Problems

1. a. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \qquad 0 < y < b,$$
$$u(x, 0) = 0, \quad u(x, b) = g(x), \qquad 0 < x < a.$$

b. Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

G c. For $a = 3$ and $b = 1$, plot u versus x for several values of y and also plot u versus y for several values of x. (Use enough terms in the Fourier series to accurately approximate the nonhomogeneous boundary condition.)

G d. Plot u versus both x and y in three dimensions. Also draw a contour plot showing several level curves of $u(x, y)$ in the xy-plane.

2. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b,$$
$$u(x, 0) = h(x), \quad u(x, b) = 0, \quad 0 < x < a.$$

3. a. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = f(y), \quad 0 < y < b,$$
$$u(x, 0) = h(x), \quad u(x, b) = 0, \quad 0 < x < a.$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a, y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x, 0) = h(x)$.

b. Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - y/b$.

G c. Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x (for a uniform sample of y values), u versus y (for a uniform sample of x values), u versus both x and y, and a contour plot.

4. Show how to find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = k(y), \quad u(a, y) = f(y), \quad 0 < y < b,$$
$$u(x, 0) = h(x), \quad u(x, b) = g(x), \quad 0 < x < a.$$

Hint: See Problem 3.

5. Find the solution $u(r, \theta)$ of Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r > a, \quad 0 < \theta < 2\pi,$$

outside the circle $r = a$, that satisfies the boundary condition

$$u(a, \theta) = f(\theta), \quad 0 \le \theta < 2\pi,$$

on the circle. Assume that $u(r, \theta)$ is single-valued and bounded for $r > a$.

6. a. Find the solution $u(r, \theta)$ of Laplace's equation in the semicircular region $r < a$, $0 < \theta < \pi$, that satisfies the boundary conditions

$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad 0 < r < a,$$
$$u(a, \theta) = f(\theta), \qquad 0 < \theta < \pi.$$

Assume that u is single-valued and bounded in the given region.

b. Find the solution if $f(\theta) = \theta(\pi - \theta)$.

G c. Let $a = 2$ and plot the solution in several ways: u versus r, u versus θ, u versus both r and θ, and a contour plot.

7. Find the solution $u(r, \theta)$ of Laplace's equation in the circular sector $0 < r < a$, $0 < \theta < \alpha$, that satisfies the boundary conditions

$$u(r, 0) = 0, \quad u(r, \alpha) = 0, \quad 0 < r < a,$$
$$u(a, \theta) = f(\theta), \qquad 0 < \theta < \alpha.$$

Assume that u is single-valued and bounded in the sector and that $0 < \alpha < 2\pi$.

8. a. Find the solution $u(x, y)$ of Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \quad y > 0,$$
$$u(x, 0) = f(x), \quad 0 < x < a$$

and the additional condition that $u(x, y) \to 0$ as $y \to \infty$.

b. Find the solution if $f(x) = x(a - x)$.

N c. Let $a = 5$. Find the smallest value of y_0 for which $u(x, y) \le 0.1$ for all $y \ge y_0$.

9. Show that equation (24) has periodic solutions only if λ is real. *Hint:* Let $\lambda = -\mu^2$, where $\mu = \nu + i\sigma$ with ν and σ real.

10. Consider the problem of finding a solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0, y) = 0, \quad u_x(a, y) = f(y), \quad 0 < y < b,$$
$$u_y(x, 0) = 0, \quad u_y(x, b) = 0, \qquad 0 < x < a.$$

This is an example of a Neumann problem.

a. Show that Laplace's equation and the homogeneous boundary conditions determine the fundamental set of solutions

$$u_0(x, y) = c_0,$$
$$u_n(x, y) = c_n \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right), \quad n = 1, 2, 3, \ldots.$$

b. By superposing the fundamental solutions of part (a), formally determine a function u satisfying the nonhomogeneous boundary condition $u_x(a, y) = f(y)$. Note that when $u_x(a, y)$ is calculated, the constant term in $u(x, y)$ is eliminated, and there is no condition from which to determine c_0. Furthermore, it must be possible to express f by means of a Fourier cosine series of period $2b$, which does not have a constant term. This means that

$$\int_0^b f(y)dy = 0$$

is a necessary condition for the given problem to be solvable. Finally, note that c_0 remains arbitrary, and hence the solution is determined only up to this additive constant. This is a property of all Neumann problems.

