LISTA 3

Mônica A Caracanhas

29/10/20

9. A exponencial de um operador é definida pela sua expansão em série de Taylor, por exemplo, $e^A = 1 + A + A^2/2! + A^3/3! + \ldots$. Demonstre, então, que $\psi(x+a,t) = e^{i\hat{p}a/\hbar}\psi(x,t)$, e por isso o operador $\hat{p}/\hbar$ é denominado *gerador das translações espaciais*. Sugestão: expanda $\psi(x+a,t)$ em Taylor, troque as derivadas por $p = -i\hbar\partial_x$ e reagrupe os termos.

$$\Rightarrow \quad f(x \pm h) = f(x) \pm hf'(x) + \frac{h^2}{2}f''(x) \pm \frac{h^3}{6}f^{(3)}(x) + O_{1\pm}(h^4)$$

$$\psi(x+a) = \psi(x) + \partial_x\psi(x)\,a + \frac{1}{2}\partial_x^2\psi(x)\,a^2 + \frac{1}{3!}\partial_x^3\psi(x)\,a^3 + \ldots$$

$$\hat{p} = \frac{\hbar}{i}\partial_x \rightarrow \partial_x = \frac{i\hat{p}}{\hbar}$$

$$\psi(x+a) = \hat{1}\,\psi(x) + \left(\frac{i\hat{p}}{\hbar}a\right)\psi(x) + \frac{1}{2}\left(\frac{i\hat{p}a}{\hbar}\right)^2\psi(x) + \frac{1}{3!}\left(\frac{i\hat{p}a}{\hbar}\right)^3\psi(x) + \ldots$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \rightarrow \underline{\psi(x+a,t) = e^{\frac{i\hat{p}a}{\hbar}}\psi(x,t)}$$

10. Suponha que $\psi(x,0) = A/(x^2+a^2)$ para $-\infty < x < \infty$. Normalize ψ. Como é mesmo que interpretamos $|\psi(x)|^2$? Calcule $\langle x\rangle$, $\langle x^2\rangle$, e Δx. Agora determine a função de onda normalizada no espaço de momento, $\phi(p)$. Qual é a interpretação de $|\phi(p)|^2$? Usando $\phi(p)$ (e não $\psi(x)$) calcule $\langle p\rangle$, $\langle p^2\rangle$ e Δp. Finalmente, verifique o princípio de incerteza de Heisenberg para esse estado.

$$\underbrace{\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx}_{} = \int_{-\infty}^{\infty} |A|^2 \frac{1}{(x^2+a^2)^2} dx = \int_{-\pi/2}^{\pi/2} \frac{|A|^2 a \sec^2 y \, dy}{a^4 \sec^4 y} = \frac{|A|^2}{a^3}\int_{-\pi/2}^{\pi/2} \cos^2 y \, dy = \frac{|A|^2}{a^3}\frac{\pi}{2}$$

$x = a\tan y$

$dx = a\sec^2 y \, dy$

$$1 = \frac{|A|^2 \pi}{2a^3} \rightarrow |A|^2 = \frac{2a^3}{\pi}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\, dx = \int_{-\infty}^{\infty} \frac{|A|^2}{(x^2+a^2)^2}\, x \, dx = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \frac{|A|^2}{(x^2+a^2)^2} x^2 \, dx = |A|^2 \int_{-\pi/2}^{\pi/2} \frac{(a\tan y)^2 (a\sec^2 y)\, dy}{a^4 \sec^4 y}$$

$$= \frac{|A|^2}{a} \int_{-\pi/2}^{\pi/2} \frac{\tan^2 y}{\sec^2 y} dy = \frac{|A|^2}{a} \underbrace{\int_{-\pi/2}^{\pi/2} \sin^2 y \, dy}_{\pi/2}$$

$$\langle x^2 \rangle = \frac{|A|^2}{a} \frac{\pi}{2} = \frac{2a^3}{\pi} \frac{1}{a} \frac{\pi}{2} = a^2$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \underline{a} \; //$$

$$\langle \phi_p | \psi \rangle = \Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{-ipx}{\hbar}} \psi(x)\, dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{-ipx}{\hbar}} \frac{A}{(x^2+a^2)}\, dx = \frac{1}{\sqrt{2\pi\hbar}} A \frac{\pi}{a} e^{\frac{-|p|a}{\hbar}}$$

