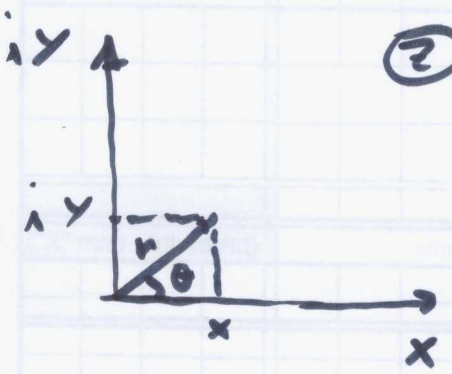
Caps 1 and 2

$$z \in \mathbb{C}$$



(z) $$z = (x, y) = x + iy$$

where $i = \sqrt{-1}$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = +\sqrt{x^2 + y^2}$$

$$z = r\left[\cos\theta + i\sin\theta\right]$$

igualdade $\quad z_1 = z_2 \iff x_1 = x_2$

$$y_1 = y_2$$

Soma: $z_1 + z_2 = x_1 + x_2 + i\,(y_1 + y_2)$

multipliz: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) =$

$$= (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$$

Properties: $\quad \mathrm{Re}\{z\} = x \in \mathbb{R}$

$$\mathrm{Im}\{z\} = y \in \mathbb{R}$$

$z_1 + z_2 = z_2 + z_1$ comutative

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ associative

$z_1(z_2 z_3) = (z_1 z_2) z_3$ "

$z_1 z_2 = z_2 z_1$ Comutative

$$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$$ distributive

$$\exists\, 0 \mid z + 0 = z$$

$$\exists\, 1 \mid z \cdot 1 = z$$

$$z + (-z) = 0$$

$$z(z^{-1}) = 1 \quad \forall z \neq 0$$

Complex Conjugate: $z^* = \bar{z} = x - iy$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z \cdot \bar{z}}$$

$$\mathrm{Re}\{z\} = \frac{z+\bar{z}}{2} \; ; \quad \mathrm{Im}\{z\} = \frac{z-\bar{z}}{2}$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*} \quad z_2 \neq 0$$

$$(z_1 z_2^*)^* = z_1^* z_2$$

3

$$z_1 z_2^* = (a+ib)(c-id) = (ac+bd) + i(bc-ad)$$

$$|z_1 z_2^*|^2 = (ac+bd)^2 + (bc-ad)^2$$

$$[Re(z_1 z_2^*)]^2 = (ac+bd)^2 \leq |z_1 z_2^*|^2$$

Therefore we have: $\begin{cases} |Re(z)| \leq |z| \\ |Im(z)| \leq |z| \end{cases}$

Triangular inequalities:

$$(z_1+z_2)(z_1+z_2)^* = z_1 z_1^* + (z_1 z_2^* + z_1^* z_2) + z_2 z_2^*$$

$$= |z_1|^2 + [z_1 z_2^* + (z_1 z_2^*)^*] + |z_2|^2$$

$$= |z_1|^2 + 2Re(z_1 z_2^*) + |z_2|^2$$

$$|z_1+z_2|^2 = |z_1|^2 + 2Re(z_1 z_2^*) + |z_2|^2$$

$$Re(z_1 z_2^*) \leq |z_1 z_2^*| = |z_1||z_2^*| = |z_1||z_2|$$

$$|z_1+z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1|+|z_2|)^2$$

$$\boxed{|z_1+z_2| \leq |z_1| + |z_2|}$$

4

$$|z_2| = |z_1 + (z_2 - z_1)| \leq |z_1| + |z_2 - z_1|$$

$$|z_2| - |z_1| \leq |z_2 - z_1|$$

Do the same for $|z_1|$ and you get:

$$\boxed{|z_1 - z_2| \geq \left| |z_1| - |z_2| \right|}$$

$$z = r(\cos\theta + i\sin\theta) = x + iy$$

$$\theta = \tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) = \arg(z) \quad : -\pi < \arg(z) < \pi$$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_2)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)]$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \sum_{n=0}^{\infty} \frac{i^{2n} y^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{(2n+1)} y^{(2n+1)}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{(2n+1)}}{(2n+1)!}$$

$$\boxed{e^{iy} = \cos(y) + i\sin(y)}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{z} = e^{(x+iy)} = e^{x}\left[\cos(y) + i\sin(y)\right]$$

$e^{z} \Rightarrow$ periodic function:

$$e^{z \pm i2\pi n} = e^{x}\left[\cos(y \pm 2\pi n) + i\sin(y \pm 2\pi n)\right]$$

$$= e^{x}\left[\cos(y) + i\sin(y)\right] = e^{z}$$

$$z = r e^{i\theta} = r e^{i(\theta \pm 2\pi n)}$$

$$z = \exp\left[\ln|z| + i(\theta \pm 2\pi n)\right] \quad \text{for } z \neq 0$$

$$|z| = r = \sqrt{x^2 + y^2} \Rightarrow |z| \geq 0$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$|z_1 z_2| = r_1 r_2 \quad ; \quad \arg(z_1 z_2) = \theta_1 + \theta_2 \pm 2\pi n$$

examples: $z_2 = 1\, e^{i\alpha} \Rightarrow z_1 z_2 = r_1 e^{i(\theta + \alpha)}$

rotation

$z_3 = R_3 \in \mathbb{R} \quad R_3 > 0$

$z_1 z_3 = r_1 R_3 e^{i\theta_1} \Rightarrow$ stretching

$$\text{for } z_2 \neq 0 \Rightarrow z_3 = \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$|z_3| = \frac{|z_1|}{|z_2|} \; ; \; \arg(z_3) = \arg(z_1) - \arg(z_2) \pm 2\pi n$$

$$z^n = \underbrace{z_1 \cdot z_1 \cdots z_1}_{n \text{ factors}} \quad \text{for } n \in \mathbb{N}$$

$$z^n = r^n e^{in\theta}$$

$$w^n = z \Rightarrow \rho^n e^{in\phi} = r e^{i\theta} \Rightarrow \begin{cases} \rho = r^{1/n} \\ n\phi = \theta \pm 2\pi k \end{cases}$$

$$\phi = \frac{\theta}{n} \pm \frac{2k\pi}{n} \Rightarrow \phi = \frac{\theta}{n} \pm k\left(\frac{2\pi}{n}\right) \quad \forall k \in \mathbb{Z}$$

$$z^{m/n} = r^{m/n} \exp\left[i\left(\frac{m}{n}\theta \pm 2k\pi \frac{m}{n}\right)\right]$$

$m, n \in \mathbb{Z}$

$n \neq 0$

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2} \; ; \; \frac{\exp(z_1)}{\exp(z_2)} = \exp(z_1 - z_2)$$

$$\exp(z^*) = e^x \cdot e^{-iy} = e^x[\cos(y) - i\sin(y)] = [\exp(z)]^*$$

$$e^{i\pi} = -1 \; ; \; -23 = 23 e^{i\pi}$$

$3^3 = 3.3.3$; $3^\pi \neq 3.3.3\ldots$?

How do we define the inverse function, namely, the logarithm?

