Gabarito 1ª Prova de Mec. Quântica

1- $\psi(x,0) = A(a^2 - x^2)\,, \quad |x| \leq a$

$$[\psi] = 1/L^{1/2} \; ; \quad 1 = \int_{-a}^{a} |\psi|^2 dx = |A|^2 \int (a^2 - x^2)^2 dx =$$

$$= |A|^2 \left\{ 2a^5 - \frac{4}{3}a^5 + \frac{2}{5}a^5 \right\} = \frac{16}{15}|A|^2 a^5 \Rightarrow A = \sqrt{\frac{15}{16}}\frac{1}{a^{5/2}}$$

$$\langle x \rangle = |A|^2 \int_{-a}^{a} x(a^2 - x^2)^2 dx = 0 \Leftarrow \text{integrando ímpar}$$

$$\langle p \rangle = |A|^2 \int_{-a}^{a} (a^2 - x^2) \frac{\hbar}{i} \frac{d}{dx}(a^2 - x^2)\, dx = 0, \text{ integrando ímpar}$$

$$\langle x^2 \rangle = |A|^2 \int_{-a}^{a} x^2 (a^2 - x^2)^2 dx = |A|^2 \int (a^4 x^2 - 2a^2 x^4 + x^6) =$$

$$= |A|^2 \left(\frac{2}{3}a^7 - \frac{4}{5}a^7 + \frac{2}{7}a^7 \right) = \frac{a^2}{7}$$

$$\langle p^2 \rangle = |A|^2 \int (a^2 - x^2) \frac{\hbar^2}{-1} \frac{d^2}{dx^2}(a^2 - x^2)\, dx =$$

$$= 2\hbar^2 |A|^2 \int (a^2 - x^2) dx = \frac{5}{2}\frac{\hbar^2}{a^2}$$

$$\Delta x \, \Delta p = \hbar \sqrt{\frac{5}{14}} \sim 0{,}5976\,\hbar > \hbar/2$$

$$\underline{P} = \int_{-a/2}^{a/2} |\psi|^2 dx = \frac{15}{16}\frac{1}{a^5} \int_{-a/2}^{a/2} (a^2 - x^2)^2 dx =$$

$$= \frac{15}{16}\frac{1}{a^5} \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{-a/2}^{a/2} = \frac{203}{256} \sim 79{,}3\,\%$$

2) $C = \int \psi_1^*(x,t)\,\psi_2(x,t)\,dx \Rightarrow \dot{C} = \int \left(\frac{\partial \psi_1^*}{\partial t}\psi_2 + \psi_1^* \frac{\partial \psi_2}{\partial t}\right)$

uso que $\partial_t \psi = -\frac{\hbar}{2mi}\frac{\partial^2 \psi}{\partial x^2} + \frac{V}{i\hbar}\psi$

$$\dot{c} = \int \left(\frac{\hbar}{2mi}\frac{\partial^2 \psi_1^*}{\partial x^2}\psi_2 - \cancel{\frac{V}{i\hbar}\psi_1^*\psi_2} - \psi_1^*\frac{\hbar}{2mi}\frac{\partial^2 \psi_2}{\partial x^2} \cancel{+\frac{V}{i\hbar}\psi_1^*\psi_2}\right)$$

$$= \frac{\hbar}{2mi}\int \left(\frac{\partial^2 \psi_1^*}{\partial x^2}\psi_2 - \psi_1^*\frac{\partial^2 \psi_2}{\partial x^2}\right)dx =$$

$$= \frac{\hbar}{2mi}\int \frac{\partial}{\partial x}\left(\frac{\partial \psi_1^*}{\partial x}\psi_2 - \psi_1^*\frac{\partial \psi_2}{\partial x}\right)dx =$$

$$= \frac{\hbar}{2mi}\left(\frac{\partial \psi_1^*}{\partial x}\psi_2 - \psi_1^*\frac{\partial \psi_2}{\partial x}\right\Big|_{-\infty}^{\infty} \rightarrow 0 \Rightarrow c \text{ cte.}$$

<u>Otra</u>

$$\frac{\dot{c}}{i\hbar} = \frac{1}{i\hbar}\int \partial_t \psi_1^*\,\psi_2 + \frac{1}{i\hbar}\int \psi_1^*\,\partial_t \psi_2\,dx =$$

uso que $i\hbar\,\partial_t \psi = H\psi$

$$= -\int (H\psi_1)^*\psi_2\,dx + \int \psi_1^* H \psi_2\,dx =$$

$$= \left(-\int \psi_2^* H \psi_1\,dx\right)^* + \int \psi_1^* H \psi_2\,dx =$$

$$= \left(-\int \psi_1^* \underset{=H}{H^+} \psi_2\,dx\right) + \int \psi_1^* H \psi_2\,dx = 0$$

3- $\phi_n(x) = \sqrt{\frac{2}{L}} \times \begin{cases} \cos\frac{n\pi}{L}x, & n = 1, 3, 5, \dots \\ \operatorname{sen}\frac{n\pi}{L}x, & n = 2, 4, \dots \end{cases}$ $E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$

$$\Psi(x,t) = \sum_n C_n \phi_n \, e^{-iE_n t/\hbar}$$

$\Psi(x,0) = A$, cte ; $1 = \int_{-L/3}^{L/3} |A|^2 = 2|A|^2\frac{L}{3} \Rightarrow A = \sqrt{\frac{3}{2L}}$

