$u : \overline{B(x_0,R)} \to \mathbb{R}$ contínua $u|_{B(x_0,R)}$ é harmônica $\qquad c_n, d_n$

Se $|x - x_0| = r \leq R$, então

$$\frac{1-\frac{r}{R}}{\left[1+\left(\frac{r}{R}\right)\right]^{n-1}} u(x_0) \leq u(x) \leq \frac{1+\frac{r}{R}}{\left[1-\left(\frac{r}{R}\right)\right]^{n-1}} u(x_0)$$

$c_n\, u(x_0) \leq u(x) \leq d_n\, u(x_0)$

$\sup u(x) \leq d_n\, u(x_0)$

$\inf u(x) \geq c_n\, u(x_0)$

$; \sup u(x) \leq d_n\, u(x_0) = \frac{d_n}{c_n} c_n\, u(x_0)$

$\leq \frac{d_n}{c_n} \inf u(x)$

$$\boxed{\sup_{B(x_0,r)} u(x) \leq C_n \inf_{B(x_0,r)} u(x)}$$

Desigualdade de Harnack para $B(x_0,r)$

Resolução:

$\partial B(x_0,R) \to u|_{\partial B}$

Sabemos que

$$\begin{cases} u(x) = \displaystyle\int_{\partial B} \frac{R^2 - |x-x_0|^2}{|\partial B(0,1)|\, R\, |x-y|^n} u(y)\, dS(y) = \int_{\partial B} \frac{R^2 - r^2}{|\partial B(0,1)|\, R\, |x-y|^n} u(y)\, dS(y) \\ u(x_0) = \displaystyle\int_{\partial B} \frac{1}{|\partial B(0,1)|\, R^{n-1}} u(y)\, dS(y) \quad (*) \end{cases}$$

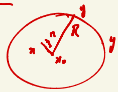
Em particular, se $x = x_0$

"óbvia" $u(x_0) = \fint_{\partial B} u\, dS(y)$.

$y \in \partial B(x_0,R) \quad |x_0 - y| = R$,

$r = |x_0 - x|$ não usamos

Precisamos estimar $|x-y|$



$$\begin{cases} |x-y| = |x - x_0 + x_0 - y| \leq |x - x_0| + |x_0 - y| = r + R, \\ |x-y| \geq |x_0 - y| - |x - x_0| \quad \left(|x_0 - y| = |x_0 - x + x - y| \leq |x_0 - x| + |x-y| \right) \\ \quad = R - r \end{cases}$$

$$\frac{R^2 - r^2}{R\,|x-y|^n} \leq \frac{R^2 - r^2}{R\,(R-r)^n} = \frac{R^2 - r^2}{R\,R^n\left(1-\frac{r}{R}\right)^n} = \frac{R^2 - r^2}{R^{n+1}\left(1-\frac{r}{R}\right)^n} = \frac{R^2\left(1-\left(\frac{r}{R}\right)^2\right)}{R^{n+1}\left(1-\frac{r}{R}\right)^n}$$

$$= \frac{1}{R^{n-1}} \frac{\left(1+\frac{r}{R}\right)\left(1-\frac{r}{R}\right)}{\left(1-\frac{r}{R}\right)^n} = \frac{1}{R^{n-1}} \frac{\left(1+\frac{r}{R}\right)}{\left(1-\frac{r}{R}\right)^{n-1}}$$

Conclusão:

$$u(x) = \int_{\partial B} \frac{R^2 - r^2}{|\partial B(0,1)|\, R\, |x-y|^n} u(y)\, dS(y) \overset{\overset{u \geq 0}{\uparrow}}{\leq} \frac{\left(1+\frac{r}{R}\right)}{\left(1-\frac{r}{R}\right)^{n-1}} \underbrace{\int_{\partial B} \frac{1}{R^{n-1}\, |\partial B(0,1)|} u(y)\, dS(y)}_{u(x_0)} \; (*) = \frac{\left(1+\frac{r}{R}\right)}{\left(1-\frac{r}{R}\right)^{n-1}} u(x_0)$$

$$\frac{R^2 - r^2}{R\,|x-y|^n} \geq \frac{R^2 - r^2}{R\,(r+R)^n} = \frac{R^2\left(1-\left(\frac{r}{R}\right)^2\right)}{R^{n+1}\left(1+\frac{r}{R}\right)^n} = \frac{R^2\left(1-\frac{r}{R}\right)\cancel{\left(1+\frac{r}{R}\right)}}{R^{n+1}\left(1+\frac{r}{R}\right)^n} = \frac{1}{R^{n-1}} \frac{\left(1-\frac{r}{R}\right)}{\left(1+\frac{r}{R}\right)^{n-1}}$$

