vi,k} - T_{i,kj})] + \right.$$
$$\left.\frac{h_+(r_j + h_+/4)}{2}[(Q_{ci+} + H_{ai+})(T_{ai,k} - T_{i,kj}) + H_{vi+}(T_{vi,k} - T_{i,kj})]\right\}. \quad (15)$$

A common factor of 2π has been cancelled out of each term. Similarly, integrating over the interval from $r_J - h_-/2$ to r_J and using equation (12) to evaluate $[k_{i-}(\delta T/\delta r)]_{r=r_J}$, one obtains

$$h_-(r_J - h_-/4)(\rho C)_{i-}\frac{\delta T_{i,kJ}}{\delta t} \cong -k_{i-}(r_J - h_-/2)\frac{\delta T_{i,k(J-1/2)}}{\delta r} -$$
$$r_J H_i(T_{i,kJ} - T_{ei}) +$$
$$\frac{h_-(r_J - h_-/4)}{2}[(Q_{ci} + H_{ai-})(T_{ai,k} - T_{i,kJ}) +$$
$$H_{vi-}(T_{vi,k} - T_{i,kJ})]. \quad (16)$$

The derivatives appearing in the preceding equations are approximated as follows:

$$\frac{\delta T_{i,(k+1/2)j}}{\delta t} = \frac{T_{i,(k+1)j} - T_{i,kj}}{\Delta t} \quad (17)$$

$$\frac{\delta T_{i,k(j+1/2)}}{\delta r} = \frac{T_{i,k(j+1)} - T_{i,kj}}{\Delta r} \quad (18)$$

$$\frac{\delta T_{i,k(j-1/2)}}{\delta r} = \frac{T_{i,kj} - T_{i,k(j-1)}}{\Delta r}. \quad (19)$$

Substituting the preceding expressions into equations (15) and (16) and using the arithmetic mean of the values of the right-hand side at times t_k and t_{k+1} to approximate the value of the right-hand side at time $(t_k + t_{k+1})/2$, one obtains a set of equations each having the form

$$A_{i,j}T_{i,(k+1)(j-1)} + B_{i,j}T_{i,(k+1)j} + C_{i,j}T_{i,(k+1)(j+1)} +$$
$$U_{i,j}T_{ai,k+1} + V_{i,j}T_{vi,k+1} = D_{i,j}, \quad (20)$$

in which $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, $U_{i,j}$, and $V_{i,j}$ are constants determined by the physical properties and the mesh size, and $D_{i,j}$ is determined by the temperatures at time t_k. It is worth noting that $A_{i,1}$ and $C_{i,J}$ are both zero.

Equation (2) was next approximated in the following way. In place of the derivative on the left-hand side use

$$\frac{\delta T_{ai,(k+1/2)}}{\delta t} = \frac{T_{ai,k+1} - T_{ai,k}}{\Delta t}, \quad (21)$$

and in place of the integral containing T_i use

$$\int_0^{a_i} H_{ai}(r)T_i(t_k, r)rdr \cong \sum_{j=1}^{J} W_{i,j}T_{i,kj}, \tag{22}$$

in which

$$W_{i,j} = \int_{r_j - h_-/2}^{r_j + h_+/2} H_{ai}(r)rdr. \tag{23}$$

Substituting the expressions given in equations (22) and (23) into equation (2) and again using the mean of the values of the right-hand side at times t_k and t_{k+1}, one obtains an equation having the form

$$\sum_{j=1}^{J} W_{i,j}T_{i,(k+1)j} + U_{i,J}T_{ai,k+1} + V_{i,J}T_{vi,k+1} = D_{i,J} + E_{i,J}T_{am,k+1}. \tag{24}$$

Similarly, equation (3) can be approximated by an algebraic equation having the form

$$\sum_{j=1}^{J} X_{i,j}T_{i,(k+1)j} + U_{i,J+1}T_{ai,k+1} + V_{i,J+1}T_{vi,k+1} = D_{i,J+1} + E_{i,J+1}T_{vn,k+1}. \tag{25}$$