$$\int_{-\infty}^{\infty} |\Phi(p)|^2 dp = \frac{|A|^2}{2\pi\hbar} \frac{\pi^2}{a^2} \int_{-\infty}^{\infty} e^{\frac{-2|p|a}{\hbar}} dp = \frac{2a^3}{\pi} \frac{1}{2\pi\hbar} \frac{\pi^2}{a^2} \left[\underbrace{\int_{-\infty}^{0} e^{\frac{2pa}{\hbar}} dp}_{\frac{\hbar}{2a}} + \underbrace{\int_{0}^{\infty} e^{\frac{-2pa}{\hbar}} dp}_{\frac{\hbar}{2a}} \right]$$

$$\int_{-\infty}^{\infty} |\Phi(p)|^2 dp = \frac{a}{\hbar} \frac{\hbar}{a} = 1$$

$$\langle p \rangle = \int_{-\infty}^{\infty} |\Phi(p)|^2 p \, dp = 0$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} |\Phi(p)|^2 p^2 \, dp = \int_{-\infty}^{\infty} \frac{|A|^2}{(2\pi\hbar)} \frac{\pi^2}{a^2} e^{-\frac{2|p|a}{\hbar}} p^2 \, dp = \frac{a}{\hbar} \int_{-\infty}^{\infty} e^{-\frac{2|p|a}{\hbar}} p^2 \, dp$$

$$\rightarrow \int_{-\infty}^{\infty} p^2 e^{-\frac{2|p|a}{\hbar}} dp = 2 \int_{0}^{\infty} p^2 e^{-\frac{2pa}{\hbar}} dp = 2 \frac{2}{\left(\frac{2a}{\hbar}\right)^3} = \frac{1}{2}\left(\frac{\hbar}{a}\right)^3$$

$$\int_0^{\infty} x^2 e^{-\beta x} dx = \frac{\partial}{\partial \beta^2} \underbrace{\int_0^{\infty} e^{-\beta x} dx}_{1/\beta}$$

$$\int_0^{\infty} x^2 e^{-\beta x} dx = \frac{2}{\beta^3}$$

$$\langle p^2 \rangle = \left(\frac{a}{\hbar}\right) \frac{1}{2} \left(\frac{\hbar}{a}\right)^3 = \frac{1}{2} \left(\frac{\hbar}{a}\right)^2$$

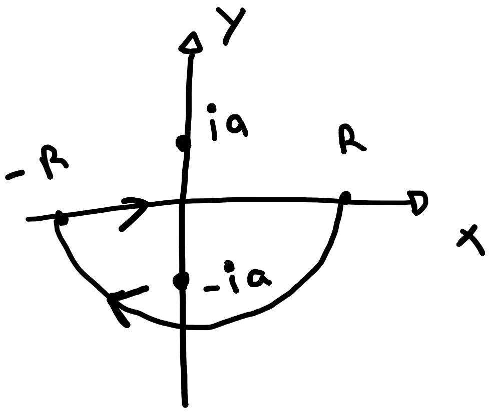
$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{1}{\sqrt{2}} \frac{\hbar}{a}$$

$$\therefore \Delta x \, \Delta p = \frac{1}{\sqrt{2}} \hbar$$

$$\int_{-\infty}^{\infty} \frac{e^{-ipx}}{(x^2+a^2)}\,dx \longrightarrow I_z = \oint \frac{e^{-ipz}}{(z+ia)(z-ia)}\,dz$$

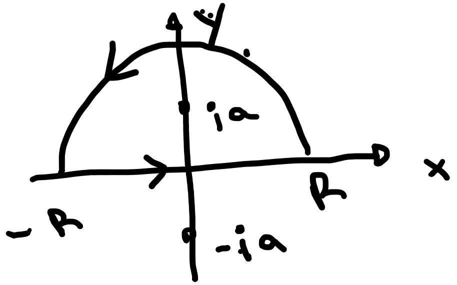
$$z = x+iy \rightarrow e^{-ip(x+iy)} = e^{-ipx}\,\underbrace{e^{py}}_{\text{FATOR DE CONVERGÊNTIA}}$$

$p > 0 \rightarrow y < 0$



$$I_z = -2\pi i \left.\frac{e^{-ipz}}{(z-ia)}\right|_{z=-ia} = \frac{\pi}{a}e^{-pa}$$

$p < 0 \rightarrow y > 0$



$$I_z = 2\pi i \left.\frac{e^{-ipz}}{(z+ia)}\right|_{z=ia} = \frac{\pi}{a}e^{pa}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-ipx}}{(x^2+a^2)}\,dx = \frac{\pi}{a}e^{-|p|a}$$