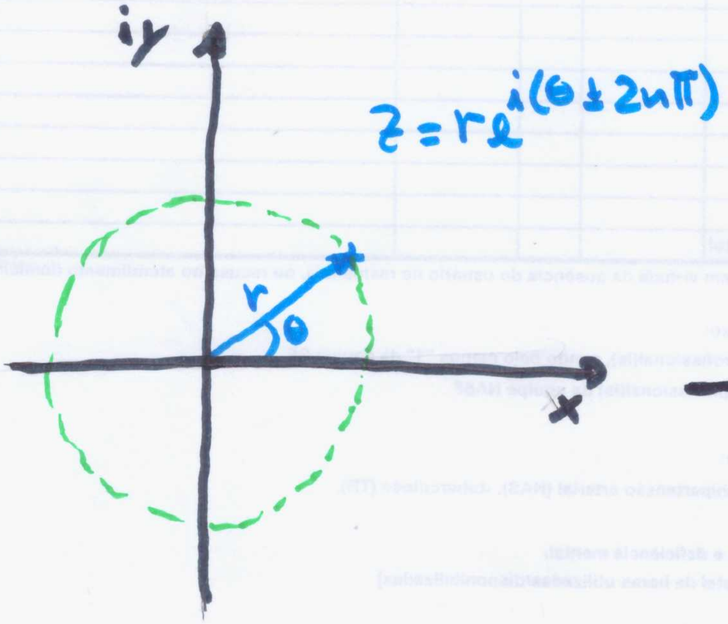
a first attempt would be:

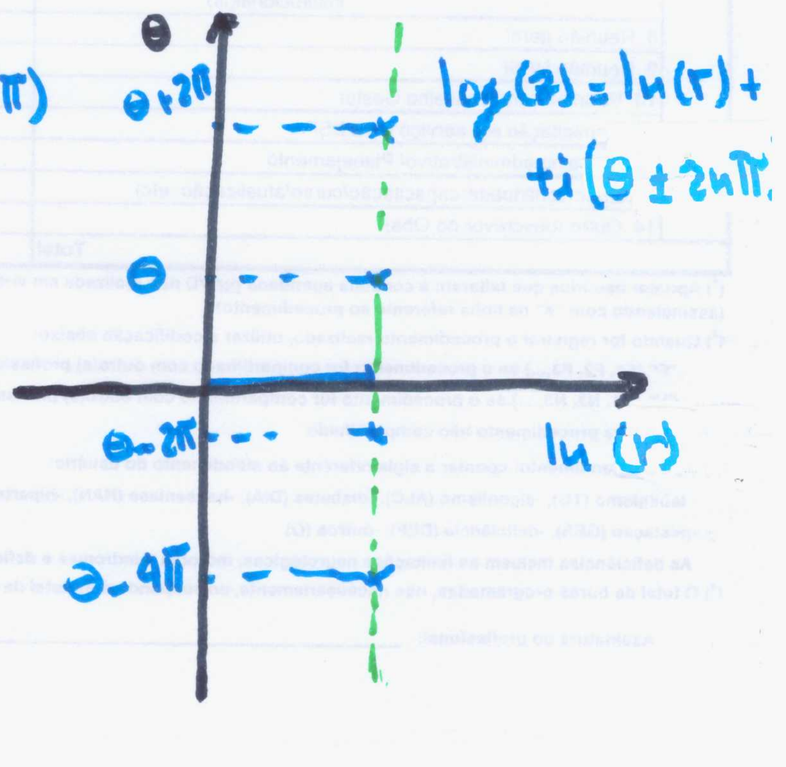
$$\log z = \log(re^{i\theta}) \equiv \ln(r) + i\theta$$

However, this definition doesn't account for $e^{i\theta}$ periodicity. So the correct definition is:

$$\log(z) = \log(re^{i\theta}) \equiv \ln(r) + i(\theta \pm 2n\pi)$$

for any $r \in \mathbb{R} \mid r > 0$ $\Rightarrow$ it is **not** defined at $z = 0$





$$z = \exp[\ln(r) + i(\Theta \pm 2n\pi)] = r e^{i\Theta}$$

$$, z \in \mathbb{C}$$

$$\text{So } \boxed{\exp[\log(z)] = z}$$

$$w = e^{z} \Rightarrow \log w = \log[e^{x+iy}] =$$

$$= \ln(e^{x}) + i(y \pm 2n\pi)$$

$$= x + i(y \pm 2n\pi)$$

$$\text{Therefore: } \boxed{\log[\exp(z)] = z \pm i2n\pi}$$

Homework set #1

Page 22, problem 18

Page 32, problem 8

Churchill's book

5th ed.

$$\log(-1) = \cancelto{0}{\ln(1)} + i(\pi \pm 2n\pi)$$

Properties: $\forall\, z_1, z_2 \in \mathbb{C} \mid z_1, z_2 \neq 0$

$$\log(z_1) + \log(z_2) = \log(z_1 . z_2)$$

$$\log(z_1) - \log(z_2) = \log\left(\frac{z_1}{z_2}\right)$$

$$\left.\begin{aligned} n\log(z) &= \log(z^n) \\ \frac{1}{n}\log(z) &= \log(z^{1/n}) \\ \frac{m}{n}\log(z) &= \log(z^{m/n}) \end{aligned}\right\} \forall\, m,n \in \mathbb{Z}$$

$$z^c \equiv \exp[c\log(z)]$$

$$z^{-c} \equiv \exp[-c\log(z)]$$

$$\left.\begin{aligned} c &= a+ib \\ z &= re^{i\theta} \end{aligned}\right\} z^c = \exp\left[(a+ib)(\ln(r) + i(\theta \pm 2n\pi))\right]$$

$$z^c = \exp\left\{\left[a\ln r - b(\theta \pm 2n\pi)\right] + i\left[b\ln(r) + a(\theta \pm 2n\pi)\right]\right.$$

24/09/2020

$$E(z) = \exp(z) = \exp(z \pm i 2n\pi) = e^{z \pm i2n\pi}$$

$$\log(z) = \ln(r) + i\,(\theta \pm 2n\pi)$$

$$\exp[\log(z)] = z$$

$$\log[\exp(z)] = z \pm i\,2k\pi$$

$$z_1 = r_1 e^{i\theta_1} \quad , z_2 = r_2 e^{i\theta_2}$$

$$\left.\begin{aligned} \log(z_1) + \log(z_2) &= \log(z_1 . z_2) \\ \log(z_1) - \log(z_2) &= \log\left(\frac{z_1}{z_2}\right) \end{aligned}\right| \begin{aligned} &\forall z \in \mathbb{C} \\ &z \neq 0 \end{aligned}$$

$$n \in \mathbb{N} \Rightarrow n\log(z) = \log(z^n)$$

$$n\log(z^{1/n}) = n\log\left[r^{1/n} e^{i\frac{(\theta \pm 2k\pi)}{n}}\right] = \ln(r) + ni\frac{(\theta \pm 2k\pi)}{n}$$

$$n\log(z^{1/n}) = \log(z) \Rightarrow \frac{1}{n}\log(z) = \log(z^{1/n})$$

$$\frac{m}{n}\log(z) = \log(z^{m/n})$$

Complex exponents :

$$\left.\begin{aligned} z^{c} &\equiv \exp\left[c \log(z)\right] \\ z^{-c} &\equiv \exp\left[-c \log(z)\right] \end{aligned}\right\} \quad z, c \in \mathbb{C}$$

$$i^{-2i} = \exp\left[-2i \log(i)\right] = \exp\left\{-2i\left[i\left(\frac{\pi}{2} \pm 2k\pi\right)\right]\right\}$$

$$i = 1\, e^{i\pi/2} \quad \Bigg| \quad i^{-2i} = \exp\left\{\pi \pm 4k\pi\right\}$$

$$\left.\begin{aligned} e^{-i\theta} &= \cos\theta - i\sin\theta \\ e^{+i\theta} &= \cos\theta + i\sin\theta \end{aligned}\right\} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

