Research Centre for Gas Innovation

Lattice Boltzmann Method: An Introductory Overview

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PME-5429 | Multiscale Methods (2020)

PME-5429 - Lattice Boltzmann Method

Summary of topics

- 20/10: Introduction & Kinetic Theory
- 27/10: Lattice Boltzmann & Hands-On
- 03/11: Dense Fluids & Hands-On

Brief review

Review

$$\boxed{\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{r}} = -\frac{1}{\tau_c}\,(f - g)}$$

$$f_0 = \frac{\rho}{m}(2\pi R\,T)^{-3/2} \exp\Big[-\frac{(\boldsymbol{\xi} - \mathbf{u})^2}{2R\,T}\Big]$$

$$g = f_0 \times \Big(1 + \tau_c\,((\boldsymbol{\xi} - \mathbf{u})/R\,T) \cdot \mathbf{g}\Big)$$

- τ_c: relaxation time (BGK collision model)
- f_0: equilibrium distribution function
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- equation of state: $p = \rho RT$ (ideal gas)
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Next steps

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Theory of the lattice Boltzmann method: From the Boltzmann equation to the lattice Boltzmann equation

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(Received 29 April 1997; revised manuscript received 26 August 1997)

- Design time-marching scheme
- Introduce discrete velocity space

Part 2:
The Lattice Boltzmann method

1. Time-marching scheme
2. Lattice scheme
3. The Lattice Boltzmann Equation
4. Boundary conditions
5. Hands-on tutorial

Time-marching scheme

Time-marching – velocity characteristics

The left-hand side of the transport equation is an *exact differential along constant velocity lines*

$$f(\mathbf{r}(t), \boldsymbol{\xi}, t) \quad \text{with} \quad \dot{\mathbf{r}}(t) = \boldsymbol{\xi}$$

that is:

$$\frac{df(\mathbf{r}(t), \boldsymbol{\xi}, t)}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{r}}$$

In the LB scheme, the transport equation is integrated along these constant velocity lines

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Time marching

Integrating over a small time interval δt we get

$$f(\mathbf{r}+\boldsymbol{\xi}\delta t, \boldsymbol{\xi}, t+\delta t) - f(\mathbf{r}, \boldsymbol{\xi}, t) = \int_t^{t+\delta t} \Omega_g(s) ds$$

The right-hand side must be approximated somehow...

$$\int_t^{t+\delta t} \Omega_g(s) ds \approx \delta t \, \Omega_g(t)$$

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$$\int_t^{t+\delta t} \Omega_g(s) ds \approx \frac{\delta t}{2}\left[\Omega_g(t+\delta t) + \Omega_g(t)\right]$$

✓second-order scheme (OK!)

Avoiding implicit scheme

... but this leads to an implicit scheme:

$$f(t+\delta t) - f(t) = \frac{\delta t}{2}\left[\Omega_g(t+\delta t) + \Omega_g(t)\right]$$

Introduce new dynamic variable

$$\tilde{f}(t) \equiv f(t) - \frac{\delta t}{2}\Omega_g(t)$$

Then, after a few manipulations,

$$\tilde{f}(t+\delta t) - \tilde{f}(t) = -\frac{1}{\tau}\left(\tilde{f}(t) - g(t)\right)$$

with the new (dimensionless) relaxation time

$$\boxed{\tau = (\tau_c/\delta t) + \tfrac{1}{2}, \quad (\tau > 0.50)}$$

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The time propagation is conducted as

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Lattice scheme

Discrete velocity space

Basic idea: introduce a discrete velocity space,

$$f(\mathbf{r}, \boldsymbol{\xi}, t) \;\rightarrow\; f(\mathbf{r}, \mathbf{e}_\alpha, t) \quad \alpha = 0, \ldots, M$$

so that flow-field integrals can be replaced by some quadrature rule :

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$\mathbf{e}_\alpha$ = *nodes*, W_α = *weight factors*, M = *order*

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Low Mach number approximation

If we assume

$$(u^2/R\,T) \ll 1 \quad (\sqrt{RT} \propto \text{sound speed})$$

$$f_0 = \frac{\rho/m}{(2\pi R\,T)^{\frac{D}{2}}} \exp\left[-\frac{(\boldsymbol{\xi}-\mathbf{u})^2}{2R\,T}\right]$$

(from now on we consider $D = 2$)

... evaluation of flow variables will involve integrals of the form:

$$\iint d\xi_x d\xi_y\ (\xi_x)^{m_1}\ (\xi_y)^{m_2}\ e^{-\frac{\xi^2}{2R\,T}}$$

Gauss-Hermite quadratures exact if $m_{max} \leq (2M-1)$

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Gauss-Hermite quadratures

General M-th order formula (one dimension):

$$\int_{-\infty}^{+\infty} \psi(x)\, e^{-x^2}\, dx \approx \sum_{\alpha} w_\alpha \psi(x_\alpha)$$

with node points and weight factors determined by

$$w_\alpha = \frac{2^{M-1} M! \sqrt{\pi}}{M^2 \left[H_{M-1}(x_\alpha)\right]^2} \quad \text{with} \quad H_M(x_\alpha) = 0$$

$H_M(x) = M$-th degree Hermite polynomial

ex: $M = 2$: $\begin{array}{lcc} x_\alpha = & -1/\sqrt{2} & +1/\sqrt{2} \\ w_\alpha = & \sqrt{\pi}/2 & \sqrt{\pi}/2 \end{array}$