Conclusão:

$$u(x) = \int_{\partial B} \frac{R^2 - r^2}{|\partial B(0,1)|\, R\, |x-y|^n} u(y)\, dS(y) \overset{\overset{u \geq 0}{\uparrow}}{\geq} \frac{1-\frac{r}{R}}{\left(1+\frac{r}{R}\right)^{n-1}} \underbrace{\int \frac{1}{R^{n-1}\, |\partial B(0,1)|} u(y)\, dS(y)}_{u(x_0)}$$

$\boxed{R}$

CONCLUSÃO: Se $u: \mathbb{R}^n \to \mathbb{R}$ e $u \geq 0$.

$$\cancel{\frac{1-\frac{r}{R}}{\left(1+\frac{r}{R}\right)^{n-1}}} u(x_0) \leq u(x) \leq \cancel{\frac{1+\frac{r}{R}}{1-\left(\frac{r}{R}\right)^{n-1}}} u(x_0)$$

$\to 1$ $\quad$ $\to 1$

TOMANDO $R \to \infty$

$u(x) \leq u(x_0)$
$u(x) \geq u(x_0)$ $\Rightarrow u(x_0) = u(x)$ $\forall x$.

$\Longrightarrow u = ct.$

EX. 48. $u \in C^2(U)$, $U \subset \mathbb{R}^n$ um aberto. E $x \in U$. Mostre que

Mostre que: $\Delta u(x) = \lim_{r \to 0} \frac{2n}{r^2} \Bigg[\underbrace{\frac{1}{|S^{n-1}|} \int_{S^{n-1}} u(x+ry)\, dS(y)}_{\underbrace{\frac{1}{r^{n-1}|S^{n-1}|}\int_{S^{n-1}} u(x+ry)\, r^{n-1} dS(y)}_{\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(z)\, dS(z) = ⨍_{\partial B(x,r)} u\, dS(z)}} - u(x) \Bigg]$

$$\frac{2n}{r^2}\left[⨍_{\partial B(x,r)} u\, dS(z) - u(x) \right] \xrightarrow{r \to 0} \Delta u(x)$$

SOLUÇÃO: $u(x+ry) = f(r)$ $\quad f(r) = f(0) + r f'(0) +$

FORMULA DE TAYLOR: $\boxed{f(x+y) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} (\partial^\alpha f)(x)\, y^\alpha + N \sum_{|\gamma| = N} \frac{y^\gamma}{\gamma!} \int_0^1 (1-\theta)^{N-1} (\partial^\gamma f)(x+\theta y)\, d\theta}$

$$f(r) = f(0) + f'(0) r + 2\frac{r^2}{2!} \underbrace{\int_0^1 (1-\theta) f''(\theta r)\, d\theta}$$

$$u(x+ry) = u(x) + \nabla u(x) \cdot y\, r + r^2 \int_0^1 (1-\theta) f''(\theta r)\, d\theta$$

$$u(x+ry)$$

$$\frac{d}{dr}u(x+ry) = \sum_j \frac{\partial u}{\partial x_j}(x+ry)\,y_j$$

$$\frac{d^2}{dr^2}u(x+ry) = \frac{d}{dr}\left(\frac{\partial u}{\partial x_j}(x+ry)\,y_j\right) = \sum_{i=1}\sum_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x+ry)\,y_i\,y_j \quad //$$

$$\frac{2n}{r^2}\Bigg[\underbrace{\frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(x)}_{u(x)} + \sum_{j=1}^{n}\frac{\partial u}{\partial x_j}(x)\,y_j + r^2\int_0^1 (1-\theta)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}(x+\theta r y)\,y_i\,y_j\,d\theta\,dS(y) - \underbrace{u(x)}_{-u(x)}\Bigg]$$