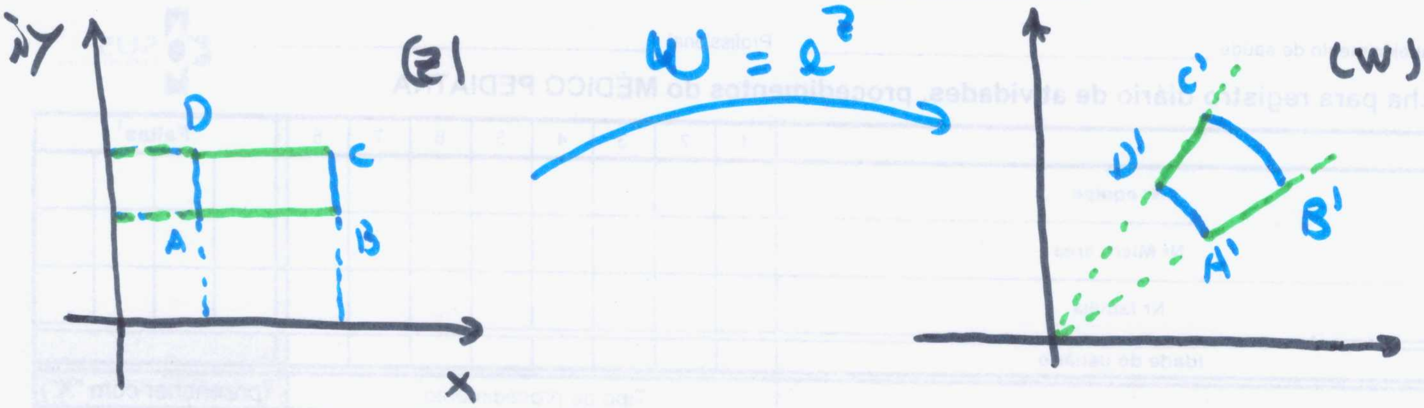
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos(z) \equiv \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad \Bigg| \quad \sinh(z) \equiv \frac{e^{z} - e^{-z}}{2}$$

$$\sin(z) \equiv \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \quad \Bigg| \quad \cosh(z) \equiv \frac{e^{z} + e^{-z}}{2}$$

$$\sinh(iz) = i \sin(z)$$

$$W = e^{z} = e^{x+iy} = e^{x} \cdot e^{iy}$$



limits in the complex plane

$$w = f(z) \qquad \forall \varepsilon > 0 \ \exists \delta > 0 \mid 0 < |z - z_0| < \delta \Rightarrow$$

$$\lim_{z \to z_0} f(z) = w_0 \qquad \Rightarrow |f(z) - w_0| < \varepsilon \ ; \ \varepsilon, \delta \in \mathbb{R}$$

$$\lim_{z \to 2i} (2x + iy^2) = 4i$$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \mid |2x + iy^2 - 4i| < \varepsilon \Leftarrow 0 < |z - 2i| < \delta$$

$$|2x + iy^2 - 4i| \le 2|x| + |y^2 - 4| = 2|x| + |y-2||y+2| < \varepsilon$$

On making $2|x| < \frac{\varepsilon}{2}$ and $|y-2||y+2| < \frac{\varepsilon}{2}$

$$|x| < \frac{\varepsilon}{4}$$

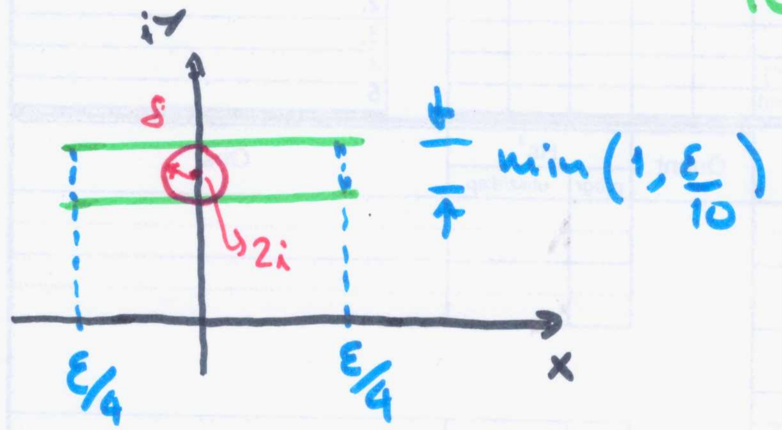
on assuming that $|y-2| < 1$

we get $|y+2| = |y-2+4| \le |y-2| + 4$

$$|y+2| \le 1 + 4 = 5$$

Then on making $|y-2| < \min\left(1, \frac{\varepsilon}{10}\right)$

$$|y-2||y+2| < \left(\frac{\varepsilon}{10} \cdot 5\right) = \frac{\varepsilon}{2}$$



Properties of Complex Limits.

1) Uniqueness

Let's assume that we have:

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1$$

$$\forall \varepsilon > 0 \ \exists \ \delta_0 > 0 \text{ and } \delta_1 > 0 \ \Big| \ \begin{cases} 0 < |z - z_0| < \delta_0 \\ 0 < |z - z_0| < \delta_1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} |f(z) - w_0| < \varepsilon \\ |f(z) - w_1| < \varepsilon \end{cases}$$

we take $\delta = \min\{\delta_0, \delta_1\}$ so that

if $0 < |z - z_0| < \delta$ both of the above inequalities are simultaneously met

Now let's consider

$$|w_0 - w_1| = |(f(z) - w_0) - (f(z) - w_1)| \leq$$

$$\leq |f(z) - w_0| + |f(z) - w_1| < 2\varepsilon$$

$$|w_0 - w_1| < 2\varepsilon \qquad \forall \delta = \min\{\delta_1, \delta_2\}$$

since this should hold for any $\varepsilon > 0$, no matter how small, I can make $\varepsilon \to 0$ and, hence, $w_0 = w_1$

for $z = x + iy$; $f(z) = u(x,y) + i v(x,y)$

$u(x,y)$ and $v(x,y)$ are two real functions in $\mathbb{R}^3$

$Re(f) = u$ $\qquad z_0 = x_0 + i y_0$

$Im(f) = v$ $\qquad w_0 = u_0 + i v_0$

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 \end{cases}$$

$(\Rightarrow)$ $\forall \varepsilon > 0 \; \exists \delta > 0 \mid 0 < |(x-x_0) + i(y-y_0)| < \delta \Rightarrow$ ⑥

$$\Rightarrow |(u-u_0) + i(v-v_0)| < \varepsilon$$

$$|u-u_0| \leq |(u-u_0) + i(v-v_0)| < \varepsilon$$

$$|v-v_0| \leq |(u-u_0) + i(v-v_0)| < \varepsilon$$

$\therefore$ $|u-u_0| < \varepsilon$ and $|v-v_0| < \varepsilon \Leftarrow$

$$\Leftarrow \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$(\Leftarrow)$ $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \Rightarrow |u-u_0| < \varepsilon/2 \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u = u_0$

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \Rightarrow |v-v_0| < \varepsilon/2 \Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} v = v_0$

$$|(u-u_0) + i(v-v_0)| \leq |u-u_0| + |v-v_0| < \varepsilon$$

$\therefore$ on picking $\delta = \min(\delta_1, \delta_2)$ we have

$$0 < |z-z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

$$\Leftrightarrow \boxed{\lim_{z\to z_0} f(z) = w_0}$$

$$\left.\begin{array}{l}\lim\limits_{z\to z_0} f(z) = w_0 \\ \lim\limits_{z\to z_0} F(z) = W_0\end{array}\right\} \Rightarrow \begin{array}{l}\lim\limits_{z\to z_0}\left[f(z)+F(z)\right] = w_0 + W_0 \\ \lim\limits_{z\to z_0}\left[f(z)F(z)\right] = w_0 W_0\end{array}$$ 7

$$W_0 \neq 0 \Rightarrow \lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$$

$$\lim_{z\to z_0} f(z) = w_0 \Longrightarrow \lim_{z\to z_0} |f(z)| = |w_0|$$

if the limit exists:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \mid 0 < |z-z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

$$\big||f(z)| - |w_0|\big| \leq |f(z) - w_0| < \varepsilon$$

$$\big||f(z)| - |w_0|\big| < \varepsilon$$

continuity:

if $\lim\limits_{z\to z_0} f(z) \;\exists$ and $f(z_0)\;\exists$ and $\lim\limits_{z\to z_0} f(z) = f(z_0)$

then, and only then we can say that $f(z)$ is continuous at z_0

Derivatives of Complex functions

$$f'(z)\Big|_{z=z_0} = \frac{df}{dz}\Big|_{z=z_0} \equiv \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} =$$

$$= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z} \frac{\Delta f}{\Delta z}$$