Gauss-Hermite quadratures

General M-th order formula (one dimension):

$$\int_{-\infty}^{+\infty} \psi(x)\, e^{-x^2}\, dx \approx \sum_{\alpha} w_\alpha \psi(x_\alpha)$$

with node points and weight factors determined by

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ex: $M = 3$: $\begin{array}{lccc} x_\alpha = & -\sqrt{6}/2 & 0 & \sqrt{6}/2 \\ w_\alpha = & \sqrt{\pi}/6 & 2\sqrt{\pi}/3 & \sqrt{\pi}/6 \end{array}$

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$$\int_{-\infty}^{+\infty} x^m\, e^{-x^2}\, dx = \sum_{\alpha} w_\alpha x_\alpha^m$$

if $m \leq 2M - 1$

Exact evaluation of flow variables

What is the minimum quadrature order needed?

$$\rho\mathbf{u} = \int d\boldsymbol{\xi}\; \overbrace{m\boldsymbol{\xi}}^{(1)} \cdot \overbrace{f^{eq}(\boldsymbol{\xi})}^{(2)} \qquad \text{(isothermal)}$$

$$\rho e = \int d\boldsymbol{\xi}\; \overbrace{\tfrac{1}{2}m\xi^2}^{(2)} \cdot \overbrace{f^{eq}(\boldsymbol{\xi})}^{(2)} \qquad \text{(thermal)}$$

But this is not enough: the relations

$$\int d\boldsymbol{\xi}\, m\boldsymbol{\xi}\, \delta f = 0; \quad \int d\boldsymbol{\xi} \tfrac{1}{2} m\xi^2 \delta f = 0$$

must also be preserved with $\delta f = \phi f^{eq}$ with $\phi(\boldsymbol{\xi})$ being of second-order in (ξ_x, ξ_y) [He-Luo (1997)].

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Thus, the minimum quadrature order for each velocity component (ξ_x, ξ_y) is

$$5 \leq 2M - 1 \ \Rightarrow\ M_{\min} = 3 \qquad \text{(isothermal)}$$

$$6 \leq 2M - 1 \ \Rightarrow\ M_{\min} = 4 \qquad \text{(thermal)}$$

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... and the minimum number of nodes in two-dimensional velocity space for each case is

$$3 \times 3 = 9 \ \text{ nodes} \qquad \text{(isothermal)}$$

$$4 \times 4 = 16 \ \text{ nodes} \qquad \text{(thermal)}$$

Isothermal vs. thermal models

Difficulties involved in thermal models:

- Not really compatible with simple BGK model (recall: $\lambda/\mu c_v \neq 2.5$ for ideal gases)
- Finer lattices are required
- Nodal positions depend on temperature (weight function involves T)
- Second-order low Mach number approximation not really adequate (thermal models should account for high compressibility)
- Generally more unstable

Isothermal lattice models

From now on we consider only *isothermal models*

Hydrodynamic moments:

$$\rho = \int m\tilde{f}d\boldsymbol{\xi}, \quad \rho\mathbf{u} = \int m\boldsymbol{\xi}\tilde{f}d\boldsymbol{\xi} + \frac{\delta t}{2}\rho\mathbf{g}$$

Equation of state (ideal gas for now):

$p = \rho RT = \rho c_s^2$

$c_s \equiv \sqrt{(\partial p/\partial \rho)_T} = \sqrt{RT}$ (isothermal sound speed)

Transport coefficient:

$\mu = \rho RT\tau_c \Rightarrow \nu \equiv \mu/\rho = c_s^2\tau_c$ (kinematic viscosity)

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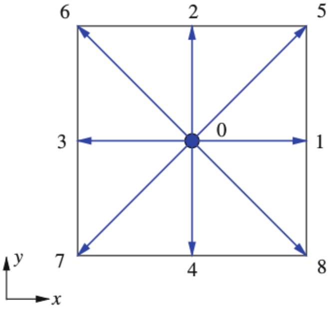
D2Q9 isothermal lattice model

Evaluation of flow variables with 3×3 quadrature:

$$\rho = \sum_\alpha W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t), \quad \rho\mathbf{u} = \sum_\alpha m\mathbf{e}_\alpha W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t)$$



$c = \sqrt{3RT}$: 'lattice speed'

$\mathbf{e}_0 = (0, 0)$
$\mathbf{e}_1 = (+c, 0)$
$\mathbf{e}_2 = (0, +c)$
$\mathbf{e}_3 = (-c, 0)$
$\mathbf{e}_4 = (0, -c)$
$\mathbf{e}_5 = (+c, +c)$
$\mathbf{e}_6 = (-c, +c)$
$\mathbf{e}_7 = (-c, -c)$
$\mathbf{e}_8 = (+c, -c)$

(figure: J. Zhang, Microfluid Nanofluid (2011) 10:1–28)

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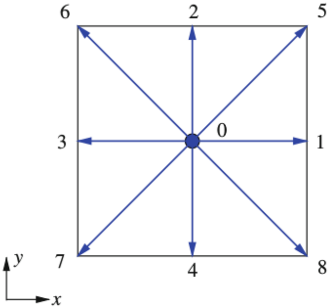

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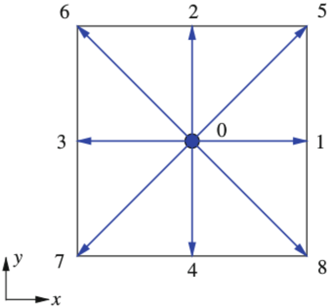

weights:

$W_\alpha = \omega_\alpha \cdot (2\pi R\,T)^{\frac{D}{2}} \cdot e^{\frac{\xi_\alpha^2}{2R\,T}}$