Given the temperatures $T_{am,k+1}$ and $T_{vn,k+1}$ of the arterial and venous blood entering an element, one can compute the tissue temperatures and blood temperatures in that element by solving simultaneously the J equations represented by equation (19) together with equations (24) and (25). Because of the particularly simple form of equation (20), a solution can be obtained with ease using a Gaussian elimination procedure. The complete set of equations is displayed below:

$$\begin{array}{ll}
B_1T_1 + C_1T_2 & + U_1T_a + V_1T_v = D_1 \\
A_2T_1 + B_2T_2 + C_2T_3 & + U_2T_a + V_2T_v = D_2 \\
\quad A_3T_2 + B_3T_3 + C_3T_4 & + U_3T_a + V_3T_v = D_3 \\
\cdots & \cdots \\
A_{J-2}T_{J-3} + B_{J-2}T_{J-2} + C_{J-2}T_{J-1} & + U_{J-2}T_a + V_{J-2}T_v = D_{J-2} \\
A_{J-1}T_{J-2} + B_{J-1}T_{J-1} + C_{J-1}T_J & + U_{J-1}T_a + V_{J-1}T_v = D_{J-1} \\
A_JT_{J-1} + B_JT_J & + U_JT_a + V_JT_v = D_J \\
W_1T_1 + W_2T_2 + \cdots + W_{J-1}T_{J-1} + W_JT_J & + U_{J+1}T_a + V_{J+1}T_v = D_{J+1} + E_{J+1}T_{am}
\end{array} \tag{26}$$

$$X_1T_1 + X_2T_2 + \cdots + X_{J-1}T_{J-1} + X_JT_J \quad + U_{J+2}T_a + V_{J+2}T_v = D_{J+2} + E_{J+2}T_{vn}.$$

The subscripts i and $k + 1$ have been dropped to conserve space. To solve this set of equations let

$$b_1 = \frac{C_1}{B_1}, \quad u_1 = \frac{U_1}{B_1}, \quad v_1 = \frac{V_1}{B_1}, \quad q_1 = \frac{D_1}{B_1}, \quad w_1 = W_1, \quad x_1 = X_1,$$

$$b_i = \frac{C_i}{B_i - A_ib_{i-1}}, \quad u_i = \frac{U_i}{B_i - A_ib_{i-1}}, \quad v_i = \frac{V_i}{B_i - A_ib_{i-1}},$$

$$q_i = \frac{D_i}{B_i - A_ib_{i-1}}, \qquad (1 < i \le J)$$

$$w_i = W_i - w_{i-1}b_{i-1} \qquad x_i = X_i - x_{i-1}b_{i-1} \tag{27}$$

$$u_{J+1} = U_{J+1} - \sum_{i=1}^{J} w_iu_i \qquad u_{J+2} = U_{J+2} - \sum_{i=1}^{J} x_iu_i$$

$$v_{J+1} = V_{J+1} - \sum_{i=1}^{J} w_iv_i \qquad v_{J+2} = V_{J+2} - \sum_{i=1}^{J} x_iv_i$$

$$q_{J+1} = D_{J+1} - \sum_{i=1}^{J} w_iq_i \qquad q_{J+2} = D_{J+2} - \sum_{i=1}^{J} x_iv_i.$$

This reduces the set of equations (26) to the simpler set given below:

$$\begin{aligned}
T_1 + b_1T_2 \qquad\qquad & + u_1T_a + v_1T_v = q_1 \\
T_2 + b_2T_3 \qquad & + u_2T_a + v_2T_v = q_2 \\
T_{J-1} + b_{J-1}T_J & + \\
& u_{J-1}T_a + v_{J-1}T_v = q_{J-1} \\
& T_J + u_JT_a + v_JT_v = q_J \\
& u_{J+1}T_a = v_{J+1}T_v = q_{J+1} + E_{J+1}T_{am} \\
& u_{J+2}T_a + v_{J+2}T_v = q_{J+2} + E_{J+2}T_{vn}.
\end{aligned} \tag{28}$$

Now the method of solution in a given element is clear. One solves the last two equations for T_a and T_v, and then computes

$$T_J = q_J - u_JT_a - v_JT_v$$

$$T_j = q_j - b_jT_{j+1} - u_jT_a - v_jT_v, \quad j = J - 1, \ldots, 1. \tag{29}$$