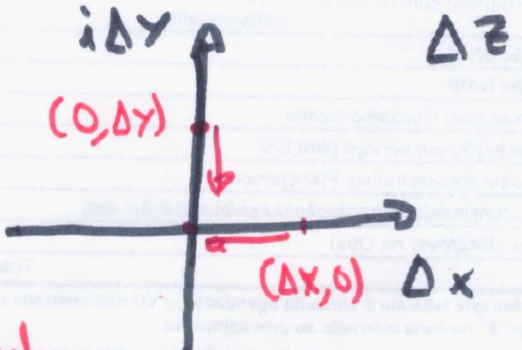
Examples:

$$f(z) = z^2 \; ; \; \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} (2z+\Delta z) = 2z$$

$$f'(z) = 2z$$

$$f(z) = |z|^2 = z\,.\,\bar{z}$$

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \bar{z} + \overline{\Delta z} + \frac{z\,\overline{\Delta z}}{\Delta z}$$



for $\Delta y = 0$, $\Delta z = \Delta x \to 0$

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta x} = \bar{z} + z$$

for $\Delta x = 0$, $\Delta z = \Delta y \to 0$ and

$$\overline{\Delta z} = -\Delta y \to 0$$

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta x} = \bar{z} - z \qquad \Delta z = -\overline{\Delta z}$$

Because in this case the limits are different, except at $z=0$, $f(z) = |z|^2$ derivative only exists at $z=0 \Rightarrow f'(0) = 0$

c = constant $c \in \mathbb{C}$

$$\frac{dc}{dz} = 0 \quad , \quad \frac{dz}{dz} = 1 \quad , \quad \frac{d}{dz}\left(c\, f(z)\right) = c f'(z)$$

$$n > 0 \quad n \in \mathbb{N} \Rightarrow \frac{dz^n}{dz} = n z^{n-1}$$

for $n < 0$ and $\forall z \neq 0$; $\frac{dz^n}{dz} = n z^{(n-1)}$

$$\frac{d}{dz}\left[f(z) + g(z)\right] = f'(z) + g'(z)$$

$$\frac{d}{dz}\left[f(z) \cdot g(z)\right] = f'(z)\, g(z) + f(z)\, g'(z)$$

$$g(z) \neq 0 \quad \frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f' g + f g'}{g^2(z)}$$

$$F(z) = g\left[f(z)\right] \Rightarrow F' = g'\left[f(z)\right] . f'(z)$$

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$$f(z) = u + i v \qquad z_0 = x_0 + i y_0$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \Delta z = \Delta x + i \Delta y$$

$\exists f'(z_0) \Rightarrow$

$$\frac{\Delta f}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i \Delta y} +$$

$$+ i \left[\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i \Delta y} \right]$$

$\Delta y = 0$ and $\Delta x \to 0$

$$\operatorname{Re}[f'(z)] = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = \left. \frac{\partial u}{\partial x} \right|_{x_0, y_0} = u_x(x_0, y_0)$$

$$\operatorname{Im}[f'(z)] = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = \left. \frac{\partial v}{\partial x} \right|_{x_0, y_0} = v_x(x_0, y_0)$$

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

Now we pick a different path:

$\Delta x = 0$; $\Delta y \to 0$

we get: $f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, v_0)$

Therefore, we get

$$\exists f'(z_0) \Longrightarrow \begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{z_0} = \left.\dfrac{\partial v}{\partial y}\right|_{z_0} \\ \left.\dfrac{\partial u}{\partial y}\right|_{z_0} = -\left.\dfrac{\partial v}{\partial x}\right|_{z_0} \end{cases}$$

C.R. conditions

$(\Longleftarrow)$

Now we assume that: $\exists\, u_x, u_y, v_x, v_y$ within a neighbourhood $\delta > |z - z_0|$, and those partial derivatives are also continuous

Furthermore, we assume that the satisfy C.R., that is: $u_x = v_y$ and $u_y = -v_x$ there.

Then, we make:

$\Delta z = \Delta x + i\,\Delta y$

$\Delta f = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\,\Delta v$

$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$

$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$

Taylor series expansion of Δu and Δv in $\mathbb{R}^3$

$$\Delta u = u_x\Big|_{x_0,y_0}\Delta x + u_y\Big|_{x_0,y_0}\Delta y + \varepsilon_1\sqrt{\Delta x^2+\Delta y^2}$$

$$\Delta v = v_x\Big|_{x_0,y_0}\Delta x + v_y\Big|_{x_0,y_0}\Delta y + \varepsilon_2\sqrt{\Delta x^2+\Delta y^2}$$

let's bear in mind that $\sqrt{\Delta x^2+\Delta y^2} \Leftrightarrow |\Delta z|$

$$\Delta f = u_x\Big|_{x_0,y_0}\Delta x + u_y\Big|_{x_0,y_0}\Delta y + \varepsilon_1\sqrt{\Delta x^2+\Delta y^2} +$$

$$+ i\left(v_x\Big|_{x_0,y_0}\Delta x + v_y\Big|_{x_0,y_0}\Delta x + \varepsilon_2\sqrt{\Delta x^2+\Delta y^2}\right)$$

And since these functions are assumed to meet C.R.; $u_x = v_y$ and $v_x = -u_y$, we get:

$$\frac{\Delta f}{\Delta z} = u_x\Big|_{x_0,y_0} + i\,v_x\Big|_{x_0,y_0} + (\varepsilon_1 + i\varepsilon_2)\frac{|\Delta z|}{\Delta z}$$

$$\left|\frac{|\Delta z|}{\Delta z}\right| = 1 \Rightarrow \frac{|\Delta z|}{\Delta z} = e^{-i\Delta\theta} \Rightarrow$$ since its magnitude is one, regardless of its argument/direction

And, moreover, we have:

$$\lim_{\Delta x, \Delta y \to 0} (\varepsilon_1 + i\varepsilon_2) = 0$$

Therefore, one gets: $\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{df}{dz}$

$$\frac{df}{dz} = u_x + i v_x = v_y - i u_y$$

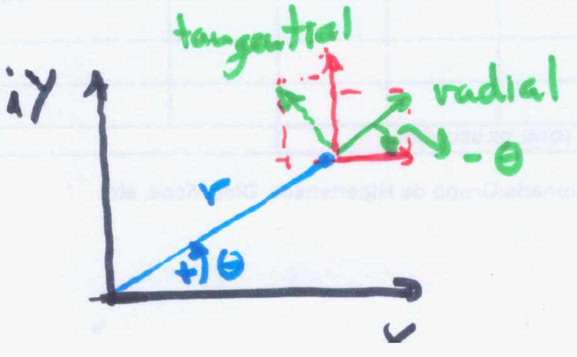
Hence the theorem:

$\exists\ u_x, u_y, v_x, v_y$ in $|z - z_0| < \delta$ and they are continuous at $z = z_0$. Then, if they satisfy C.R. $u_x = v_y$ and $u_y = -v_x$ at $z_0 \Rightarrow \exists\ f'(z)\big|_{z_0}$ and $f'(z)\big|_{z_0} = u_x\big|_{z_0} + i v_x\big|_{z_0}$

C.R. in polar form

$$f'(z) = (u_{,r} + i v_{,r})\, e^{-i\theta}$$

$$\text{C.R.} \begin{cases} u_{,r} = \dfrac{1}{r} v_{,\theta} \\[2ex] v_{,r} = -\dfrac{u_{,\theta}}{r} \end{cases}$$



$$\frac{d(z^n)}{dz} = n z^{n-1}$$

$n \in \mathbb{N}$: $$\frac{d(z^n)}{dz} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \sum_{k=0}^{n} \binom{n}{k} z^k \Delta z^{n-k} - z^k \right\}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \cancel{\binom{n}{n} z^n} + \sum_{k=0}^{n-1} \binom{n}{k} z^k \Delta z^{n-k} - \cancel{z^n} \right\}$$

$$= \lim_{\Delta z \to 0} \sum_{k=0}^{(n-1)} \binom{n}{k} z^k \frac{\Delta z^{n-k}}{\Delta z} = \binom{n}{n-1} z^{n-1} =$$

$$= n z^{n-1}$$

Now by making use of C.R. : $z = r e^{i\theta}$

$$f(z) = z^n = r^n e^{in\theta} = \underbrace{r^n \cos(n\theta)}_{u(r,\theta)} + i \underbrace{r^n \sin(n\theta)}_{v(r,\theta)}$$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos(n\theta) \longleftrightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = n r^{(n-1)} \cos(n\theta)$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{r^n n \sin(n\theta)}{r} = -r^{n-1} n \sin(n\theta) \longleftrightarrow -\frac{\partial v}{\partial r} = -n r^{(n-1)} \sin(n\theta)$$

