The SST Model

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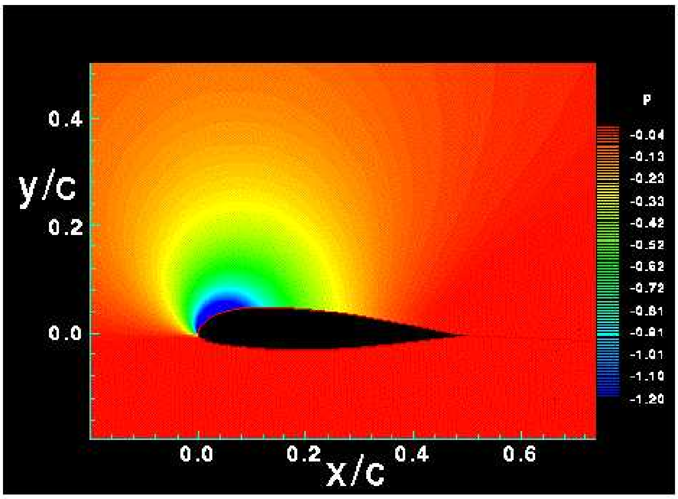
The SST (Shear Stress Transport) model of Menter (1994) is an eddy-viscosity model which includes two main novelties:

1. It is combination of a $k - \omega$ model (in the inner boundary layer) and $k - \varepsilon$ model (in the outer region of and outside of the boundary layer);
2. A limitation of the shear stress in adverse pressure gradient regions is introduced.

The $k - \varepsilon$ model has two main weaknesses: it over-predicts the shear stress in adverse pressure gradient flows because of too large length scale (due to too low dissipation) and it requires near-wall modification (i.e. low-Re number damping functions/terms)



Pressure contours. Red: high pressure; blue: low pressure

One example of adverse pressure gradient is the flow along the surface of an airfoil, see figure above. Consider the upper surface (suction side). Starting from the leading edge, the pressure decreases because the velocity increases. At the crest (at $x/c \simeq 0.15$) and further downstream, the pressure increases since the velocity decreases. This region is called the *adverse pressure gradient* (APG) region.

The $k-\omega$ model is better at predicting predicting adverse pressure gradient flow and the standard model of Wilcox (1988) does not use any damping functions. However, the disadvantage of the standard $k-\omega$ model is that it is dependent on the free-stream value of ω (Menter, 1992)

In order to improve both the $k-\varepsilon$ and the $k-\omega$ model, Menter (1994) suggested to combine the two models. Before doing this, it is convenient to transform the $k-\varepsilon$ model into a $k-\omega$ model using the relation $\omega = \varepsilon/(\beta^* k)$, where $\beta^* = c_\mu$. The transformation of the left-hand side (LHS) reads

$$\frac{D\omega}{Dt} = \frac{D\varepsilon/(\beta^* k)}{Dt} = \frac{1}{\beta^* k}\frac{D\varepsilon}{Dt} - \frac{\varepsilon}{\beta^* k^2}\frac{Dk}{Dt} = \frac{1}{\beta^* k}\frac{D\varepsilon}{Dt} - \frac{\omega}{k}\frac{Dk}{Dt} \tag{69}$$

where $D/Dt = U_j \partial/\partial x_j$ denotes the material derivative. Inserting the modelled equations for k and ε yields:

$$\begin{aligned}\frac{D\omega}{Dt} = &\underbrace{\left[\frac{1}{\beta^* k}P_\varepsilon - \frac{\omega}{k}P_k\right]}_{\text{Production, } P_\omega} - \underbrace{\left[\frac{1}{\beta^* k}\Psi_\varepsilon - \frac{\omega}{k}\Psi_k\right]}_{\text{Destruction, } \Psi_\omega} + \\ &\underbrace{\left[\frac{1}{\beta^* k}D^T_\varepsilon - \frac{\omega}{k}D^T_k\right]}_{\text{Turbulent diffusion, } D^T_\omega} + \underbrace{\left[\frac{\nu}{\beta^* k}\frac{\partial^2 \varepsilon}{\partial x_j^2} + \frac{\nu\omega}{k}\frac{\partial^2 k}{\partial x_j^2}\right]}_{\text{Viscous diffusion, } D^\nu_\omega}\end{aligned} \tag{70}$$

