Lista 13: Equação de difusão do calor 2 (extraídos do livro de Boyce e DiPrima)

In each of Problems 1 through 8, find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

1. $u(0, t) = 10, \quad u(50, t) = 40$

2. $u(0, t) = 30, \quad u(40, t) = -20$

3. $u_x(0, t) = 0, \quad u(L, t) = 0$

4. $u_x(0, t) = 0, \quad u(L, t) = T$

5. $u(0, t) = 0, \quad u_x(L, t) = 0$

6. $u(0, t) = T, \quad u_x(L, t) = 0$

7. $u_x(0, t) - u(0, t) = 0, \quad u(L, t) = T$

8. $u(0, t) = T, \quad u_x(L, t) + u(L, t) = 0$

9. Let an aluminum rod of length 20 cm be initially at the uniform temperature of 25°C. Suppose that at time $t = 0$, the end $x = 0$ is cooled to 0°C while the end $x = 20$ is heated to 60°C, and both are thereafter maintained at those temperatures.

15. Consider a uniform bar of length L having an initial temperature distribution given by $f(x)$, $0 \le x \le L$. Assume that the temperature at the end $x = 0$ is held at 0°C, while the end $x = L$ is insulated so that no heat passes through it.

a. Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n(x, t) = e^{-(2n-1)^2\pi^2\alpha^2 t/4L^2} \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, 3, \ldots .$$

b. Find a formal series expansion for the temperature $u(x, t)$

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t)$$

that also satisfies the initial condition $u(x, 0) = f(x)$.

Hint: Even though the fundamental solutions involve only the odd sines, it is still possible to represent f by a Fourier series involving only these functions. See Problem 39 of Section 10.4.

20. Consider the problem

$$\begin{aligned} &\alpha^2 u_{xx} = u_t, && 0 < x < L, t > 0; \\ &u(0, t) = 0, \quad u_x(L, t) + \gamma u(L, t) = 0, && t > 0; \\ &u(x, 0) = f(x), && 0 \le x \le L. \end{aligned}$$

a. Let $u(x, t) = X(x)T(t)$, and show that

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(L) + \gamma X(L) = 0, \quad (48)$$

and

$$T' + \lambda\alpha^2 T = 0,$$

where λ is the separation constant.

b. Assume that λ is real, and show that problem (48) has no nontrivial solutions if $\lambda \le 0$.

c. If $\lambda > 0$, let $\lambda = \mu^2$ with $\mu > 0$. Show that problem (48) has nontrivial solutions only if μ is a solution of the equation

$$\mu \cos(\mu L) + \gamma \sin(\mu L) = 0. \quad (49)$$

d. Rewrite equation (49) as $\tan(\mu L) = -\mu/\gamma$. Then, by drawing the graphs of $y = \tan(\mu L)$ and $y = -\mu/\gamma$ for $\mu > 0$ on the same set of axes, show that equation (49) is satisfied by infinitely many positive values of μ; denote these by $\mu_1, \mu_2, \ldots, \mu_n, \ldots$, ordered in increasing size.

e. Determine the set of fundamental solutions $u_n(x, t)$ corresponding to the values μ_n found in part d.

An External Heat Source. Consider the heat conduction problem in a bar that is in thermal contact with an external heat source or sink. Then the modified heat conduction equation is

$$u_t = \alpha^2 u_{xx} + s(x), \quad (50)$$

where the term $s(x)$ describes the effect of the external agency; $s(x)$ is positive for a source and negative for a sink. Suppose that the boundary conditions are

$$u(0, t) = T_1, \quad u(L, t) = T_2 \quad (51)$$

and the initial condition is

$$u(x, 0) = f(x). \quad (52)$$

Problems 21 through 23 deal with this kind of problem.

21. Write $u(x, t) = v(x) + w(x, t)$, where v and w are the steady-state and transient parts of the solution, respectively. State the boundary value problems that $v(x)$ and $w(x, t)$, respectively, satisfy. Observe that the problem for w is the fundamental heat conduction problem discussed in Section 10.5, with a modified initial temperature distribution.