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(x)\,dS(y) = \cancel{|S^{n-1}|}$$

$$= u(x)\,\frac{1}{\cancel{|S^{n-1}|}}\int_{S^{n-1}} dS(y)$$

$$= u(x)$$

$$\int_{S^{n-1}}\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}(x)\,y_j\,dS(y) = \sum_{j=1}^{n}\frac{\partial u}{\partial x_j}(x)\,\cancel{\int_{S^{n-1}} y_j\,dS(y)} \to 0$$

$$\boxed{\underbrace{\int_{S^{n-1}\cap\{y_j>0\}} y_j\,dS(y) + \int_{S^{n-1}\cap\{y_j<0\}} y_j\,dS(y)}_{S^{n-1}} = 0}$$

$= -$

y_j é FUNÇÃO ÍMPAR

S^n é

$n=3$

$y_1 = \cos\theta \sin\varphi$

$y_2 = \sin\theta \sin\varphi$

$y_3 = \cos\varphi$

$$\int y_1\,dS(y) = \int y_2\,dS(y) = \int y_3\,dS(y)$$

$$\int \cos\theta \sin\varphi \sin^2\varphi\,d\theta\,d\varphi = 0, \quad \int_0^{2\pi}\cos\theta = 0$$

$$\int_0^{2\pi}\sin\theta = 0$$

$$\int_0^{\pi}\cos\varphi\,\cancel{\sin^2\varphi}\,d\varphi = 0 \quad x = \sin\varphi,\ dx = \cos\varphi\,d\varphi$$

$$\frac{2n}{\cancel{r^2}} \frac{1}{|S^{n-1}|} \underbrace{\int_{S^{n-1}} \int_0^1}_{} \cancel{r^2}(1-\theta) \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x+r\theta)\, y_i y_j}_{S^{n-1}\times[0,1]} \, d\theta\, dS(y)$$

CONTÍNUA

$$\lim_{r\to 0} = \frac{2n}{|S^{n-1}|} \int_{S^{n-1}} \int_0^1 (1-\theta) \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x)\, y_i y_j \, d\theta\, dS(y)$$

PODEMOS IGNORAR $i \neq j$.

$$\int_0^1 (1-\theta)\, d\theta \overset{!}{=} \int_0^1 \psi\, d\psi = \frac{\psi^2}{2}\Big|_0^1 = \frac{1}{2}$$

$\psi = 1-\theta$

$d\psi = d\theta$

$$\frac{n}{|S^{n-1}|} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x) \underbrace{\int_{S^{n-1}} y_j^2\, dS(y)}_{=\frac{1}{n}|S^{n-1}|}$$

$$= \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x) = \Delta u(x).$$

$$= \frac{n}{|S^{n-1}|} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \int_{S^{n-1}} y_i y_j\, dS(y).$$

PODE FAZER EXPLICITAMENTE PARA $n=3$.

Se $i \neq j$

$$\int_{S^{n-1}} y_i y_j = \int_{S^{n-1}\cap\{y_i>0,\, y_j>0\}} y_i y_j + \int_{S^{n-1}\cap\{y_i<0,\, y_j>0\}} y_i y_j + \int_{S^{n-1}\cap\{y_i<0,\, y_j>0\}} y_i y_j + \int_{S^{n-1}\cap\{y_i<0,\, y_j<0\}} y_j y_i = 0$$

(>0, <0, <0, >0)

Se $i = j$

$$\int_{S^{n-1}} y_i^2 = \int y_j^2\, dS(y)$$

$$\int y_1^2\, dS(y) + \dots + \int y_n^2\, dS(y)$$

Logo

$$\int_{S^{n-1}} y_i^2 = \frac{1}{n}\, n \int_{S^{n-1}} y_i^2 = \frac{1}{n} \sum_{j=1}^{n} \int_{S^{n-1}} y_j^2\, dS(y) = \frac{1}{n} \int_{S^{n-1}} \underbrace{\sum_{j=1}^{n} y_j^2}_{|y|^2=1}\, dS(y)$$

$$= \frac{1}{n} \int_{S^{n-1}} dS(y) = \frac{1}{n}|S^{n-1}|.$$

$$\Delta u(x) = \lim \frac{2n}{r^2}\left[\frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(x+ry)\,dS(y) - u(x)\right]$$

Se $u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y)$, então $\Delta u(x) = 0$.

