4. Autovalores e Autovetores

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Slide 3 - Exemplos

1) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x,y) = (ax+by,\ cx+dy) \longrightarrow T(\vec{v}) = A\vec{v}$

$T(\vec{v}) = A_{m\times n}\ \vec{v}$

$$[\ ]_{2\times 1} = [\ ]_{2\times 2}\ [\ ]_{2\times 1} \longrightarrow \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\substack{T(\vec{v}) \\ \vec{w}\in\mathbb{R}^2}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{\vec{v} \\ \vec{v}\in\mathbb{R}^2}}$$

2) $\mathbb{D}: P_3(x) \longrightarrow P_2(x)$ ou $\mathbb{D}(p(x)) = p'(x)$

$B = \{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}, \underset{v_4}{x^3}\}$ ... base de $P_3(x)$

$B' = \{\underset{w_1}{1}, \underset{w_2}{x}, \underset{w_3}{x^2}\}$ ... base de $P_2(x)$

Os elementos do conjunto $B \in V = P_3(x)$, portanto, podem passar pela TL. Assim:

$\mathbb{D}(v_1) = 0$

$\mathbb{D}(v_2) = 1$

$\mathbb{D}(v_3) = 2x$

$\mathbb{D}(v_4) = 3x^2$

Escrevendo $\mathbb{D}(v_i)$ ⟶ como CL dos vetores de B'

$\mathbb{D}(v_1) = 0w_1 + 0w_2 + 0w_3$

$\mathbb{D}(v_2) = 1w_1 + 0w_2 + 0w_3$

$\mathbb{D}(v_3) = 0w_1 + 2w_2 + 0w_3$

$\mathbb{D}(v_4) = 0w_1 + 0w_2 + 3w_3$

A transposta da matriz de coeficientes deste sistema é $[T]^B_{B'}$, chamada de matriz da TL $\mathbb{D}(\cdot)$ em relação às bases B e B':

$$[T]^B_{B'} = \left[[\mathbb{D}(v_1)]_{B'} \cdots [\mathbb{D}(v_4)]_{B'} \right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4} = A$$

Observe que $\mathbb{D}(\cdot)$ passa a ser a TL associada à matriz A e às bases B e B'.

** Essa matriz A deriva qualquer polinômio $\in P_3(x)$ quando se executa a operação $A\,p(x)$, $p(x) \in P_3(x)$.

Ex: $p(x) = \pi - 8x + ex^2 - \sqrt{3}\,x^3$

$p'(x) = -8 + 2ex - 3\sqrt{3}\,x^2$

ou

$$\mathbb{D}(p(x)) = A\,p(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \pi \\ -8 \\ e \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} -8 \\ 2e \\ -3\sqrt{3} \end{bmatrix}$$

vetor coluna, composto pelos coeficientes do polinômio $p(x)$, do grau menor para o maior

O vetor $\begin{bmatrix} -8 \\ 2e \\ -3\sqrt{3} \end{bmatrix}$ corresponde ao polinômio derivado

$$p'(x) = -8 + 2ex - 3\sqrt{3}\,x^2$$

→ É desta maneira que um computador deriva polinômios!

3) $\mathbb{I} : P_3(x) \longrightarrow \mathbb{R}$ ou $\mathbb{I}(p(x)) = \int_0^1 p(x)\,dx$

$B = \{ \underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}, \underset{v_4}{x^3} \}$ ... base de $P_3(x)$

$B' = \{ \underset{w_1}{1} \}$ ... base de $\mathbb{R}$

Os elementos do conjunto $B \in V = P_3(x)$, portanto, podem passar pela TL. Assim:

$\mathbb{I}(v_1) = \int_0^1 1\,dx = 1$

$\mathbb{I}(v_2) = \int_0^1 x\,dx = \frac{1}{2}$

$\mathbb{I}(v_3) = \int_0^1 x^2\,dx = \frac{1}{3}$

$\mathbb{I}(v_4) = \int_0^1 x^3\,dx = \frac{1}{4}$

Escrevendo $\mathbb{I}(v_i)$ como CL dos vetores de B'

$\mathbb{I}(v_1) = 1\,w_1$

$\mathbb{I}(v_2) = \frac{1}{2}\,w_1$

$\mathbb{I}(v_3) = \frac{1}{3}\,w_1$

$\mathbb{I}(v_4) = \frac{1}{4}\,w_1$

TRANSPOSTA

$$[T]^{B}_{B'} = A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \end{bmatrix}$$

matriz A integra qualquer polinômio $\in P_3(x)$

Ex: $p(x) = -1 + 8x + 9x^2$

$$\mathbb{I}(p(x)) = \int_0^1 (-1 + 8x + 9x^2)\,dx = \left[-x + 4x^2 + 3x^3\right]_0^1 = 6$$

ou

$$\mathbb{I}(p(x)) = A\ p(x) = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \end{bmatrix} \begin{bmatrix} -1 \\ 8 \\ 9 \\ 0 \end{bmatrix} = 6$$

$\longrightarrow$ É assim que um computador resolve a integral definida de um polinômio.