$\omega_0 = \frac{4}{9}$

$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \frac{1}{9}$

$\omega_5 = \omega_6 = \omega_7 = \omega_8 = \frac{1}{36}$

(obs: $\sum_\alpha \omega_\alpha = 1$)

(figure: J. Zhang, Microfluid Nanofluid (2011) 10:1–28)

D2Q9: Lattice equilibrium distribution

Then:

$$\rho = \sum_{\alpha=0}^{8} m \times W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t)$$

$$\rho\mathbf{u} = \sum_{\alpha=0}^{8} m\mathbf{e}_\alpha \times W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t)$$

$$W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t) = \frac{\rho\omega_\alpha}{m}\left[1 + \frac{\mathbf{e}_\alpha \cdot \mathbf{u}}{RT} + \frac{(\mathbf{e}_\alpha \cdot \mathbf{u})^2}{2RT} - \frac{u^2}{2RT}\right]$$

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Then:

$$\rho = \sum_{\alpha=0}^{8} m \, n_\alpha^{eq}(\mathbf{r}, t)$$

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$$n_\alpha^{eq}(\mathbf{r}, t) \equiv \frac{\rho\omega_\alpha}{m}\left[1 + \frac{3\mathbf{e}_\alpha \cdot \mathbf{u}}{c^2} + \frac{9(\mathbf{e}_\alpha \cdot \mathbf{u})^2}{2c^4} - \frac{3u^2}{2c^2}\right]$$

(*D2Q9 lattice equilibrium distribution*)

Fractional populations

D2Q9 lattice quadrature:

$$\rho = \sum_{\alpha=0}^{8} m \left[W_\alpha f^{eq}(\mathbf{r}, \mathbf{e}_\alpha, t)\right]$$

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Same formulas apply to the dynamic variables f:

$$\rho = \sum_{\alpha=0}^{8} m \left[W_\alpha f(\mathbf{r}, \mathbf{e}_\alpha, t)\right]$$

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And also to the time-marching variables $\tilde{f}$:

$$\rho = \sum_{\alpha=0}^{8} m \left[W_\alpha \tilde{f}(\mathbf{r}, \mathbf{e}_\alpha, t) \right]$$

$$\rho\mathbf{u} = \sum_{\alpha=0}^{8} m\mathbf{e}_\alpha \left[W_\alpha \tilde{f}(\mathbf{r}, \mathbf{e}_\alpha, t) \right] + \frac{\delta t}{2} \rho\mathbf{g}$$

Fractional populations

New dynamic variables:

$$n_\alpha(\mathbf{r},t) \equiv W_\alpha \tilde{f}(\mathbf{r},\mathbf{e}_\alpha,t) \qquad \text{(obs: } [W_\alpha] = [d\boldsymbol{\xi}]\text{)}$$

Time-marching scheme translates to

$$n_\alpha(\mathbf{r}+\mathbf{e}_\alpha\delta t, t+\delta t) - n_\alpha(\mathbf{r},t) = -\frac{1}{\tau}(n_\alpha(\mathbf{r},t) - g_\alpha(\mathbf{r},t))$$

$$g_\alpha(\mathbf{r},t) = n_\alpha^{eq}(\mathbf{r},t)\left[1 + 3(\delta t/c^2)\left(\tau - \tfrac{1}{2}\right)(\mathbf{e}_\alpha - \mathbf{u})\cdot\mathbf{g}\right]$$

Evaluation of flow fields

$$\rho(\mathbf{r},t) = \sum_\alpha m\, n_\alpha(\mathbf{r},t)$$

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just one step away from the LBE ...

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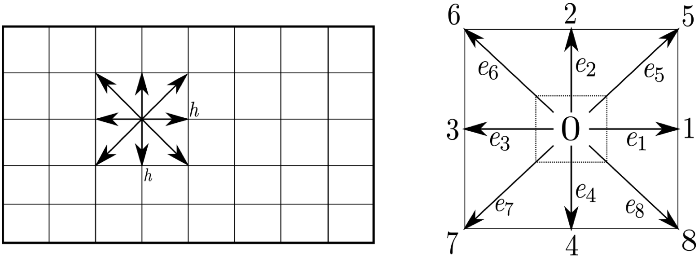
just one step away from the LBE ...

The Lattice Boltzmann Equation

Adjusting the spatial grid

Final step: spatial grid is chosen so that *updated populations are mapped to a new site* (instead of ending up in an arbitrary point)



(image downloaded from Krishna Kumar's site: kks32-slides.github.io)

The grid becomes a regular lattice with parameter

$$\delta x = c \cdot \delta t = \sqrt{3RT} \cdot \delta t$$

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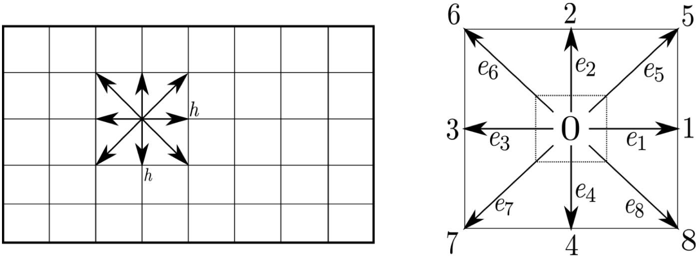