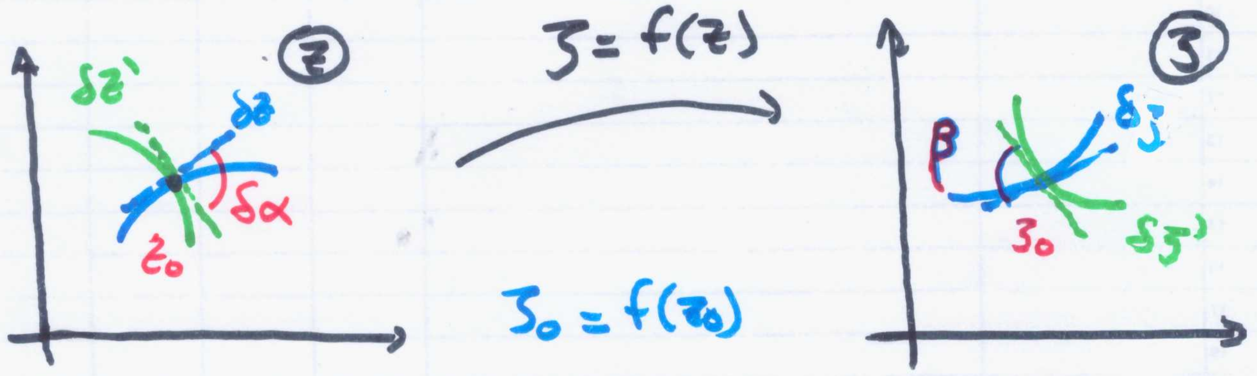
C.R. okay !!

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = n r^{(n-1)}\left[\cos(n\theta) + i\sin(n\theta)\right] e^{-i\theta}$$

$$= n r^{(n-1)} e^{in\theta} \cdot e^{-i\theta} = n r^{(n-1)} e^{i(n-1)\theta}$$

$$= n z^{(n-1)} \qquad \forall\, n \in \mathbb{I}$$

Geometric interpretation of the derivative of an analytic function



$$\delta\zeta = \left.\frac{df}{dz}\right|_{z_0} \delta z \quad \text{where I assume that } \frac{df}{dz} \neq 0$$

$$\left\{ \begin{array}{l} |\delta\zeta| = \left|\left.\dfrac{df}{dz}\right|_{z_0}\right| \, |\delta z| \implies \text{isotropic stretching or isotropic shrinking} \\ \arg(\delta\zeta) = \arg\left(\left.\dfrac{df}{dz}\right|_{z_0}\right) + \arg(\delta z) \implies \text{homogeneous rotation} \end{array} \right.$$

$$\begin{cases} \arg(\delta\zeta') = \arg\left(\left.\dfrac{df}{dz}\right|_{z_0}\right) + \arg(\delta z') \\[2ex] \arg(\delta\zeta) = \arg\left(\left.\dfrac{df}{dz}\right|_{z_0}\right) + \arg(\delta z) \end{cases}$$

$$\arg(\delta\zeta) - \arg(\delta\zeta') = \arg(\delta z) - \arg(\delta z')$$

$$\boxed{\beta = \alpha}$$

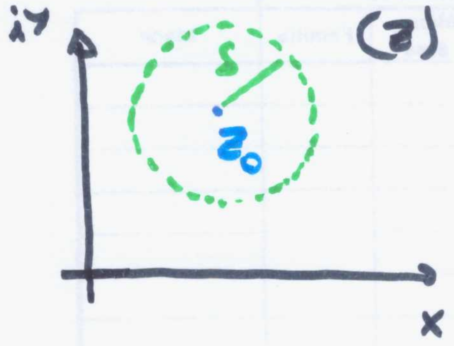
For their preserving angles, we call them <u>conformal</u> mappings or transformation.

Note: whenever $\left.\dfrac{df}{dz}\right|_{z_0} = 0$ at $z = z_0$, the function and the mapping are still analytic, but it is no longer conformal. For the angles are <u>not</u> preserved there.

Read sec 19. Pages 52 – 54

P 7. P. 8 $\Rightarrow$ still the 1st homework set

Defining an analytic function



An analytic function has a derivative in an open neighbourhood S of z_0

$$S: |z - z_0| < \delta \Rightarrow \exists f'(z)$$

$f(z) = \frac{1}{z}$ is analytic over the whole complex plane (z), except at the origin ($z = 0$)

On the other hand:

$f(z) = |z|^2$ is <u>NOT</u> analytic anywhere on the complex plane. Precisely because it only has a derivative defined at an isolated point, that is, the origin ($z = 0$), itself.

Entire functions are those that are analytic over the whole complex plane examples, e^z, complex polynomials.

Properties of Analytic functions:

Sum, product and division (ratio) between analytic functions yield analyti functions (in particular for the ratio $\Rightarrow$ the rule only applies to points where the denominator is nonzero: $F(z) = \dfrac{f(z)}{g(z)} \quad \forall z \in \mathbb{C} \mid g(z) \neq 0$

$H(z) = f[g(z)]$ where f and g are analytic at $z = z_0 \Rightarrow H(z_0) = f[g(z_0)]$ is also analytic.

Harmonic functions are those that have continuous partial derivatives and which satisfy Laplace equation

$h(x,y) \in \mathbb{R}^3 \mid h_x, h_{xx}, h_y, h_{yy} \ \exists$ and are continuous. Also $\nabla^2 h = 0$

Let us test to see whether analytic functions involve harmonic function or not:

$$f(z) = u(x,y) + i\,v(x,y)$$

where $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\left.\begin{array}{l} u_{xy} = v_{yy} \\ u_{yx} = -v_{xx} \end{array}\right\} v_{xx} + v_{yy} = 0 \Rightarrow \boxed{\nabla^2 v = 0}$$

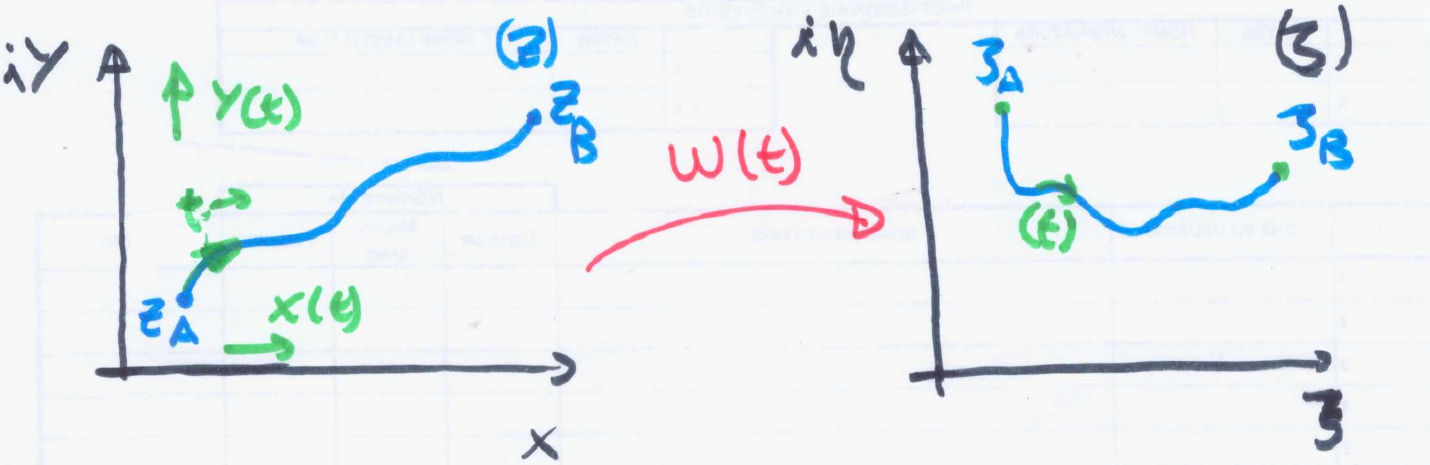
$$\left.\begin{array}{l} u_{xx} = v_{xy} \\ u_{yy} = -v_{xy} \end{array}\right\} u_{xx} + u_{yy} = 0 \Rightarrow \boxed{\nabla^2 u = 0}$$