Finally, we must devise a scheme for computing simultaneously the blood temperatures in all of the elements. These temperatures are defined by pairs of equations similar to the last two of equations (28). To see how these equations can be solved easily, consider the distal and medial segments of an arm or leg as shown in Figure 3. The equations for these two segments are written below:

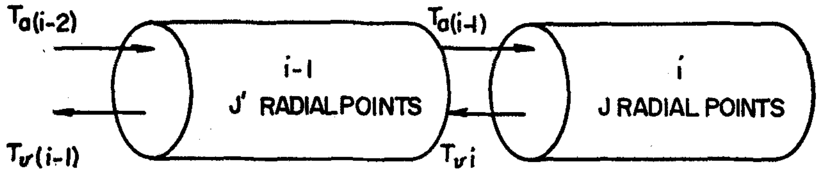

Figure 3. Diagram showing the blood flows into and out of two adjacent elements

$$u_{i-1,\,J'+1}T_{a(i-1)} + v_{i-1,\,J'+1}T_{v(i-1)} = q_{J'+1} + E_{J'+1}T_{a(i-2)} \tag{30}$$
$$u_{i-1,\,J'+2}T_{a(i-1)} + v_{i-1,\,J'+2}T_{v(i-1)} = q_{J'+2} + E_{J'+2}T_{vi}$$

$$u_{i,\,J+1}T_{ai} + v_{i,\,J+1}T_{vi} = q_{J+1} + E_{J+1}T_{a(i-1)} \tag{31}$$
$$u_{i,\,J+2}T_{ai} + v_{i,\,J+2}T_{vi} = q_{J+2}.$$

Note that $E_{J+2} = 0$ because there is no venous flow into a distal element. Eliminate T_{ai} from the second of equations (31) to obtain

$$T_{vi} = g_i + S_i T_{a(i-1)}, \tag{32}$$

in which

$$g_i = \frac{1}{d}\left(\frac{q_{J+1}}{u_{i,\,J+1}} - \frac{q_{J+2}}{u_{i,\,J+2}}\right)$$
$$S_i = \frac{1}{d}\left(\frac{E_{J+1}}{u_{i,\,J+1}}\right)$$
$$d = \frac{v_{i,\,J+1}}{u_{i,\,J+1}} - \frac{v_{i,\,J+2}}{u_{i,\,J+2}}.$$

Substituting the expression for T_{v1} computed in equation (32) into the second of equations (30), one obtains

$$(u_{i-1,\,J'+1} - S_i E_{J'+1})T_{a(i-1)} + v_{i-1,\,J'+1}T_{v(i-1)} = q_{J'+1} + S_i E_{J'+1}. \tag{33}$$

Since this equation has the same form as the second of equations (31), one can obviously obtain another equation of the form

$$T_{v(i-1)} = g_{i-1} + S_{i-1}T_{a(i-2)}. \tag{34}$$

In this way one can work his way back to the thoracic section where all venous streams terminate and all arterial streams originate. The last two of equations (28) written for the thoracic section have the form

$$u_{1,\,J}T_{a1} + v_{1,\,J}T_{v1} = q_J$$
$$u_{1,\,J+1}T_{a1} + v_{1,\,J+1}T_{v1} = q_{J+1} + \sum_n E_{J+1,\,n}T_{vn}, \tag{35}$$

in which the summation extends over those elements connected to the thoracic segment. For each of these segments an equation corresponding to equation (34) can be written:

$$T_{vn} = g_n + S_n T_{a1}. \tag{36}$$

Thus, T_{a1} and T_{v1} can be computed using equations (35) and (36), and then all of the remaining temperatures can be computed.

A program was written in Fortran language for performing the previously described calculations on the CDC 1604 computer located at The University of Texas. Since the numerical procedure used only gives an approximate answer and since it is easy to make mechanical mistakes in preparing a program of this size, a great deal of effort was devoted to checking the accuracy of the results.