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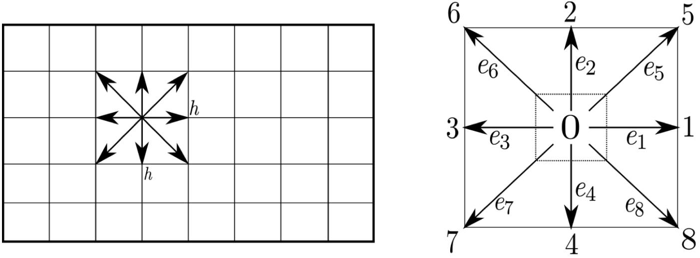
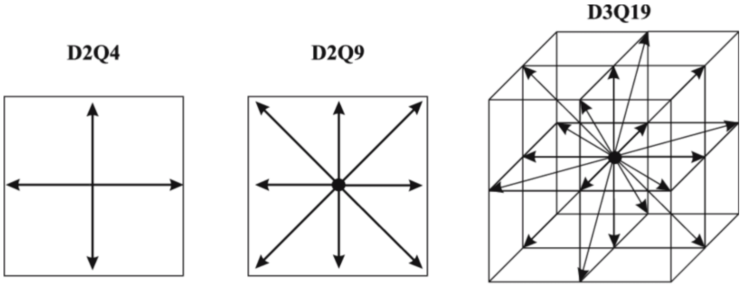


(image downloaded from Krishna Kumar's site: kks32-slides.github.io)

(spatial labels: $\mathbf{r}_k$, functions: $\psi_k = \psi(\mathbf{r}_k)$)

Other lattices



from: C. Körner *et. al.*, J. Stat. Phys, 121, 179-196 (2005)

- Not all lattices are physically acceptable
- Rotational properties of the Navier-Stokes equations must be preserved

The Lattice Boltzmann Equation

$$\boxed{n_\alpha(\mathbf{r}_k + \mathbf{e}_\alpha \delta t, t + \delta t) = n_\alpha(\mathbf{r}_k, t) + \Omega_\alpha(\mathbf{r}_k, t)}$$

- $\Omega_\alpha(\mathbf{r}_k, t) = -\dfrac{1}{\tau}\big(n_\alpha(\mathbf{r}_k, t) - g_\alpha(\mathbf{r}_k, t)\big)$
- $g_\alpha(\mathbf{r}_k, t) = n_\alpha^{eq}(\mathbf{r}_k, t)\big[1 + 3(\delta t/c^2)\big(\tau - \tfrac{1}{2}\big)(\mathbf{e}_\alpha - \mathbf{u}_k) \cdot \mathbf{g}_k\big]$
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- $\nu = \frac{1}{3}c^2\delta t(\tau - \frac{1}{2}), \quad p = \frac{1}{3}c^2\rho, \quad c_s = c/\sqrt{3}$

D2Q9-isothermal Lattice Boltzmann model

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D2Q9-isothermal Lattice Boltzmann model

The Lattice Boltzmann Equation

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D2Q9-isothermal Lattice Boltzmann model

Lattice units

The formulas can be put in dimensionless form by employing *lattice units*:

$$\mathbf{r} = (\delta x)\,\bar{\mathbf{r}}\ ,\ t = (\delta t)\,\bar{t}\ ,\ \mathbf{u} = (\delta x/\delta t)\,\bar{\mathbf{u}}\ ,\ m = (m)\,\bar{m}$$

(obs: in these units: $\bar{c} = 1$ and $\bar{m} = 1$)

Other quantities:

$$\begin{aligned}
& n_\alpha = \bar{n}_\alpha \times (1/\delta x^3) \\
& \rho = \bar{\rho} \times (m/\delta x^3) \\
& \mathbf{g} = \bar{\mathbf{g}} \times (\delta x/\delta t^2) \\
& \nu = \bar{\nu} \times (\delta x^2/\delta t) \\
& \text{etc.}
\end{aligned}$$

LBE - dimensionless form

In lattice units (dimensionless form, bars omitted):

$$\boxed{n_\alpha(\mathbf{r}_k + \mathbf{e}_\alpha, t+1) = n_\alpha(\mathbf{r}_k, t) + \Omega_\alpha(\mathbf{r}_k, t)}$$

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LBE - fluid modeling

Fluid properties $(\nu, c_s, \rho_0)|_T$ provide natural scales:

$$\text{time: } \nu/c_s^2, \quad \text{length: } \nu/c_s, \quad \text{mass: } \rho_0\,(\nu/c_s)^3$$

The dimensionless parameter τ then sets the model's physical time and length scales:

$$\tau = \frac{3\nu}{c_s^2 \delta t} + \frac{1}{2} \;\Rightarrow\; \delta t = \frac{(\nu/c_s^2)}{\tau - \frac{1}{2}}, \quad \delta x = (\sqrt{3} c_s)\delta t$$

Conflict:

- For most fluids: $10^{-13}s < (\nu/c_s^2) < 10^{-9}s$
- In simulations we would like: $0.50 < \tau < 1.00$

LBE - fluid modeling

$[T \sim 20^o C]$	ν $(10^{-6} m^2/s)$	c_s (m/s)	ν/c_s^2 (s)
Air	15	343	1.275×10^{-10}
Glycerine	648	1920	1.758×10^{-10}
Castor Oil	292	1474	1.344×10^{-10}
Water	1	1482	4.550×10^{-13}

(https://www.engineeringtoolbox.com)

Example: $\tau = 0.51 \;\Rightarrow\; \delta t = 100 \times (\nu/c_s^2)\,,\; \delta x = \sqrt{3} c_s \delta t$

	δt	δx
Air	12.8 ns	7.6 μm
Glycerine	17.6 ns	58.5 μm
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- In this introductory course we will simply choose τ so as to ensure the stability of our simulations.