Solução: $\frac{1}{|S^{n-1}|}\int u(x+ry)\,dS(y) = \frac{1}{r^{n-1}|S^{n-1}|}\int_{\partial B(0,1)} u(x+ry)\,r^{n-1}\,dS(y)$

$$= \frac{1}{\partial B(0,r)}\int_{\partial B(x,r)} u(z)\,dS(z) = \fint_{\partial B(x,r)} u\,dS(z)$$

Logo

$$\Delta u(x) = \lim_{r\to 0}\frac{2n}{r^2}\underbrace{\left[\underbrace{\fint_{\partial B(x,r)} u(z)\,dS(z)}_{=u(x)} - u(x)\right]}_{=0}$$

$$\Rightarrow \Delta u(x) = 0.$$

SEJA $\phi : \mathbb{R}^3 \to \mathbb{R}$ DE CLASSE C^2.

$\phi(x) = 0$, $|x| \geq R$. $(\phi \in C_c^2(\mathbb{R}^3))$.

MOSTRE QUE

$$\phi(0) = -\frac{1}{4\pi} \int_U \frac{1}{|x|} \Delta\phi(x)\,dx.$$

U É ABERTO C^2 QUE CONTÉM $\overline{B(0,R)}$.

SOLUÇÃO:

FORMA SIMPLES.

$\Phi(x)$ É SOLUÇÃO FUNDAMENTAL DE $-\Delta$.

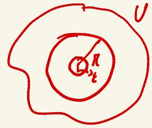
LOGO $-\int \Phi(x) \Delta\phi(x)\,dx = \phi(0)$

MAS $\Phi(x) = \frac{1}{n(n-2)|B(0,1)|} \frac{1}{|x|^{n-2}}$

$n=3$ $\quad \frac{1}{3(3-2)\frac{4}{3}\pi} \frac{1}{|x|^{3-2}} = \frac{1}{4\pi} \frac{1}{|x|}$

$$\Rightarrow \underbrace{\int_{\mathbb{R}^n} \frac{1}{4\pi} \frac{1}{|x|} \Delta\phi(x)\,dx}_{= \int_U \frac{1}{4\pi} \frac{1}{|x|} \Delta\phi(x)\,dx} = -\phi(0)$$

Outra forma:



2ª identidade de Green.

$$\int_V (\Delta f\, g - f\,\Delta g)\,dx = \int_{\partial V}\left(\frac{\partial f}{\partial \nu} g - f\frac{\partial g}{\partial \nu}\right) dS(y)$$

Seja $V = U \setminus B(0,\varepsilon)$. $f=\phi$ e $g = \frac{1}{|x|}$.

Logo

$$\int_{U\setminus B(0,\varepsilon)} \left(\Delta\phi \frac{1}{|x|} - \phi\, \cancel{\Delta \frac{1}{|x|}}^{=0}\right)dx = \int_{\partial U\cup \partial B(0,\varepsilon)} \left(\frac{\partial\phi}{\partial\nu}\frac{1}{|x|} - \phi\frac{\partial}{\partial\nu}\frac{1}{|x|}\right)dS(y)$$

$\Delta\left(\frac{1}{|x|}\right) = 0$ $\quad \partial_{x_i}\left(\frac{1}{|x|}\right) = -\frac{x_i}{|x|^3}$ $\quad \partial^2_{x_i}\left(\frac{1}{|x|}\right) = -\frac{1}{|x|^3} + \frac{3x_i^2}{|x|^5}$

$(x_1^2 + \dots + x_3^2)^{-1/2}$ $\quad |x|^{-3/2}$

$-\frac{1}{2}(\quad)^{-3/2}\, 2x_i$ $\quad -\frac{3}{2}\partial_{x_i}|x|^{-3/2}$

$\Delta\left(\frac{1}{|x|}\right) = -\frac{3}{|x|^3} + 3\frac{|x|^2}{|x|^5} = 0.$

$$\int_{U\setminus B(0,\varepsilon)} \Delta\phi \frac{1}{|x|}\,dx \xrightarrow{\varepsilon\to 0} \int_U \Delta\phi \frac{1}{|x|}\,dx$$

$$\int_{U\setminus B(0,\varepsilon)} \Delta\phi \frac{1}{|x|}\,dx = \underbrace{\int_{\partial U}\left(\frac{\partial\phi}{\partial\nu}\frac{1}{|x|} - \phi\frac{\partial}{\partial\nu}\frac{1}{|x|}\right)dS(y)}_{=0} + \int_{\partial B(0,\varepsilon)}\left(\frac{\partial\phi}{\partial\nu}\frac{1}{|x|} - \phi\frac{\partial}{\partial\nu}\frac{1}{|x|}\right)dS(y) \qquad (*)$$