Então, $C_n = \int \Psi(x,0)\phi_n^* \, dx$

n par: $C_n = \sqrt{\frac{3}{2L}} \int_{-L/3}^{L/3} \operatorname{sen}\frac{n\pi}{L}x \, dx = 0$, integrando ímpar

n ímpar $C_n = \sqrt{\frac{3}{2L}} \int_{-L/3}^{L/3} \cos\frac{n\pi}{L}x \, dx = \frac{2\sqrt{3}}{n\pi} \operatorname{sen}\frac{n\pi}{3}$

$$\Psi(x\,t) = \frac{2\sqrt{3}}{\pi}\sqrt{\frac{2}{L}} \sum_{n\ \text{ímpar}} \frac{\operatorname{sen} n\pi/3}{n} \operatorname{sen}\frac{n\pi}{L}x \, e^{-iE_n t/\hbar}$$

Probabilidade de medir E_n é $|C_n|^2$:

$P_1 = |C_1|^2 = \frac{4.3}{\pi^2}\operatorname{sen}^2\frac{\pi}{3} = \frac{9}{\pi^2}$

$P_2 = P_4 = 0$; $P_3 = \frac{4.3}{9\pi^2}\operatorname{sen}^2\frac{3.\pi}{3} = 0$

4 - $\Psi(x,0) = (a^+)^2 \phi_0 + a^2 \phi_2 = \sqrt{2}\,\phi_2 + \sqrt{2}\,\phi_0$

normalizando:

$$\int |\Psi(x,0)|^2 dx = 2\int \Big(\overset{=1}{\phi_2^*\phi_2} + \overset{=1}{\phi_0^*\phi_0} + \overset{0}{\cancel{\phi_2^*\phi_0}} + \overset{0}{\cancel{\phi_0^*\phi_2}} \Big) dx = 4$$

$$\Rightarrow \Psi(x,0) = \frac{\phi_0 + \phi_2}{\sqrt{2}}$$

$$\Psi(x,t) = \sum_n C_n \phi_n e^{-iE_n t/\hbar}, \quad E_n = \hbar\omega(n+\tfrac{1}{2})$$

Por mais óbvio que seja, ñ escreva $\Psi(x,t)$ direto, isto é, sem calcular C_n; Você está numa prova e deve calcular os C_n:

$$C_n = \int \Psi(x,0)\,\phi_n^*(x)\,dx = \int \frac{\phi_0+\phi_2}{\sqrt{2}}\,\phi_n^*\,dx =$$

$$= \frac{1}{\sqrt{2}}\left(\delta_{n,0} + \delta_{n,2}\right). \text{ Assim,}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_0 e^{-iE_0 t/\hbar} + \phi_2 e^{-iE_2 t/\hbar}\right)$$

$$\bar{E} = \langle H \rangle = \int \Psi^*(x,t)\, H\, \Psi(x,t)\, dx, \quad \text{ou}$$

$$= \sum_n E_n |C_n|^2 = E_0|C_0|^2 + E_2|C_2|^2 = \frac{3}{2}\hbar\omega$$

5) $\langle \phi_n | p\cdot x | \phi_m \rangle = \int \phi_n^* \, p\cdot x \, \phi_m \, dx =$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i\,p}{\sqrt{2m\hbar\omega}} \Rightarrow x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$p = \sqrt{\frac{m\hbar\omega}{2}}\, i\,(a^+ - a)$$

$$= \frac{i\hbar}{2}\int \phi_n^* (a^+ - a)(a + a^+)\phi_m \, dx =$$

$$= \frac{i\hbar}{2}\int \phi_n^* (a^+ - a)\left(\sqrt{m}\,\phi_{m-1} + \sqrt{m+1}\,\phi_{m+1}\right) dx$$

$$= \frac{i\hbar}{2}\int \phi_n^* \Big\{ \sqrt{m}\left(\sqrt{m}\,\phi_m - \sqrt{m-1}\,\phi_{m-2}\right) +$$

$$+ \sqrt{m+1}\left(\sqrt{m+2}\,\phi_{m+2} - \sqrt{m+1}\,\phi_m\right)\Big\} =$$

$$= \frac{i\hbar}{2}\left(\sqrt{(m+1)(m+2)}\,\delta_{n,m+2} - \delta_{n,m} - \sqrt{m(m-1)}\,\delta_{n,m-2}\right)$$

$$\langle \phi_n | xp | \phi_m \rangle = \langle \phi_m | (xp)^+ | \phi_n \rangle^* =$$

$$= \langle \phi_m | p\cdot x | \phi_n \rangle^*$$

ou usando que $[x, p] = i\hbar \Rightarrow xp = i\hbar + px$,

$$\langle \phi_n | xp | \phi_m \rangle = \underbrace{\langle \phi_n | i\hbar | \phi_m \rangle}_{= i\hbar\,\delta_{nm}} + \langle \phi_n | px | \phi_m \rangle$$