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The basic LB algorithm

Given an initial state $\{n_\alpha(\mathbf{r}_k, 0)\}$ time propagation is carried out by iterating two simple operations:

- Collide: compute collision term and update local populations (using a buffer array)

$$n^*_\alpha(\mathbf{r}_k, t) = n_\alpha(\mathbf{r}_k, t) + \Omega_\alpha(\mathbf{r}_k, t)$$

(obs: hydrodynamic moments updated here)

- Stream: copy updated populations to neighboring sites

$$n_\alpha(\mathbf{r}_k + \mathbf{e}_\alpha, t + 1) = n^*_\alpha(\mathbf{r}_k, t)$$

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The basic LB algorithm

**shock waves in a periodic domain*

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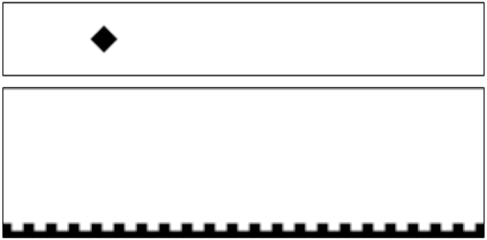
(obs: special rules apply at boundary nodes)

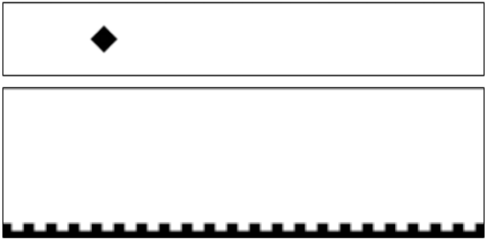
Boundary conditions

Boundary conditions in the LB method



- Some populations on *boundary nodes* are left unspecified after a streaming step
- Bottom-up: a more fundamental theory could tell us how to assign the missing values
- Top-down: unknown populations are adjusted so that standard fluid-dynamics boundary conditions are obtained

Boundary conditions in the LB method



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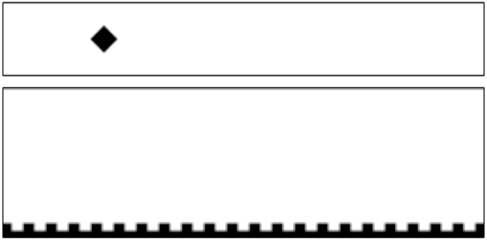

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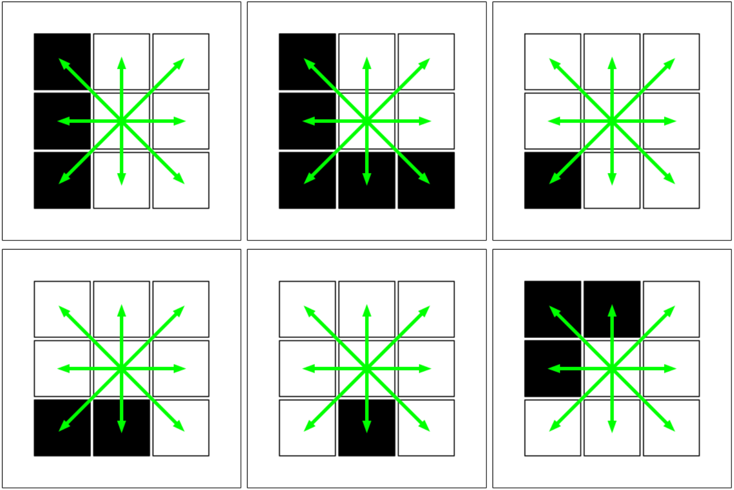
Boundaries: undetermined populations



- the simplest way to specify the unknown populations is through the *bounce-back scheme*

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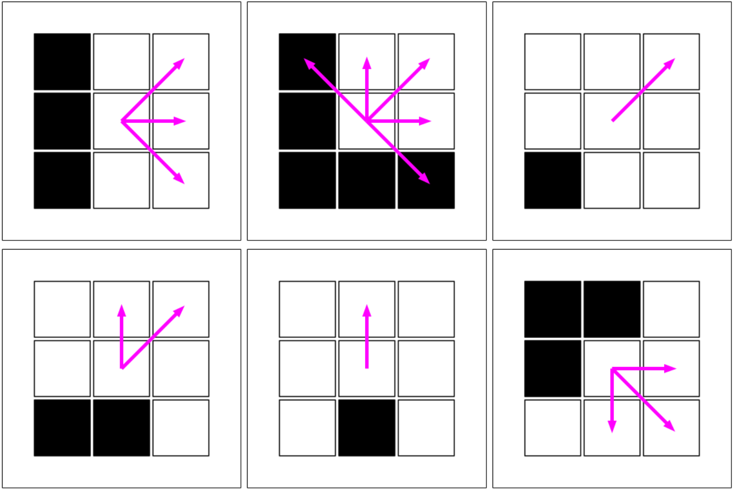

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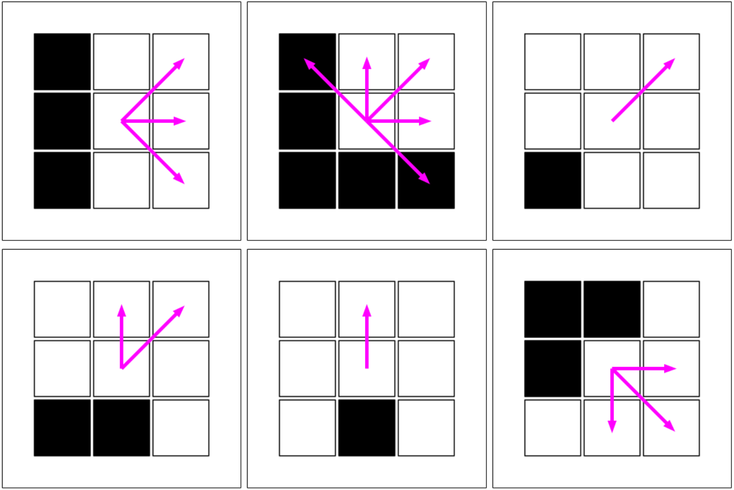