Under these conditions, $u(x,y)$ and $v(x,y)$ are called harmonic conjugate

Read Section 21. Harmonic functions.

$$W(z) = u(x,y) + i v(x,y)$$

$$w(t) = W[z(t)] = u[x(t), y(t)] + i v[x(t), y(t)]$$



$$\frac{dw}{dt} = \frac{du}{dt} + i\frac{dv}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\left[\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right]$$

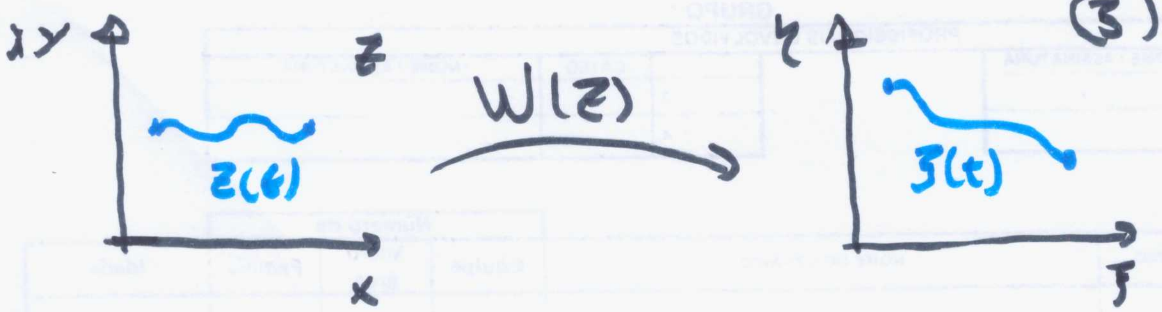
$$\frac{dw}{dt} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\dot{x} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\dot{y}$$

$$= \frac{dW}{dz}\dot{x} + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\dot{y} = \frac{dW}{dz}\dot{x} + i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\dot{y}$$

$$= \frac{dW}{dz}\dot{x} + i\frac{dW}{dz}\dot{y} = \frac{dW}{dz}(\dot{x} + i\dot{y})$$

$$\frac{dw}{dt} = \frac{dW}{dz}\frac{dz}{dt} = \frac{dW}{dz}\dot{z}$$

$$W(z) \Rightarrow W[z(t)] = \zeta(t)$$

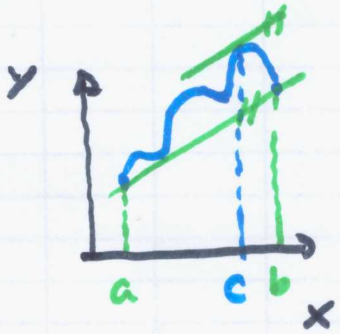


$$\frac{d\zeta}{dt} = \frac{dW}{dt} = \frac{dW}{dz}\dot{z} \quad \text{where} \quad \dot{z} = \frac{dz}{dt}$$

$z, W, \zeta \in \mathbb{C}$ and $t \in \mathbb{R}$

Example $\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}$

For real variables we have:

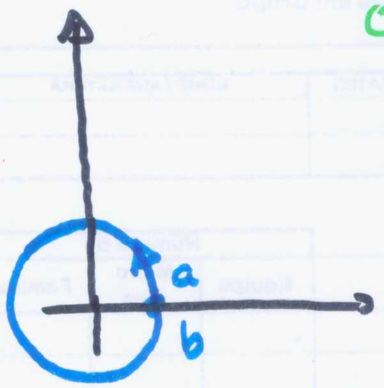


$$W(b) - W(a) = W'(c)(b-a)$$

$$\exists c \mid a \le c \le b$$

Mean Value Theorem from Basic Calculus.

But in the complex plane, things are different...



$0 \leq t \leq 2\pi$

$$w(t) = e^{it} = 1(\cos t + i \sin t)$$

$$w'(t) = i e^{it} \neq 0$$

$$\left.\begin{aligned} t_A &= 0 \\ t_B &= 2\pi \end{aligned}\right\} w(2\pi) - w(0) = 0$$

But we have: $i e^{it_c}(t_\beta - t_a) = i 2\pi e^{it_c}$

and this quantity never vanishes.

Hence we'll take an alternative approach to define complex integrals...

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

$u(t) = u[x(t), y(t)]$ and $v(t) = v[x(t), y(t)]$

Under these conditions, we can write:

$$\mathrm{Re}\left\{\int_a^b w(t)\,dt\right\} = \int_a^b \mathrm{Re}[w(t)]\,dt = \int_a^b u(t)\,dt$$

$$\mathrm{Im}\left\{\int_a^b w(t)\,dt\right\} = \int_a^b \mathrm{Im}[w(t)]\,dt = \int_a^b v(t)\,dt$$

$$\forall c \mid a \le c \le b : \int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$$

Let us define: $w(t) \equiv u(t) + i v(t)$ and

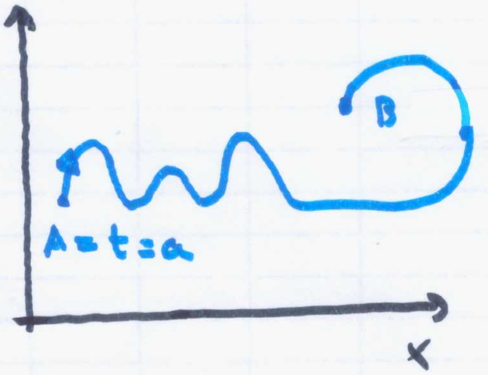
$W(t) = U(t) + i V(t)$ so that

$\dot{W}(t) = w(t)$ under these conditions, and on making use of the integral definition, one gets:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt = \Big[U(t)\Big]_a^b + i \Big[V(t)\Big]_a^b$$

Since both $u(x,y)$ and $v(x,y) \in \mathbb{R}^3$, we have:

$$\int_a^b w(t)\,dt = \Big[U(t) + i V(t)\Big]_a^b = \Big[W(t)\Big]_a^b$$



B ⇒ t = b

— Jordan arc.

$a \le t \le b$

$z(t) = x(t) + i y(t)$

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0} \implies r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$$

multiply eq. through by $e^{-i\theta_0}$

$$r_0 = \int_a^b e^{-i\theta_0}\, w(t)\,dt \in \mathbb{R}$$

$$r_0 = \left|\int_a^b w(t)\,dt\right| \therefore r_0 = \mathrm{Re}\left[\int_a^b e^{-i\theta_0}\, w(t)\,dt\right]$$

$$r_0 = \int_a^b \mathrm{Re}\left[e^{-i\theta_0}\, w(t)\right] dt$$

And since we know that $\mathrm{Re}\{z\} \leq |z|$

we can write:

$$\mathrm{Re}\left[e^{-i\theta_0}\, w(t)\right] \leq \left|e^{-i\theta_0}\, w(t)\right| = \cancelto{1}{\left|e^{-i\theta_0}\right|}\,|w(t)|$$

$$\mathrm{Re}\left[e^{-i\theta_0}\, w(t)\right] \leq |w(t)|$$

And this leads to:

$$\boxed{\left|\int_a^b w(t)\,dt\right| \leq \left|\int_a^b |w(t)|\,dt\right| \qquad (a<b)}$$

Example:

$$\int_0^{2\pi} e^{i(m-n)\theta}\,d\theta = \begin{cases} \displaystyle\int_0^{2\pi} e^{i0\theta}\,d\theta = \int_0^{2\pi} d\theta = 2\pi & (m=n) \\ \displaystyle\left.\frac{e^{i(m-n)\theta}}{i(m-n)}\right|_0^{2\pi} = 0 & \forall\, m \neq n \end{cases}$$