- Production term

$$
\begin{aligned}
P_\omega &= \frac{1}{\beta^* k} P_\varepsilon - \frac{\omega}{k} P_k = C_{\varepsilon 1} \frac{\varepsilon}{\beta^* k^2} P_k - \frac{\omega}{k} P_k = \\
&= (C_{\varepsilon 1} - 1) \frac{\omega}{k} P_k
\end{aligned}
\tag{71}
$$

- Destruction term

$$
\begin{aligned}
\Phi_\omega &= \frac{1}{\beta^* k} \Psi_\varepsilon - \frac{\omega}{k} \Psi_k = C_{\varepsilon 2} \frac{\varepsilon^2}{k} - \frac{\omega}{k} \varepsilon = \\
&= (C_{\varepsilon 2} - 1) \beta^* \omega^2
\end{aligned}
\tag{72}
$$

- Viscous diffusion term

$$
\begin{aligned}
D_\omega^\nu =& \frac{\nu}{\beta^* k} \frac{\partial^2 \varepsilon}{\partial x_j^2} - \frac{\nu \omega}{k} \frac{\partial^2 k}{\partial x_j^2} = \frac{\nu}{k} \frac{\partial^2 \omega k}{\partial x_j^2} - \frac{\nu \omega}{k} \frac{\partial^2 k}{\partial x_j^2} = \\
=& \frac{\nu}{k} \left[\frac{\partial}{\partial x_j} \left(\omega \frac{\partial k}{\partial x_j} + k \frac{\partial \omega}{\partial x_j} \right) \right] - \nu \frac{\omega}{k} \frac{\partial^2 k}{\partial x_j^2} = \\
=& \frac{\nu}{k} \left[\frac{\partial \omega}{\partial x_j} \frac{\partial k}{\partial x_j} + \omega \frac{\partial^2 k}{\partial x_j^2} + \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} + k \frac{\partial^2 \omega}{\partial x_j^2} \right] - \nu \frac{\omega}{k} \frac{\partial^2 k}{\partial x_j^2} = \\
=& \frac{2\nu}{k} \frac{\partial \omega}{\partial x_j} \frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \omega}{\partial x_j} \right)
\end{aligned}
\tag{73}
$$

The turbulent diffusion term is obtained as (the derivation can be downloaded (Bredberg, 2000))

$$
\begin{aligned}
D_\omega^T =& \frac{2\nu_t}{\sigma_\varepsilon k} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j} + \frac{\omega}{k} \left(\frac{\nu_t}{\sigma_\varepsilon} - \frac{\nu_t}{\sigma_k} \right) \frac{\partial^2 k}{\partial x_j^2} + \\
&+ \frac{\omega}{k} \left(\frac{1}{\sigma_\varepsilon} - \frac{1}{\sigma_k} \right) \frac{\partial \nu_t}{\partial x_j} \frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \omega}{\partial x_j} \right)
\end{aligned}
\tag{74}
$$

In the standard $k - \varepsilon$ model we have $\sigma_k = 1$ and $\sigma_\varepsilon = 1.3$. If we assume that $\sigma_k = \sigma_\varepsilon$ in the second and third term of the right-hand side, we can considerably simplify the turbulence diffusion so that

$$D_\omega^T = \frac{2\nu_t}{\sigma_\varepsilon k}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \omega}{\partial x_j}\right) \tag{75}$$

We can now finally write the ε equation formulated as an equation for ω

$$\begin{aligned}\frac{\partial}{\partial x_j}(\bar{U}_j\omega) = \frac{\partial}{\partial x_j}&\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right] + \alpha\frac{\omega}{k}P_k - \beta\omega^2\\ &+ \frac{2}{k}\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i}\\ &\alpha = C_{\varepsilon 1} - 1 = 0.44, \beta = (C_{\varepsilon 2} - 1)\beta^* = 0.0828\end{aligned} \tag{76}$$

Since this equation will be used for the outer part of the boundary layer, the viscous part in the last term is omitted.