One good test of the accuracy is to solve a problem for which an analytical solution can be obtained and compare the two results. The distal segment of an arm or leg can be caused to cool like a section of an infinite homogeneous cylinder by setting the rate of blood flow into the segment equal to some negligibly small value (zero leads to division by zero in the program) and setting the metabolic heat generation rate equal to zero. The analytical solution for this case is discussed in many books, such as the one by H. I. Carslaw and J. C. Jaeger (1959). In the test calculation, a uniform initial temperature of 37°C and an environmental temperature of 20°C were used. The physical properties were such that the surface temperature of the cylinder fell to about 24°C during 3,000 seconds of cooling. After 1,000 seconds of cooling, the analytical solution gave a surface temperature of 25.57°C while the numerical solution gave 25.58°C; and after 3,000 seconds of cooling the corresponding temperatures were 24.07°C and 24.06°C. The mean tissue temperature at $t = 3{,}000$ seconds computed analytically was 27.81°C while the numerically computed temperature of the venous blood leaving the element, which should be very close to the mean tissue temperature, was 27.84°C. It appears that the numerical procedure used is sufficiently accurate to produce useful results. The above results were obtained using 15 radial points and taking time steps of 5 seconds. Under these conditions, computing the temperatures in all 15 elements of the body requires about 15 minutes of computer time.

Since it is not convenient to obtain analytical solutions for the heat conduction equation applied to a nonhomogeneous cylinder, some other checking procedure had to be devised. It proved to be fairly convenient (and informative since several errors were found in this way) to check the over-all energy balances. For instance, in a given element the net rate at which heat is transported into the element by circulating blood, plus the rate of heat generation by metabolic reactions, minus the rate at which heat is lost to the environment must equal the rate of accumulation of heat in the element. It must also be true that the rate at which heat is carried into the arterial pool in an element by incoming arterial blood, minus the rate at which it is carried out of the pool by arterial blood entering the capillaries or flowing into an adjacent element must equal the rate at which heat is transferred from the arterial pool to the surrounding tissue plus the rate of accumulation of heat in the arterial pool. Making such over-all checks on the computed results indicated that the program was quite free from error.

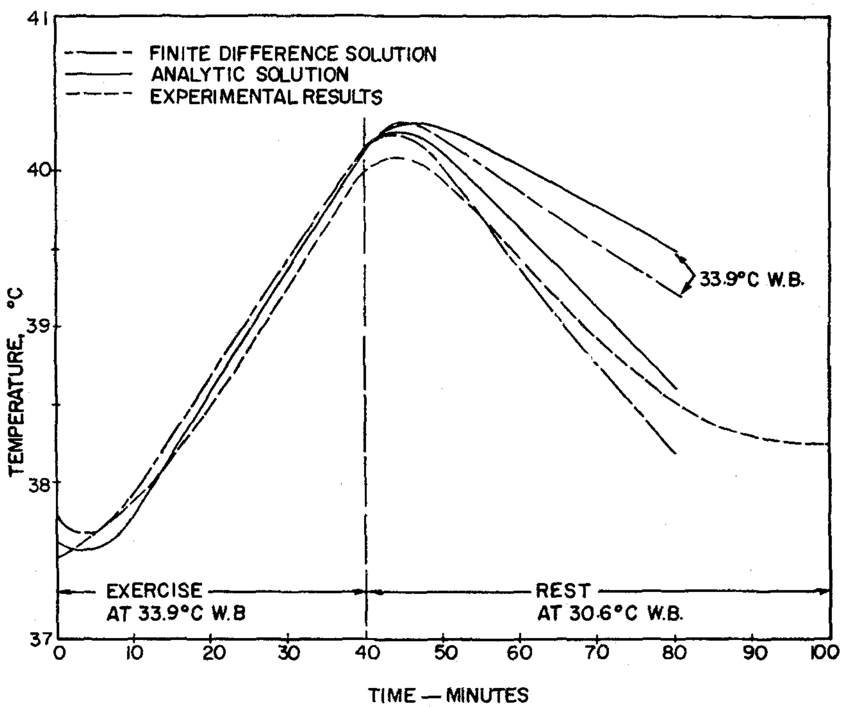

Figure 4. Comparison of rectal temperatures obtained (1) experimentally, (2) by computation using an analytical solution, and (3) by computation using the finite difference scheme presented in this paper