$\phi|_{\partial U} = 0.$

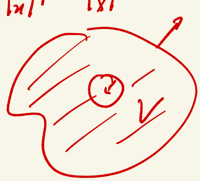
$$\left|\int_{\partial B(0,\varepsilon)} \frac{\partial\phi}{\partial\nu}\frac{1}{|x|}\,dS(x)\right| \le \|\nabla\phi\|_{L^\infty}\int_{\partial B(0,\varepsilon)} \frac{1}{|y|}\,dS(y) = \|\nabla\phi\|_{L^\infty}\,\frac{1}{\varepsilon}\,\varepsilon^2|S^{n-1}| \xrightarrow[\varepsilon\to 0]{} 0$$

$\frac{1}{|y|} = \frac{1}{\varepsilon}$, $\quad |y| = \varepsilon$, $\quad dS(y) = \varepsilon^2\, d\theta\, d\psi\, \sin^2\psi$

$$-\int_{\partial B(0,\varepsilon)} \phi\frac{\partial}{\partial\nu}\left(\frac{1}{|x|}\right)dS(x) = -\int_{\partial B(0,\varepsilon)} \phi(x)\frac{1}{|x|^2}\,dS(x) = -\int_0^{2\pi}\int_0^{\pi} \phi(\varepsilon\ldots)\frac{1}{\varepsilon^2}\,\varepsilon^2 \sin^2\psi\, d\theta\, d\psi \xrightarrow[\varepsilon\to 0]{} -\phi(0)4\pi$$

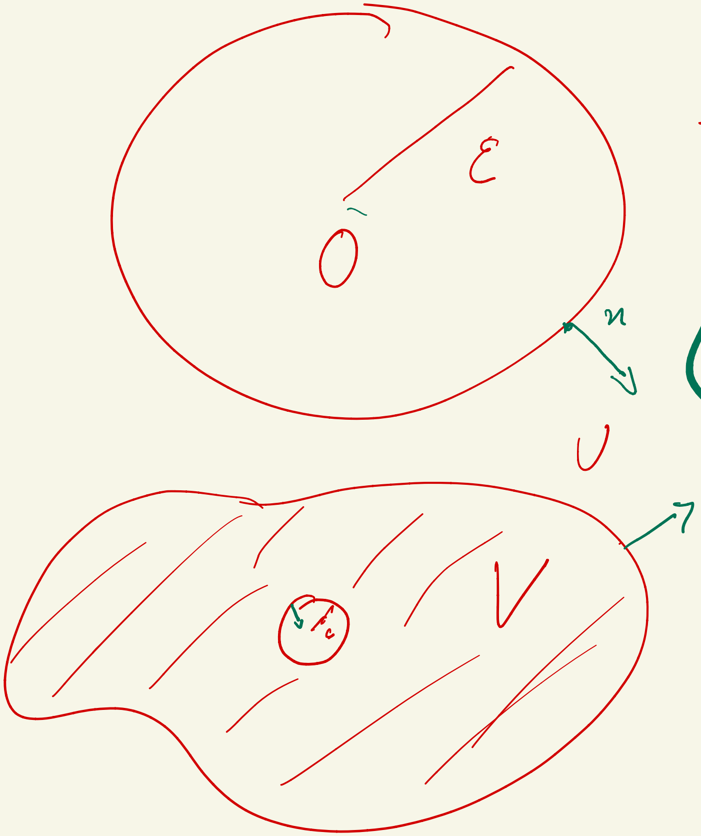
$$\int_0^{2\pi}\left(\int_0^{\pi}\sin^2\psi\, d\psi\right)d\theta = 4\pi$$

$$\left.\begin{aligned} &\partial_{x_i}\frac{1}{|x|} = -\frac{x_i}{|x|^3} \Rightarrow \nabla\frac{1}{|x|} = -\frac{x}{|x|^3} \\ &\nu = -\frac{x}{|x|} \end{aligned}\right\} \quad \frac{\partial}{\partial\nu}\left(\frac{1}{|x|}\right) = \frac{|x|^2}{|x|^4} = \frac{1}{|x|^2}$$