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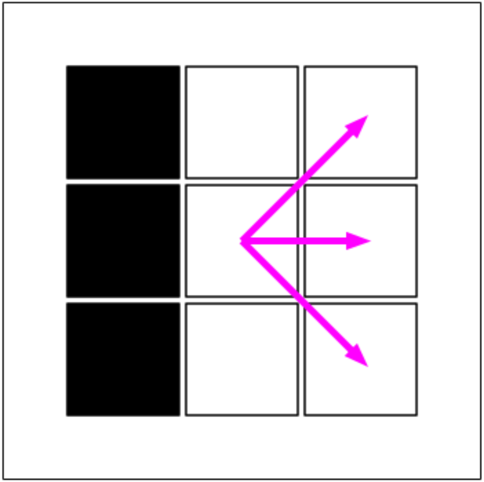
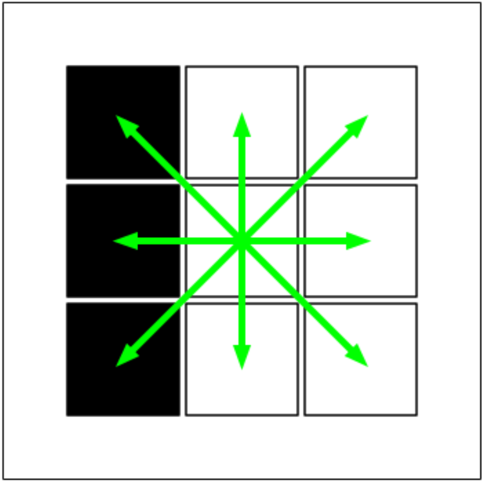
Bounce back scheme



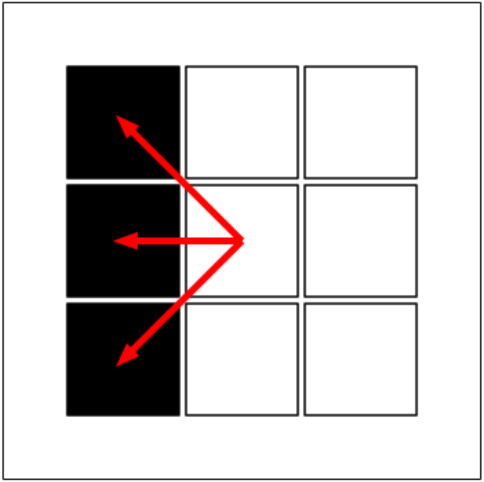
west wall/inlet: undetermined populations

Bounce back scheme



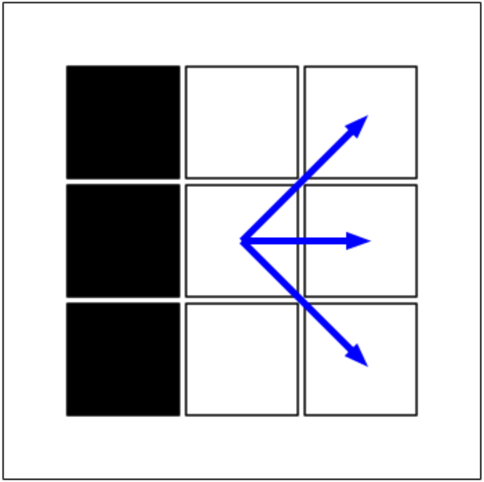
after colliding; before streaming

Bounce back scheme



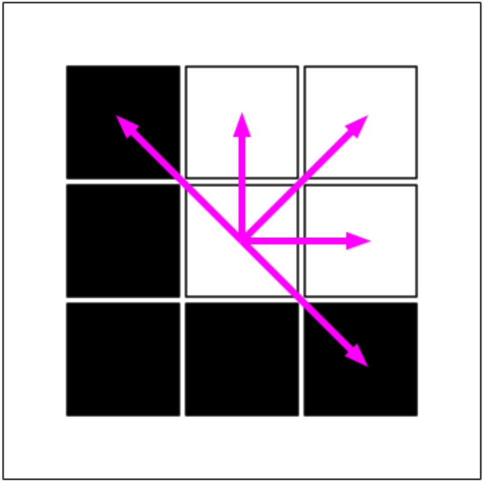
populations going into the wall...

Bounce back scheme



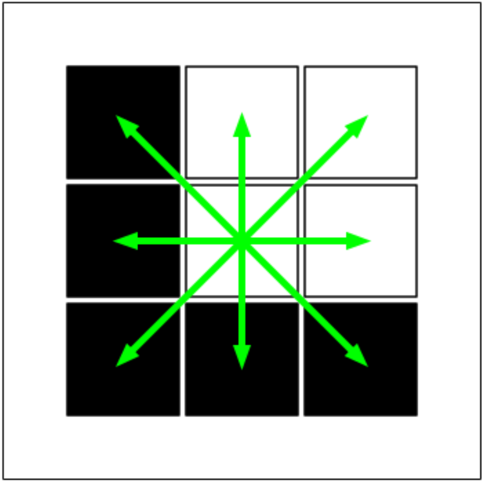
... are bounced to their opposite direction