Finally, results computed using the numerical procedure were compared with roughly equivalent results computed using the analytical procedure reported in a previously published paper (Wissler, 1961b). Although inherent differences in the two programs precluded making an exact check, the agreement as shown in Figure 4 was acceptable in the sense that the differences between corresponding curves could be explained logically. The most striking difference is that the central abdominal temperature computed numerically falls much more rapidly during the early resting period than the corresponding temperature computed using the analytical procedure. An explanation for this can be found by studying the temperature profiles existing at the beginning

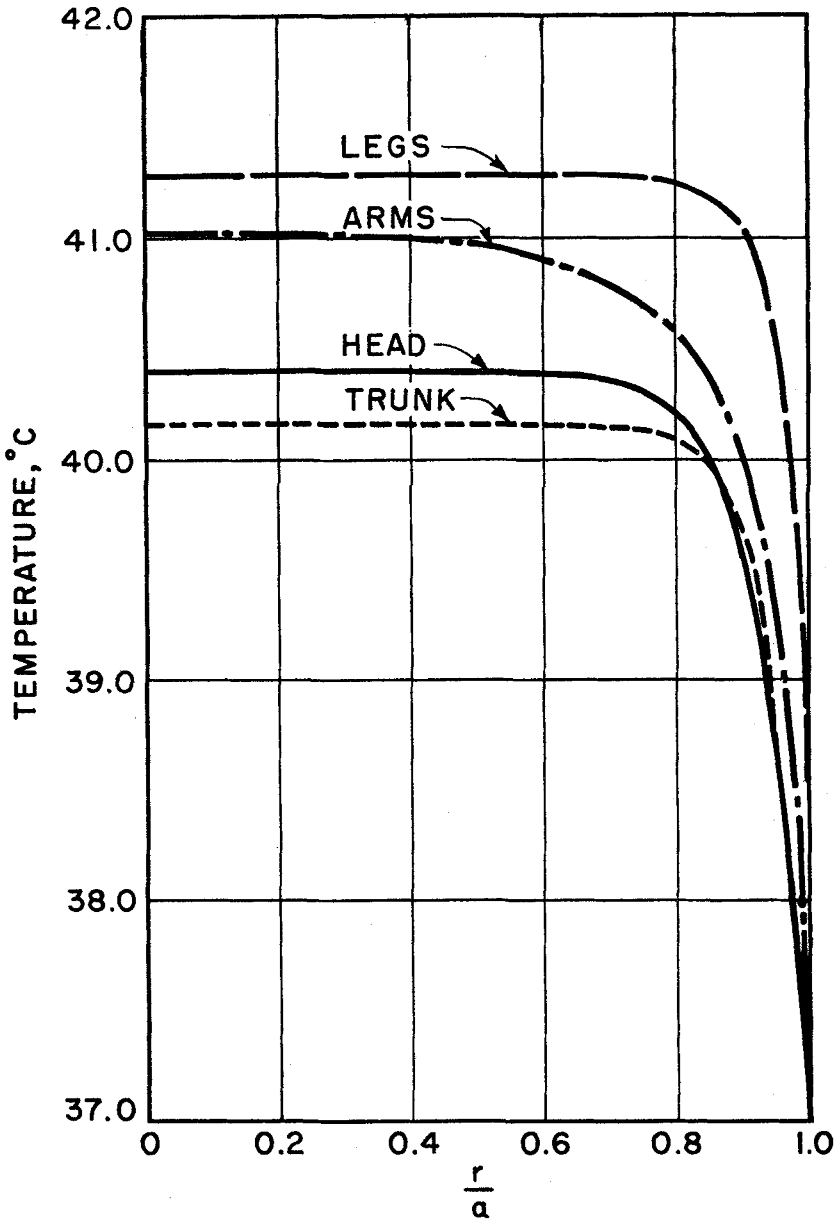

Figure 5. Temperature profiles existing at the end of the exercise period as computed using the analytical solution

of the period of cooling as shown in Figures 5 and 6. One interesting feature is that the mean temperature of the abdomen computed numerically is higher than the corresponding temperature computed using the earlier analytical