$m, n \in \mathbb{Z}$

$$f(z) = u + iv$$

$$f(t) = u[x(t), y(t)] + i\,v[x(t), y(t)] = f[z(t)]$$

$$\oint_C f(z)\,dz = \int_a^b f[z(t)]\left(\dot{x} + i\dot{y}\right)dt \quad \Bigg| \quad C:\ z = z(t),\quad a \le t \le b$$

$$dz = \frac{dz}{dt}\,dt \ ; \quad \dot{x} = \frac{dx}{dt} \ , \quad \dot{y} = \frac{dy}{dt}$$

$$\oint_C f(z)\,dz = \oint_C [u\dot{x} - v\dot{y}]\,dt + i\oint_C [u\dot{y} + v\dot{x}]\,dt$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

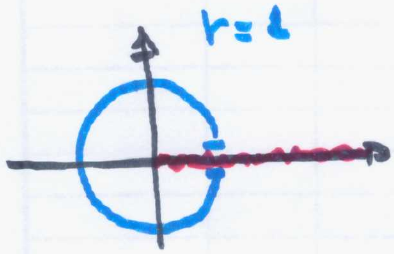
Example: P100 ex. 6 | path $\Rightarrow C:\ |z| = 1 \Rightarrow z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$

$$f(z) = z^{-1+i} = \exp[(-1+i)\log(z)]$$

$$= \exp\{(-1+i)[\cancelto{0}{\ln(1)} + i\theta]\} = e^{-\theta(1+i)}$$

at $\theta = 0$, $f[z(0)] = 1$

at $\theta = 2\pi$, $f[z(2\pi)] = e^{-2\pi}$



$0 < \theta < 2\pi$

$$\int_0^{2\pi} f[z(\theta)]\,z'(\theta)\,d\theta = \int_0^{2\pi} e^{-\theta(1+i)}\, i e^{i\theta}\,d\theta =$$

$$= \int_0^{2\pi} i e^{-\theta}\,d\theta = -i e^{-\theta}\Big|_0^{2\pi} = i\left(1 - e^{-2\pi}\right)$$

Antiderivatives . P. 102

$\exists\, F(z) \;\Big|\; \dfrac{dF}{dz} = F'(z) = f(z) \;\Big|$ F(z) is analytic in a neighborhood of z

Theorem: Assume that f is continuous in a domain $\mathcal{D}$. then, if any of the following statement holds, the so do the other two.

a) f has an antiderivative in $\mathcal{D}$ ($F' = f$)

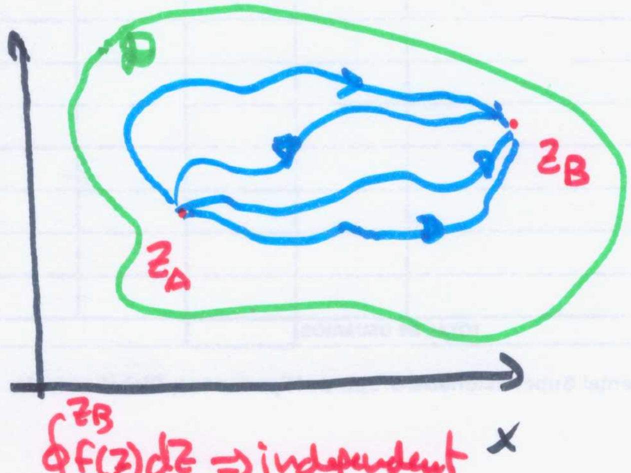
b) $\oint_c f\,dz \;\Big|\; c \subset \mathcal{D}$ and $c: z(t) \;\Big|\; a \le t \le b$, z_a, z_B

all have the same value

c) all integrals around closed contours $c \subset \mathcal{D}$ are zero ($z_a \equiv z_B$)

a) $\Rightarrow \exists\, F(z) \;\Big|\; F'(z) = f(z) \quad \forall z \in \mathcal{D}$

b) $\Rightarrow$



$\int_{z_A}^{z_B} f(z)\,dz \Rightarrow$ independent of path

c)

$\oint_c f(z)\,dz = 0$

On assuming that statement (a) is true $\exists F(z) \mid F'(z) = f(z) \quad \forall z \in \mathcal{D}$

$$\frac{dF[z(t)]}{dt} = \frac{dF}{dz}\frac{dz}{dt} = f(z)\frac{dz}{dt}$$

$$\oint_C f(z)\,dz = \int_a^b f[z(t)]\frac{dz}{dt}\,dt = \int_a^b \frac{F(t)}{dt}\,dt =$$

$$= \Big\{F[z(t)]\Big\}_a^b = F[z(a)] - F[z(b)] =$$

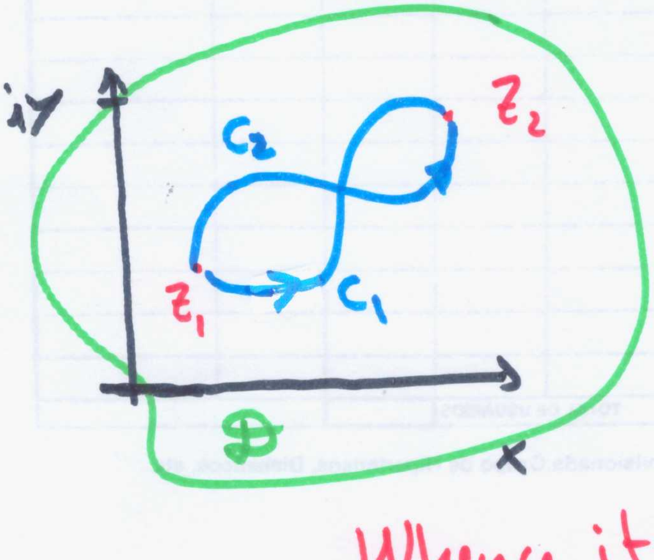
$$\boxed{\oint_C f(z)\,dz = F(z_A) - F(z_B)}$$

which is the same for any $C \subset \mathcal{D}$

Therefore, (b) follows from (a)

Both arcs go from z_1 to z_2

So we'll make: $C = C_1 - C_2$ thus reversing C_2



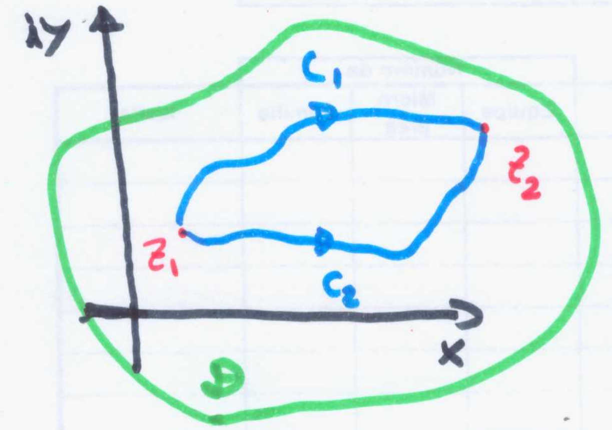
from (b), we know that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Whence it comes: $\oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0 = \oint_C f\,d$

Therefore we see that if (a) holds, both (b) and (c) follow.