Bounce back scheme



Southwest edge: undetermined populations

Bounce back scheme



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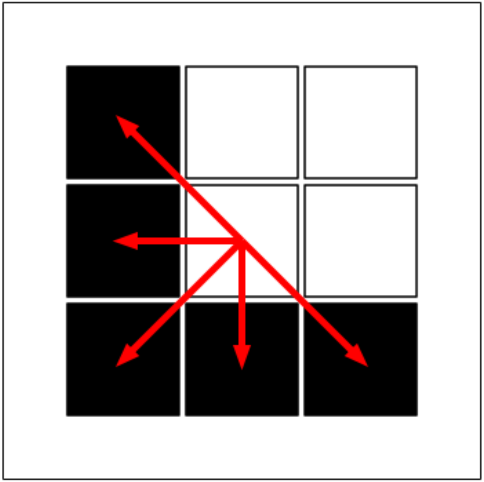

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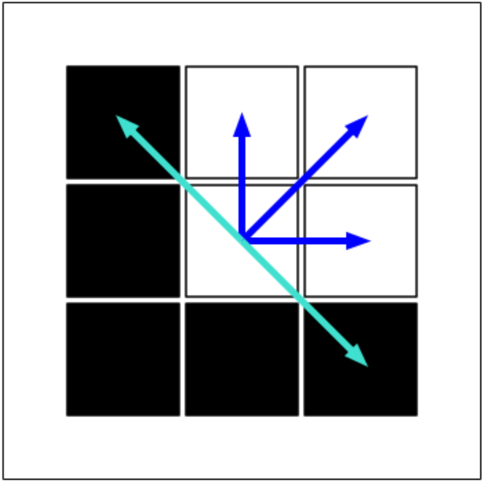

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Bounce back scheme

Single time-step bounce back:

- Completely general
- Extremely simple to implement
- No-slip velocity condition holds on average
- No-slip boundary lies midway between solid and fluid nodes

Zou-He boundary conditions

On pressure and velocity boundary conditions for the lattice Boltzmann BGK model

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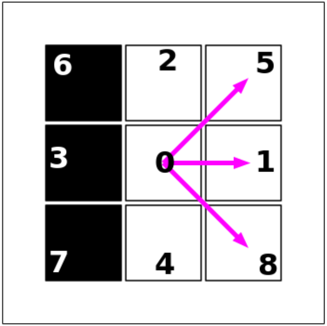
(Received 10 August 1995; accepted 24 February 1997)

- Fix inlet/outlet pressure (density)
- Fix inlet/outlet velocity
- Fix wall velocity

Zou-He: fix inlet density (bulk)



4 unknowns:
(u_x, n_1, n_5, n_8)

$$\rho = \sum_\alpha n_\alpha$$

$$\rho u_x = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_x$$

$$0 = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_y$$

$$\delta n_1 = \delta n_3$$

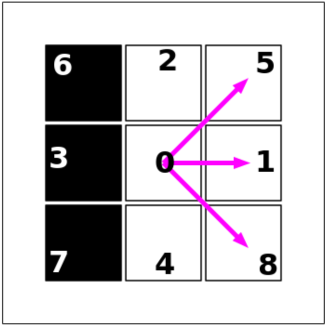
$$n_1 + n_5 + n_8 = n_0 + n_2 + n_3 + n_4 + n_6 + n_7 - \rho$$

$$n_1 + n_5 + n_8 - \rho u_x = n_3 + n_6 + n_7$$

$$n_5 - n_8 = n_4 - n_2 + n_7 - n_6$$

$$n_1 - n_1^{eq} = n_3 - n_3^{eq} \quad \leftarrow \quad \text{non-equilibrium bounce-back}$$

Zou-He: fix inlet density (bulk)



4 unknowns:
(u_x, n_1, n_5, n_8)

$$\rho = \sum_\alpha n_\alpha$$

$$\rho u_x = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_x$$

$$0 = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_y$$

$$\delta n_1 = \delta n_3$$

$$n_1 + n_5 + n_8 = n_0 + n_2 + n_3 + n_4 + n_6 + n_7 - \rho$$

$$n_1 + n_5 + n_8 - \rho u_x = n_3 + n_6 + n_7$$

$$n_5 - n_8 = n_4 - n_2 + n_7 - n_6$$

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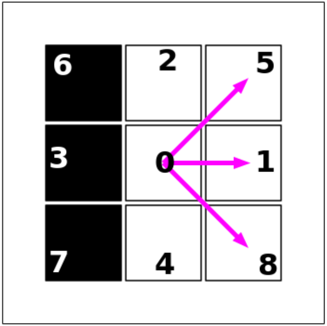

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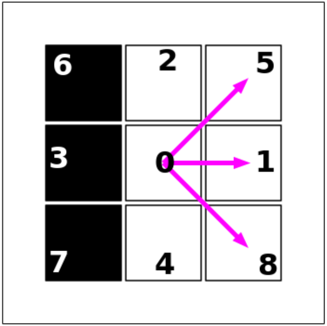

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$$\rho u_x = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_x$$

$$0 = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_y$$

$$\delta n_1 = \delta n_3$$

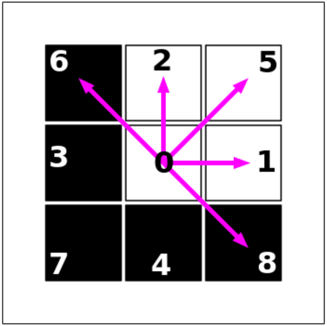
$$u_x = 1 - [n_0 + n_2 + n_4 + 2(n_3 + n_6 + n_7)]/\rho$$

$$n_1 = n_3 + \tfrac{2}{3}\rho u_x$$

$$n_5 = n_7 - \tfrac{1}{2}(n_2 - n_4) + \tfrac{1}{6}\rho u_x$$

$$n_6 = n_8 + \tfrac{1}{2}(n_2 - n_4) + \tfrac{1}{6}\rho u_x$$

Zou-He: fix inlet density (edge)