In the SST model the coefficients are smoothly switched from $k-\omega$ values in the inner region of the boundary layer and $k-\varepsilon$ values in the outer region. Functions of the form

$$F_1 = \tanh(\xi^4), \quad \xi = \min\left[\max\left\{\frac{\sqrt{k}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right\}, \frac{4\sigma_{\omega 2}k}{CD_\omega y^2}\right] \tag{77}$$

are used. $F_1 = 1$ in the near-wall region and $F = 0$ in the outer region.

At p. 42 it was mentioned that the $k-\omega$ model is better than the $k-\varepsilon$ model in predicting adverse pressure-gradient flows because it predicts a smaller shear stress. Still, the predicted shear stress is too large. This brings us to the second modification (see p. 42). When introducing this second modification, Menter (1994) noted that a model (the Johnson - King model [JK]) which is based on transport of the main shear stress $\overline{uv}$, predicts adverse pressure gradient flows much better than the $k-\omega$ model. In the JK model, the $\overline{uv}$ transport equation is built on Bradshaw's assumption (Bradshaw *et al.*, 1967)

$$-\overline{uv} = a_1 k \tag{78}$$

where $a_1 = c_\mu^{1/2} = \beta^{*1/2}$. In boundary layer flow, the Boussinesq assumption can be written as

$$\begin{aligned} -\overline{uv} = \frac{k}{\omega}\frac{\partial U}{\partial y} = \frac{k}{\omega}\Omega = \frac{c_\mu k^2}{\varepsilon}\Omega = \\ c_\mu^{1/2} k \left(\frac{c_\mu k^2}{\varepsilon^2}\Omega^2\right)^{1/2} = c_\mu^{1/2} k \left(\frac{P_k}{\varepsilon}\right)^{1/2} = a_1 k \left(\frac{P_k}{\varepsilon}\right)^{1/2} \end{aligned} \tag{79}$$

where Ω is the vorticity (in boundary layer flow $\Omega = \partial U/\partial y$). It is found from experiments that in adverse pressure gradient flow the production is much larger than the dissipation ($P_k > \varepsilon$), which explains why Eq. 79 over-predicts the shear stress; the model works poorly in this type of flow. To reduce $|\overline{uv}|$ in Eq. 79 in adverse pressure gradient flow, Menter (1994) proposed to re-define the turbulent eddy viscosity as (cf. Eq. 78)

$$-\overline{uv} = \nu_t \Omega \Rightarrow -\nu_t = \frac{a_1 k}{\max(a_1\omega, F_2\Omega)} \tag{80}$$

where F_2 is a damping function (similar to F_1) which is 1 near walls and zero elsewhere. When the production is large (i.e. when Ω is large), Eq. 80 reduces ν_t. It is important to ensure that this limitation is not active in usual boundary layer flows where $P_k \simeq \varepsilon$. It can be seen that the shear stress is reduced only in regions where $P_k > \varepsilon$, because if $P_k > \varepsilon$ then $\Omega > a_1\omega$ since

$$\Omega^2 = \frac{\omega P_k}{k} > \frac{\omega\varepsilon}{k} = c_\mu \omega^2 = a_1^2 \omega^2 \tag{81}$$

In regions where $P_k \leq \varepsilon$, Eq. 80 returns to $\nu_t = k/\omega$ as it should.

Presently, the SST model has been slightly further developed. Two modifications have been introduced (Menter *et al.*, 2003*b*). The absolute vorticity Ω in Eq. 80 has been replaced by $S = 2S_{ij}S_{ij}$. This limits ν_t in stagnation regions similar to Eq. 42. The production term is in the new SST model limited by 10ε, i.e.

$$P_{k,new} = \min\left(P_k, 10\varepsilon\right) \tag{82}$$

The final form of the SST model is given in Eq. 165 at p. 95.