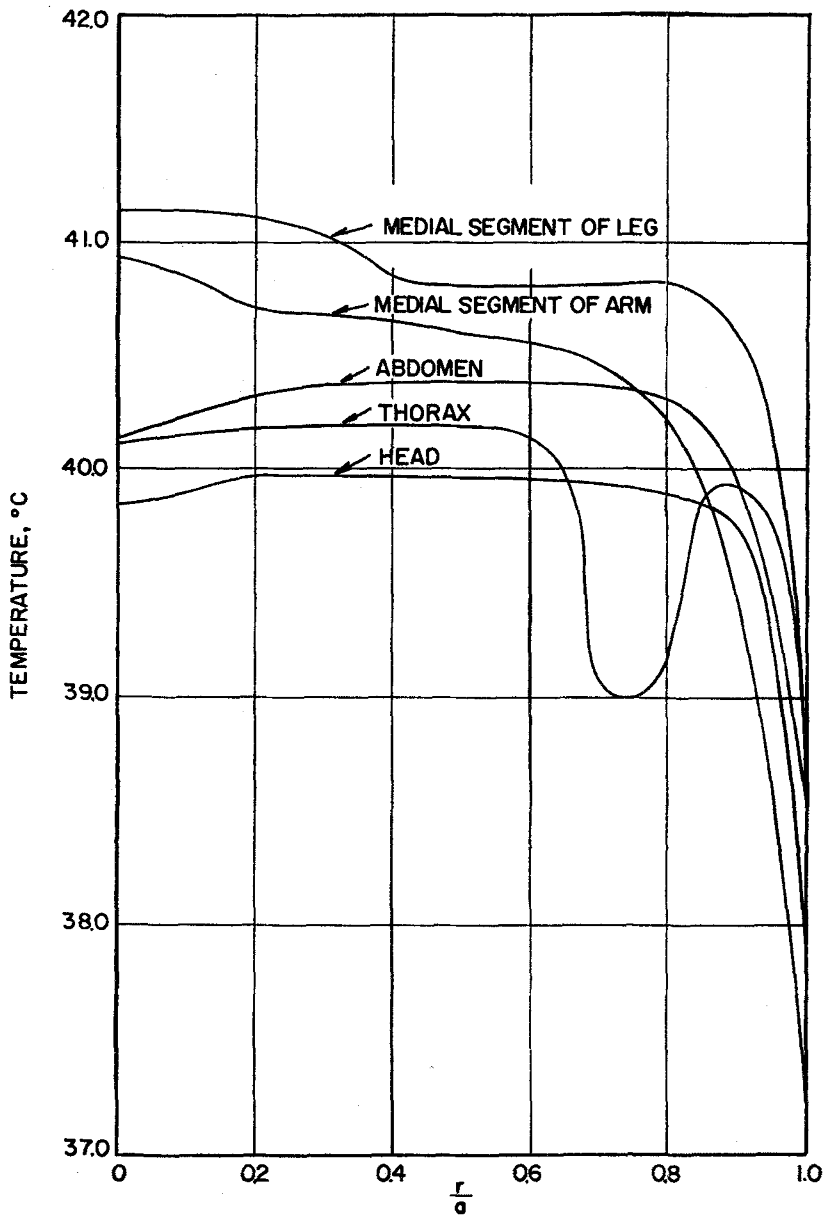

Figure 6. Temperature profiles existing at the end of the exercise period as computed using the finite difference scheme

solution. This is reasonable because the metabolic heat generated in the trunk was concentrated in the abdomen in the latest model but was distributed uniformly throughout the entire trunk in the earlier model. Although there is not much heat generation in the thoracic section, the temperature is still relatively high. This is due to the very high blood flow rate that exists in the lungs. The pronounced dip in the temperature profile of the thorax occurs in the region just outside of the lungs where a capillary blood flow rate of 0.0004 cc of blood/cc of tissue-second was used. Since this value is much lower than the value of 0.0055 assigned to the subcutaneous region, the temperature of the region just outside of the lungs does not rise as rapidly as the temperature of the lungs or the subcutaneous tissue. Finally, it should be noted that the temperatures of the arms and legs are somewhat lower in the latest model than they were in the previous model. This is due to the fact that the temperature of the arterial blood entering these regions is lower in the latest model. It seems

