Lista 12: Equação de difusão do calor (extraídos do livro de Boyce e DiPrima)

Problems

In each of Problems 1 through 6, determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

1. $xu_{xx} + u_t = 0$

2. $tu_{xx} + xu_t = 0$

3. $u_{xx} + u_{xt} + u_t = 0$

4. $(p(x)u_x)_x - r(x)u_{tt} = 0$

5. $u_{xx} + (x + y)u_{yy} = 0$

6. $u_{xx} + u_{yy} + xu = 0$

7. Find the solution of the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, & 0 < x < 1, \quad t > 0; \\ u(0, t) &= 0, & u(1, t) = 0, \quad t > 0; \\ u(x, 0) &= \sin(2\pi x) - \sin(5\pi x), & 0 \le x \le 1. \end{aligned}$$

8. Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, \quad 0 < x < 2, \quad t > 0; \\ u(0, t) &= 0, \quad u(2, t) = 0, \quad t > 0; \\ u(x, 0) &= 2\sin(\pi x/2) - \sin(\pi x) + 4\sin(2\pi x), \quad 0 \le x \le 2. \end{aligned}$$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. In each of Problems 9 through 12, find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

9. $u(x, 0) = 50, \quad 0 < x < 40$

10. $u(x, 0) = \begin{cases} x, & 0 \le x < 20, \\ 40 - x, & 20 \le x \le 40 \end{cases}$

11. $u(x, 0) = \begin{cases} 0, & 0 \le x < 10, \\ 50, & 10 \le x \le 30, \\ 0, & 30 < x \le 40 \end{cases}$

12. $u(x, 0) = x, \quad 0 \le x < 40$

24. In solving differential equations, the computations can almost always be simplified by the use of **dimensionless variables**.

a. Show that if the dimensionless variable $\xi = x/L$ is introduced, the heat conduction equation becomes

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{L^2}{\alpha^2}\frac{\partial u}{\partial t}, \quad 0 < \xi < 1, \quad t > 0.$$

b. Since L^2/α^2 has the units of time, it is convenient to use this quantity to define a dimensionless time variable $\tau = (\alpha^2/L^2)t$. Then show that the heat conduction equation reduces to

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial \tau}, \quad 0 < \xi < 1, \quad \tau > 0.$$

25. Consider the equation

$$au_{xx} - bu_t + cu = 0, \tag{25}$$

where a, b, and c are constants.

a. Let $u(x, t) = e^{\delta t}w(x, t)$, where δ is constant, and find the corresponding partial differential equation for w.

b. If $b \neq 0$, show that δ can be chosen so that the partial differential equation found in part a has no term in w. Thus, by a change of dependent variable, it is possible to reduce equation (25) to the heat conduction equation.

26. The heat conduction equation in two space dimensions is

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, find ordinary differential equations that are satisfied by $X(x)$, $Y(y)$, and $T(t)$.

27. The heat conduction equation in two space dimensions may be expressed in terms of polar coordinates as

$$\alpha^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_t.$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations that are satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.