Slide 4 - Exercícios

1) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $T(x,y,z) = (x+y, 2z)$

$B = \{ \underset{\vec{v}_1}{(1,0,0)}, \underset{\vec{v}_2}{(0,1,0)}, \underset{\vec{v}_3}{(0,0,1)} \}$ ... base do $\mathbb{R}^3$

$B' = \{ \underset{\vec{w}_1}{(1,0)}, \underset{\vec{w}_2}{(0,1)} \}$ ... base do $\mathbb{R}^2$

Os elementos do conjunto $B \in V$; portanto, podem passar pela TL.

Assim:

$T(\vec{v}_1) = (1,0)$

$T(\vec{v}_2) = (1,0)$

$T(\vec{v}_3) = (0,2)$

Escrevendo $T(v_i)$ como CL dos vetores de B'

$T(\vec{v}_1) = 1\vec{w}_1 + 0\vec{w}_2$

$T(\vec{v}_2) = 1\vec{w}_1 + 0\vec{w}_2$

$T(\vec{v}_3) = 0\vec{w}_1 + 2\vec{w}_2$

TRANSPOSTA: $[T]^{B}_{B'} = A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $T(x,y,z) = (x+y, 2z)$

$B = \{ \underset{\vec{v}_1}{(1,1,0)}, \underset{\vec{v}_2}{(1,0,1)}, \underset{\vec{v}_3}{(0,0,-1)} \}$ ... base do $\mathbb{R}^3$

$B' = \{ \underset{\vec{w}_1}{(1,0)}, \underset{\vec{w}_2}{(1,1)} \}$ ... base do $\mathbb{R}^2$

Os elementos do conjunto $B \in V$; portanto, podem passar pela TL.

Assim:

$T(\vec{v_1}) = (2,0) = \alpha_1 \vec{w_1} + \alpha_2 \vec{w_2}$ (i)

$T(\vec{v_2}) = (1,2) = \beta_1 \vec{w_1} + \beta_2 \vec{w_2}$ (ii)

$T(\vec{v_3}) = (0,-2) = \delta_1 \vec{w_1} + \delta_2 \vec{w_2}$ (iii)

(i) $(2,0) = \alpha_1 (1,0) + \alpha_2 (1,1)$

$$\begin{cases} \alpha_1 + \alpha_2 = 2 \\ \alpha_2 = 0 \end{cases} \longrightarrow \alpha_1 = 2 \; ; \; \alpha_2 = 0$$

(ii) $(1,2) = \beta_1 (1,0) + \beta_2 (1,1)$

$$\begin{cases} \beta_1 + \beta_2 = 1 \\ \beta_2 = 2 \end{cases} \longrightarrow \beta_1 = -1 \; ; \; \beta_2 = 2$$

(iii) $(0,-2) = \delta_1 (1,0) + \delta_2 (1,1)$

$$\begin{cases} \delta_1 + \delta_2 = 0 \\ \delta_2 = -2 \end{cases} \longrightarrow \delta_1 = 2 \; ; \; \delta_2 = -2$$

Portanto, para $T(\vec{v_i})$ escritos como CL dos vetores de B' tem-se:

$T(\vec{v_1}) = 2\vec{w_1} + 0\vec{w_2}$

$T(\vec{v_2}) = -1\vec{w_1} + 2\vec{w_2}$ $\xrightarrow{\text{TRANSPOSTA}}$ $[T]^{B}_{B'} = A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & -2 \end{bmatrix}$

$T(\vec{v_3}) = 2\vec{w_1} - 2\vec{w_2}$

3) $T: P_2(x) \longrightarrow P_3(x)$, $T(p(x)) = (x+1)\,p(x)$

$B = \{1, x-1, (x-1)^2\} = \{\underset{v_1}{1}, \underset{v_2}{x-1}, \underset{v_3}{x^2-2x+1}\}$ ... base de $P_2(x)$

$B' = \{\underset{w_1}{1}, \underset{w_2}{x}, \underset{w_3}{x^2}, \underset{w_4}{x^3}\}$ ... base de $P_3(x)$

Os elementos do conjunto $B \in V$; portanto, podem passar pela TL.

