Lista 11: Séries de Fourier (extraídos do livro de Boyce e DiPrima)

Problems

In each of Problems 1 through 8, determine whether the given function is periodic. If so, find its fundamental period.

1. $\sin(5x)$

2. $\cos(2\pi x)$

3. $\sinh(2x)$

4. $\sin(\pi x/L)$

5. $\tan(\pi x)$

6. x^2

7. $f(x) = \begin{cases} 0, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

8. $f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n, \\ 1, & 2n \le x < 2n+1; \end{cases} \quad n = 0, \pm 1, \pm 2, \ldots$

9. If $f(x) = -x$ for $-L < x < L$, and if $f(x+2L) = f(x)$, find a formula for $f(x)$ in the interval $L < x < 2L$; in the interval $-3L < x < -2L$.

10. If $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ x, & 0 < x < 1, \end{cases}$ and if $f(x+2) = f(x)$, find a formula for $f(x)$ in the interval $1 < x < 2$; in the interval $8 < x < 9$.

11. If $f(x) = L - x$ for $0 < x < 2L$, and if $f(x+2L) = f(x)$, find a formula for $f(x)$ in the interval $-L < x < 0$.

12. Verify equations (6) and (7) in this section by direct integration.

In each of Problems 13 through 18:

a. Sketch the graph of the given function for three periods.
b. Find the Fourier series for the given function.

13. $f(x) = -x, \quad -L \le x < L; \quad f(x+2L) = f(x)$

14. $f(x) = \begin{cases} 1, & -L \le x < 0, \\ 0, & 0 \le x < L; \end{cases} \quad f(x+2L) = f(x)$

15. $f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \quad f(x+2\pi) = f(x)$

16. $f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1; \end{cases} \quad f(x+2) = f(x)$

17. $f(x) = \begin{cases} x+L, & -L \le x \le 0, \\ L, & 0 < x < L; \end{cases} \quad f(x+2L) = f(x)$

18. $f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ x, & -1 < x < 1, \\ 0, & 1 \le x < 2; \end{cases} \quad f(x+4) = f(x)$

In each of Problems 19 through 24:

a. Sketch the graph of the given function for three periods.
b. Find the Fourier series for the given function.
G **c.** Plot the partial sum $s_m(x)$ versus x for $m = 5$, 10, and 20.
d. Describe how the Fourier series seems to be converging.

19. $f(x) = \begin{cases} -1, & -2 \le x < 0, \\ 1, & 0 \le x < 2; \end{cases} \quad f(x+4) = f(x)$

20. $f(x) = x, \quad -1 \le x < 1; \quad f(x+2) = f(x)$

21. $f(x) = x^2/2, \quad -2 \le x \le 2; \quad f(x+4) = f(x)$

22. $f(x) = \begin{cases} x+2, & -2 \le x < 0, \\ 2-2x, & 0 \le x < 2; \end{cases} \quad f(x+4) = f(x)$

23. $f(x) = \begin{cases} -\frac{1}{2}x, & -2 \le x < 0, \\ 2x - \frac{1}{2}x^2, & 0 \le x < 2; \end{cases} \quad f(x+4) = f(x)$

24. $f(x) = \begin{cases} 0, & -3 \le x \le 0, \\ x^2(3-x), & 0 < x < 3; \end{cases} \quad f(x+6) = f(x)$

25. Consider the function f defined in Problem 21, and let $e_m(x) = f(x) - s_m(x)$.

a. Plot $|e_m(x)|$ versus x for $0 \le x \le 2$ for several values of m.
b. Find the smallest value of m for which $|e_m(x)| \le 0.01$ for all x.

26. Consider the function f defined in Problem 24, and let $e_m(x) = f(x) - s_m(x)$.

G **a.** Plot $|e_m(x)|$ versus x for $0 \le x \le 3$ for several values of m.
N **b.** Find the smallest value of m for which $|e_m(x)| \le 0.1$ for all x.

27. Suppose that g is an integrable periodic function with period T.

a. If $0 \le a \le T$, show that

$$\int_0^T g(x)dx = \int_a^{a+T} g(x)dx.$$

Hint: Show first that $\int_0^a g(x)dx = \int_T^{a+T} g(x)dx$. Then, in the second integral, consider the change of variable $s = x - T$.

b. Show that for any value of a, not necessarily in $0 \le a \le T$,

$$\int_0^T g(x)dx = \int_a^{a+T} g(x)dx.$$

c. Show that for any values of a and b,

$$\int_a^{a+T} g(x)dx = \int_b^{b+T} g(x)dx.$$

28. If f is differentiable and is periodic with period T, show that f' is also periodic with period T. Determine whether

$$F(x) = \int_0^x f(t)dt$$

is always periodic.