Tomando $\varepsilon \to 0$ na expressão (*), concluímos que

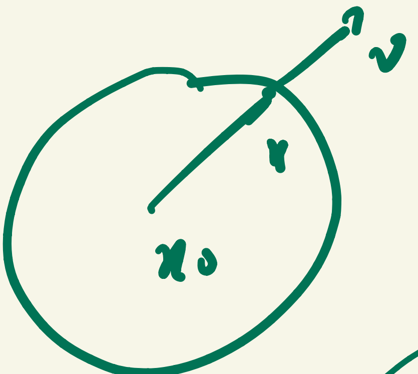
$$\int_U \Delta\phi(x)\frac{1}{|x|}\,dx = -\phi(0)4\pi \Longrightarrow \int_U \Delta\phi(x)\frac{1}{4\pi|x|}\,dx = -\phi(0)$$



$$\frac{\partial f}{\partial \nu} = \nabla f \cdot \nu$$

$$\nu = -\frac{x}{|x|}$$

$\frac{x}{|x|}$ → ESTA FÓRMULA SÓ VALE PARA $\partial B(0,\varepsilon)$. BOLAS CENTRADAS EM 0.



$$\nu = \frac{x - x_0}{|x - x_0|}$$

$g : U \subset \mathbb{R}^n \to \mathbb{R}$.

VALOR REGULAR:

SE $g(x) = c$, ENTÃO $\nabla g(x) \neq 0$.

$$\boxed{\begin{gathered} S = g^{-1}(c) \\ \nu = \frac{\nabla g(x)}{|\nabla g(x)|} \end{gathered}}$$

$S = (x_1, \dots, x_n, f(x_1, \dots, x_n))$

$$\nu = \frac{(\nabla f(x), -1)}{|(\nabla f(x), -1)|}$$

$\sigma_1 = (1, 0, \dots, 0, \frac{\partial f}{\partial x_1})$

$(0, 1, 0, \dots, 0, \frac{\partial f}{\partial x_n})$.

$x^2 + y^2 = 1$

$g(x,y) = x^2 + y^2$

$\nabla g(x,y) = 2(x,y)$

$$\nu = \frac{2(x,y)}{|2(x,y)|} = \frac{(x,y)}{|(x,y)|}.$$

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha .$$

TEOREMA: $\exists \; E \in \mathcal{D}'(\mathbb{R}^n) = \{ \Lambda : C_c^\infty(\mathbb{R}^n) \to \mathbb{R} \}$

T. Q. $\underline{P(D) \underset{\exists}{E} = \delta_0}$.

$E \in \mathcal{D}'(\mathbb{R}^n)$

$E = T_\Phi \qquad -\Delta \qquad \Phi = \frac{1}{n(n-2)|B|} \frac{1}{|x|^{n-2}}$

$\frac{\partial}{\partial t} - \Delta \qquad \Phi = H(t) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$

$\frac{\partial^2}{\partial t^2} - \Delta \qquad E$ NÃO É FUNÇÃO
$n > 2$

$$\Rightarrow \quad E\Big(\underbrace{\sum_{|\alpha| \leq m} a_\alpha (-1)^{|\alpha|} \partial^\alpha \phi}\Big) = \phi(0)$$

$$\sum_{|\alpha| \leq m} a_\alpha (-1)^{|\alpha|} E(\partial^\alpha \phi) = \phi(0)$$

$$\delta_0(\phi) = \phi(0) \qquad , \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

$$(P(D)E)(\phi) = E(P^T(D)\phi)$$

~~n>2~~ $\underline{n=1}$

$n=2. \quad E(\phi) \overset{?}{=} \int_{S^{n-1}} \phi \, dS(y) \neq \int_{\mathbb{R}^n} f(x)\phi(x)\,dx$

$$E(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx$$

CALOR
LAPLACE
ONDA $n>2$ X

$\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$