11. Find a solution $u(r, \theta)$ of Laplace's equation inside the circle $r = a$ that satisfies the boundary condition on the circle

$$u_r(a, \theta) = g(\theta), \quad 0 < \theta < 2\pi.$$

Note that this is a Neumann problem and that its solution is determined only up to an arbitrary additive constant. State a necessary condition on $g(\theta)$ for this problem to be solvable by the method of separation of variables (see Problem 10).

12. a. Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \qquad 0 < y < b,$$
$$u_y(x, 0) = 0, \quad u(x, b) = g(x), \quad 0 < x < a.$$

Note that this is neither a Dirichlet nor a Neumann problem, but a mixed problem in which u is prescribed on part of the boundary and its normal derivative on the rest.

b. Find the solution if

$$g(x) = \begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

G c. Let $a = 3$ and $b = 1$. By drawing suitable plots, compare this solution with the solution of Problem 1.

13. **a.** Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = f(y), \quad 0 < y < b,$$
$$u(x, 0) = 0, \quad u_y(x, b) = 0, \quad 0 < x < a.$$

Hint: Eventually, it will be necessary to expand $f(y)$ in a series that makes use of the functions $\sin(\pi y/2b)$, $\sin(3\pi y/2b)$, $\sin(5\pi y/2b)$, ... (see Problem 39 of Section 10.4).

b. Find the solution if $f(y) = y(2b - y)$.

G **c.** Let $a = 3$ and $b = 2$; plot several different views of the solution.

14. **a.** Find the solution $u(x, y)$ of Laplace's equation in the rectangle $0 < x < a$, $0 < y < b$, that satisfies the boundary conditions

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad 0 < y < b,$$
$$u(x, 0) = 0, \quad u(x, b) = g(x), \quad 0 < x < a.$$

b. Find the solution if $g(x) = 1 + x^2(x - a)^2$.

G **c.** Let $a = 3$ and $b = 2$; plot several different views of the solution.

15. Show that Laplace's equation in polar coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Hint: Use $x = r\cos\theta$ and $y = r\sin\theta$ and the chain rule.

16. Show that Laplace's equation in cylindrical coordinates is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0.$$

Hint: Use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and the chain rule.

17. Show that Laplace's equation in spherical coordinates is

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta} + \frac{\cot\phi}{r^2}u_\phi = 0.$$

Hint: Use $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\theta$, and the chain rule.

18. **a.** Laplace's equation in cylindrical coordinates was found in Problem 15. Show that axially symmetric solutions (i.e., solutions that do not depend on θ) satisfy

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0.$$

b. Assuming that $u(r, z) = R(r)Z(z)$, show that R and Z satisfy the equations

$$rR'' + R' + \lambda^2 rR = 0, \quad Z'' - \lambda^2 Z = 0.$$

Note: The equation for R is Bessel's equation of order zero with independent variable λr.

19. **Flow in an Aquifer.** Consider the flow of water in a porous medium, such as sand, in an aquifer. The flow is driven by the hydraulic head, a measure of the potential energy of the water above the aquifer. Let $R : 0 < x < a, 0 < z < b$ be a vertical section of an aquifer. In a uniform, homogeneous medium, the hydraulic head $u(x, z)$ satisfies Laplace's equation

$$u_{xx} + u_{zz} = 0 \quad \text{in } R. \tag{39}$$

If water cannot flow through the sides and bottom of R, then the boundary conditions there are

$$u_x(0, z) = 0, \quad u_x(a, z) = 0, \quad 0 < z < b \tag{40}$$
$$u_z(x, 0) = 0, \quad 0 < x < a. \tag{41}$$

Finally, suppose that the boundary condition at $z = b$ is

$$u(x, b) = b + \alpha x, \quad 0 < x < a, \tag{42}$$

where α is the slope of the water table.

a. Solve the given boundary value problem for $u(x, z)$.

G **b.** Let $a = 1000$, $b = 500$, and $\alpha = 0.1$. Draw a contour plot of the solution in R; that is, plot some level curves of $u(x, z)$.

G **c.** Water flows along paths in R that are orthogonal to the level curves of $u(x, z)$ in the direction of decreasing u. Plot some of the flow paths.