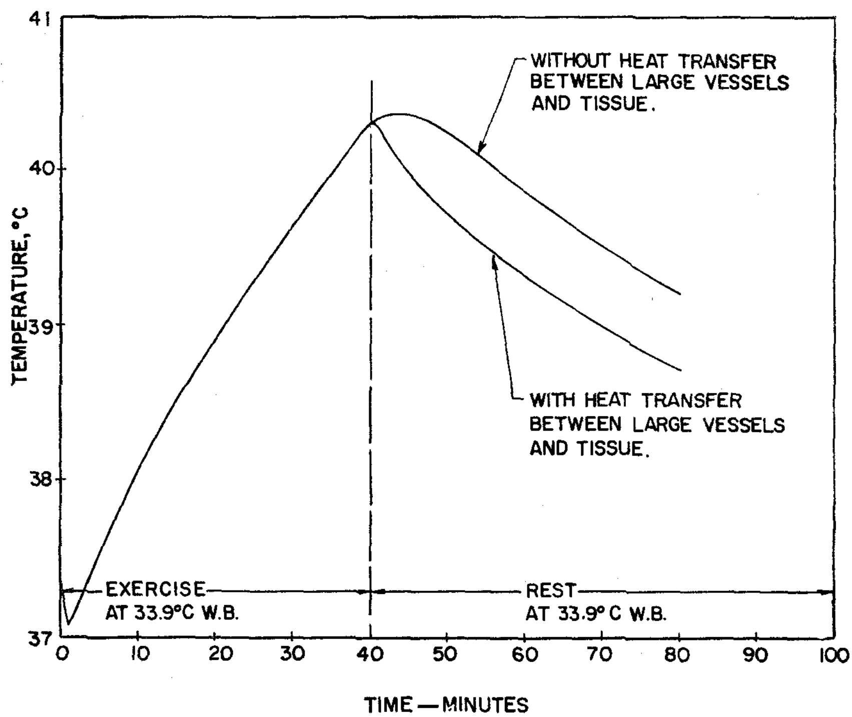

Figure 7. Two curves which show the influence of heat transfer between the blood in large vessels and the surrounding tissue on the rectal temperature during cooling

reasonable that this, in turn, can be attributed to the higher blood flow rates used in the abdomen and thorax. After all, the amount of heat generated in the entire body during the period of exercise is very nearly the same in the two models; and since the abdominal temperature increases more rapidly in the latest model than in the previous model, the temperatures of the extremities must increase less rapidly. Finally, to return to the original point, one would expect the central abdominal temperature computed using the latest model to decrease more rapidly during the first part of the cooling period because the abdominal region is at a higher temperature and, therefore, more susceptible to heat loss than it was when the earlier model was used. This is particularly true because the peripheral regions are at a lower temperature than previously and, hence, they serve as heat sinks.

It was found using the earlier model that allowing heat transfer between adjacent large arteries and veins did not affect the rate of heating or cooling significantly because of the very large blood flow rates used (Wissler, 1961b). In contrast, it was found in this study that permitting heat transfer between the large arteries and veins and the surrounding tissue does have a pronounced effect on the rate of heating or cooling. This is illustrated in Figure 7 where there are presented two cooling curves, one obtained with no heat transfer between the large vessels and tissue and the other obtained with what was evidently too much heat transfer. It is hoped that observations of this kind can be used to determine appropriate values for those parameters which cannot be measured directly.

It is felt that this model contains as much information as the currently available experimental data warrant. The next task is to study the characteristics of the model in order to determine what kind of experiments might be useful in determining those parameters which cannot be measured directly. For example, one can ask whether useful information can be obtained by the measurement of transient temperatures in the brachial veins during periodic heating of a distal portion of the arm. If the calculations show that measurable variations should exist, then a measurement of the amplitude and phase of the variations should prove to be very worth-while. The results of such calculations will be reported in future papers.

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