Assim:

$$T(v_1) = (x+1)\,1 = x+1$$
$$T(v_2) = (x+1)(x-1) = x^2-1$$
$$T(v_3) = (x+1)(x^2-2x+1) = x^3-x^2-x+1$$

Mas:

$$T(v_1) = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 \quad (i)$$
$$T(v_2) = \beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 + \beta_4 w_4 \quad (ii)$$
$$T(v_3) = \delta_1 w_1 + \delta_2 w_2 + \delta_3 w_3 + \delta_4 w_4 \quad (iii)$$

(i) $x+1 = \alpha_1(1) + \alpha_2(x) + \alpha_3(x^2) + \alpha_4(x^3)$

$\alpha_1 = \alpha_2 = 1 \;;\; \alpha_3 = \alpha_4 = 0$

(ii) $x^2-1 = \beta_1(1) + \beta_2(x) + \beta_3(x^2) + \beta_4(x^3)$

$\beta_1 = -1 \;;\; \beta_3 = 1 \;;\; \beta_2 = \beta_4 = 0$

(iii) $x^3-x^2-x+1 = \delta_1(1) + \delta_2(x) + \delta_3(x^2) + \delta_4(x^3)$

$\delta_1 = \delta_4 = 1 \;;\; \delta_2 = \delta_3 = -1$

Portanto, para $T(v_i)$ escritos como CL dos vetores de B' tem-se:

$$T(v_1) = 1w_1 + 1w_2 + 0w_3 + 0w_4$$
$$T(v_2) = -1w_1 + 0w_2 + 1w_3 + 0w_4$$
$$T(v_3) = 1w_1 - 1w_2 - 1w_3 + 1w_4$$

TRANSPOSTA: $[T]^{B}_{B'} = A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

4) $T: M(2,2) \longrightarrow M(2,2)$, $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a+b & 2b \\ 2c & 3d \end{bmatrix}$

$B = B' = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M_4} \right\}$

base canônica de $M(2,2)$

Os elementos do conjunto $B \in V$; portanto, podem passar pela TL.

Assim:

$T(M_1) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$T(M_2) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$T(M_3) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$T(M_4) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

Escrevendo $\mathbb{T}(v_i)$ como CL dos vetores de B'

$T(M_1) = 2M_1 + 0M_2 + 0M_3 + 0M_4$

$T(M_2) = 1M_1 + 2M_2 + 0M_3 + 0M_4$

$T(M_3) = 0M_1 + 0M_2 + 2M_3 + 0M_4$

$T(M_4) = 0M_1 + 0M_2 + 0M_3 + 0M_4$

TRANSPOSTA: $[T]^{B}_{B'} = A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

5) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $T(x,y,z) = (x-y, 2y, y+z) \longrightarrow T(\vec{v}) = A\vec{v}$

* $\exists\, T^{-1}$ se T for um isomorfismo. Como $\dim(V) = \dim(W)$, mostrar que T é isomorfismo implica mostrar que T é injetora, isto é, $N(T) = \{\vec{0}\}$:

$$N(T) = \{\vec{v} \in V \,/\, T(\vec{v}) = \vec{0}\}$$

$$T(\vec{v}) = (x-y, 2y, y+z) = (0,0,0) \quad \begin{cases} x-y=0 \\ 2y=0 \\ y+z=0 \end{cases}$$

$\therefore \quad x = y = z = 0 \quad$ e $\quad N(T) = \{\vec{0}\}$.

Desta forma, T é injetora e isomorfismo; portanto, $\exists\, T^{-1}$.

Encontrando $A = [T]^{\beta}_{\beta'}$, $\beta = \beta'$... base canônica do $\mathbb{R}^3$, a partir de T:

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{T(\vec{v})} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{v}}$$

$\det A = 2 \neq 0 \quad \therefore \quad \exists\, A^{-1}$

Determinando a matriz inversa A^{-1}:

$$A^{-1} = \frac{1}{\det A} (\operatorname{cof} A)^T$$

$$\operatorname{cof} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad A^{-1} = [T^{-1}]_{B}^{B'} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

E :

$$T^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \frac{1}{2} y \\ \frac{1}{2} y \\ -\frac{1}{2} y + z \end{bmatrix}$$

$$\rightarrow \boxed{T^{-1}(x, y, z) = \left(\frac{2x + y}{2}, \frac{y}{2}, \frac{-y + 2z}{2} \right)}$$