5 unknowns:

$(n_1, n_2, n_5, n_6, n_8)$

$$\rho = \sum_\alpha n_\alpha$$

$$0 = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_x$$

$$0 = \sum_\alpha n_\alpha (\mathbf{e}_\alpha)_y$$

$$\delta n_1 = \delta n_3, \;\; \delta n_2 = \delta n_4$$

$$n_1 + n_2 + n_5 + n_6 + n_8 = n_3 + n_4 + n_7 + n_8 - \rho$$

$$n_1 + n_5 - n_6 + n_8 = n_3 + n_7$$

$$n_2 + n_5 + n_6 - n_8 = n_4 + n_7$$

$$n_1 - n_1^{eq} = n_3 - n_3^{eq} \quad \leftarrow \quad \text{non-equilibrium bounce-back}$$

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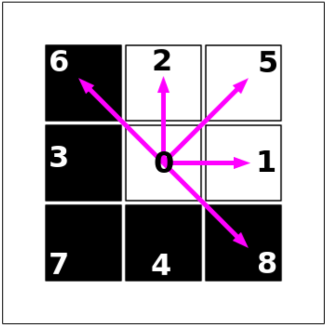

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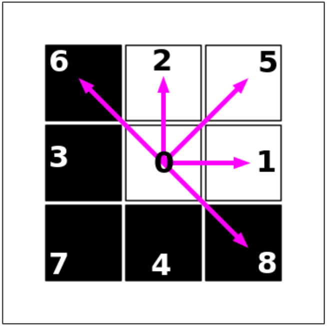

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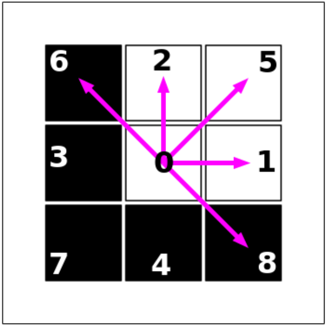

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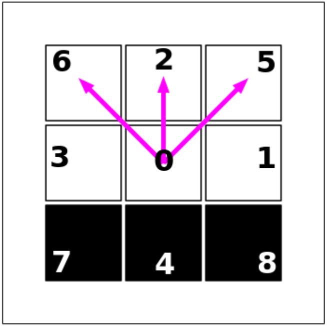
$$n_5 = n_7$$

$$n_6 = \tfrac{1}{2}[\rho - n_0 - 2(n_3 + n_4 + n_7)]$$

$$n_8 = \tfrac{1}{2}[\rho - n_0 - 2(n_3 + n_4 + n_7)]$$

Zou-He: fix velocity (wall)



4 unknowns: (ρ, n_2, n_5, n_6)

3 equations

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$$\delta n_2 = \delta n_4$$

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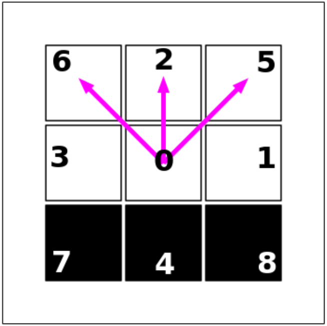

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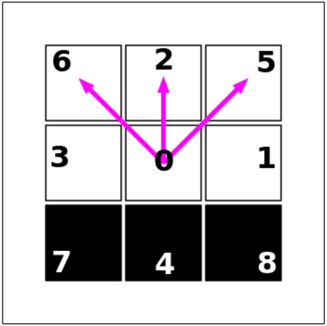

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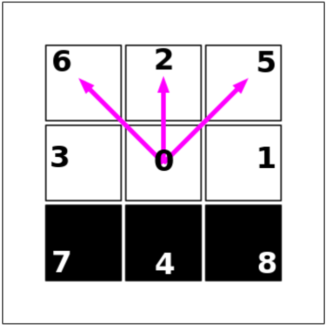

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$$\rho = [n_0 + n_1 + n_3 + 2(n_4 + n_7 + n_8)]/(1 - u_y)$$

The complete LB algorithm (1 timestep)

- Fix boundary conditions: stream-unspecified populations are set to their target values

- Collide:

$$n_\alpha^*(\mathbf{r}_k, t) = n_\alpha(\mathbf{r}_k, t) + \Omega_\alpha(\mathbf{r}_k, t)$$

(hydrodynamic moments updated, *forces computed*)

- Stream and bounce-back:

$$\begin{aligned} n_\alpha(\mathbf{r}_k + \mathbf{e}_\alpha, t+1) &= n_\alpha^*(\mathbf{r}_k, t) \quad \text{if } \mathbf{r}_k + \mathbf{e}_\alpha \text{ is fluid} \\ n_{\alpha''}(\mathbf{r}_k + \mathbf{e}_\alpha'', t+1) &= n_\alpha^*(\mathbf{r}_k, t) \quad \text{if } \mathbf{r}_k + \mathbf{e}_\alpha \text{ is solid} \end{aligned}$$

(no conflict with boundary conditions)

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Hands-on tutorial

Hands on

LB-lab-0:
program usage

LB-lab-1:
profiling a channel flow

LB-lab-2:
visualizing flow past obstacles

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