Now on assuming that (c) is true:



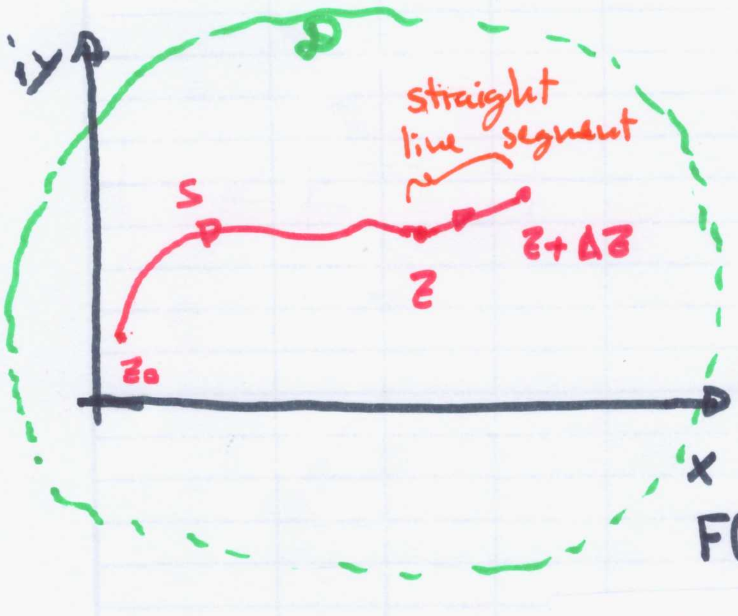
$C = C_1 - C_2$

since (c) holds, we have:

$$\oint_C f\,dz = 0 \Rightarrow \oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0$$

$$\therefore \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \Rightarrow \text{(b) holds}$$

How about statement (a), in this case?



$$F(z) \equiv \int_{z_0}^{z} f(s)\,ds$$

On the basis of this definition, let's compute:

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(s)\,ds$$

$$\int_z^{z+\Delta z} 1\,dz = \left[z\right]_z^{z+\Delta z} = \Delta z \qquad \text{Also, } \int_z^{z+\Delta z} f(z)\,ds = f(z)\,\Delta z$$

bear in mind that this integral is in "s", not "z"

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z}\left[f(s)-f(z)\right]ds$$

Since $f(z)$ is continuous in $\mathcal{D}$, by hypothesis, we have:

$$\forall \varepsilon > 0 \; \exists \delta > 0,\ \varepsilon, \delta \in \mathbb{R} \;\Big|\; |s-z| < \delta \Longrightarrow |f(s)-f(z)| < \varepsilon$$

Hence we have:

$$|\Delta z| < \delta \Longrightarrow \left|\int_z^{z+\Delta z}\left[f(s)-f(z)\right]ds\right| \leq \int_z^{z+\Delta z}|f(s)-f(z)|\,ds$$

$$\leq \varepsilon \int_z^{z+\Delta z}\left|\frac{ds}{dt}\right|dt = \varepsilon\,|\Delta z|$$

Which implies that:

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\,\varepsilon\,|\Delta z| = \varepsilon$$

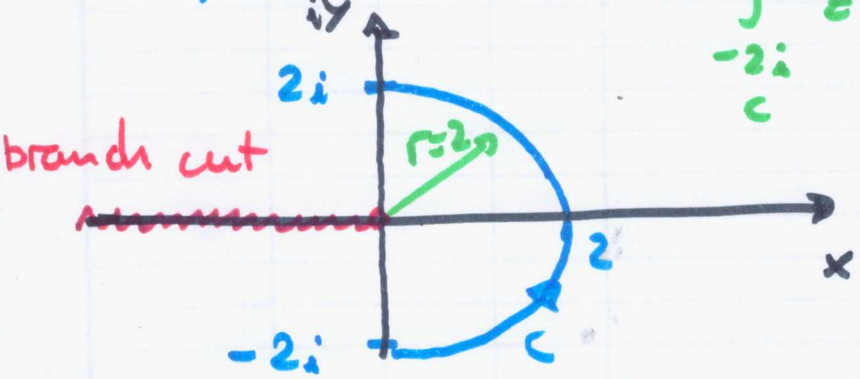
Therefore, in the limit as $\Delta z \to 0$ we must have

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) \Rightarrow F'(z) = f(z)$$

Therefore statement (a) holds:

$$[(b) \Leftrightarrow (c)] \Rightarrow (a)$$

Example:



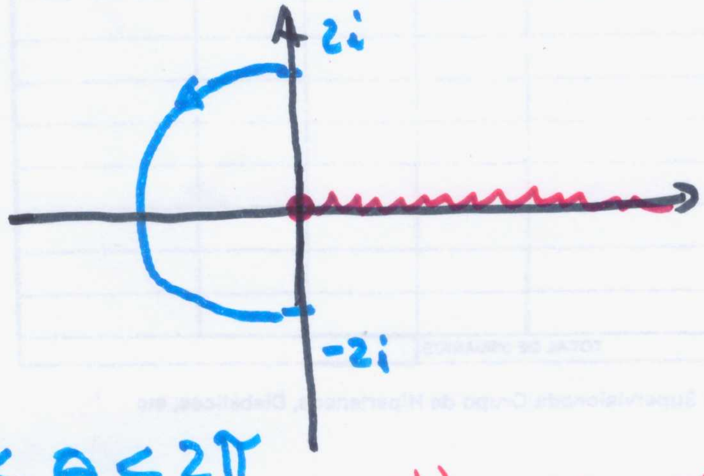
$$\int_{-2i,\,C}^{i} \frac{dz}{z} = \Big[\log(z)\Big]_{-2i}^{2i}$$

$$= \cancel{\Big[\ln(r)\Big]_{|-2|}^{|2|}}^{\,0} + \Big[i\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= i\pi$$

$\text{Arg}(z) = \theta \Rightarrow -\pi < \theta < \pi$

$$[\log z]' = \frac{1}{z}$$

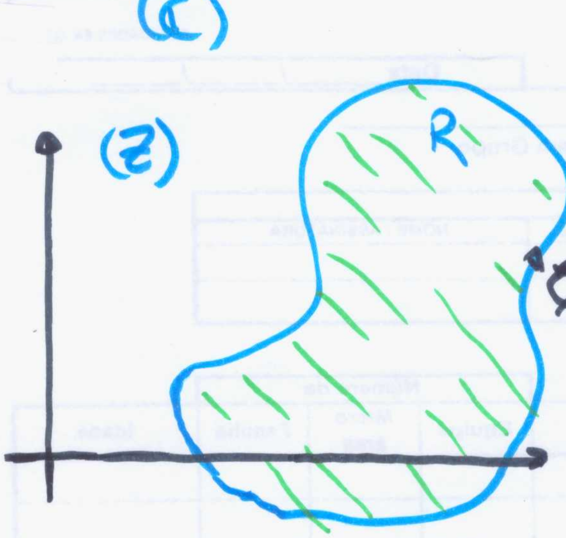


$$\int_{2i}^{-2i} \frac{dz}{z} = \Big[\log(z)\Big]_{2i}^{-2i} =$$

$$= \Big[i\theta\Big]_{\frac{\pi}{2}}^{3\pi/2} = i\pi$$

$0 < \theta < 2\pi$

it's independent of "r" and Full circle yield $i2\pi$



$\exists f(z)$ is analytic within (R) and on the contour $\mathcal{C}$

analytic $\Rightarrow$ meets C.R. conditions

$$\oint_{\mathcal{C}} f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt = \int_a^b [u(t)+i v(t)][\dot{x}(t)+i\dot{y}(t)]dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})dt + i\int_a^b (v\dot{x} + u\dot{y})dt = \oint_c (u+iv)(dx+idy)$$

Let's assume we have $P(x,y)$ and $Q(x,y)$ in $\mathbb{R}^3$ with are • C^1 (continuous up to and including 1st order partial derivatives).

Green's 1st identity:

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Then, if we take $P = u + iv$ and $Q = -v + iu$ where $u = u(x,y)$ and $v = v(x,y)$ are both C^1.

that is, $\exists\ u_x, u_y, v_x, v_y$ and they are continuous.

Since by hypothesis $f(z) = u + iv$ is analytic within and on $\mathcal{C}$ $(R \cup \mathcal{C})$, we have (C.R.): $u_x = v_y$ and $u_y = -v_x$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= u_y + i v_y \\ \frac{\partial Q}{\partial x} &= -v_x + i u_x \end{aligned}\right\} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \cancel{-v_x} + \cancel{i u_x} - \cancel{u_y} + \cancel{- i v_y} = 0$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \quad \forall z \in R \cup \mathcal{C}$$

with that result we can write:

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy) = \oint_C (u + iv)\,dx + \oint_C (-v + iu)\,dy$$

$$= \oint_C P\,dx + Q\,dy = 0 \quad \forall z \in R \cup \mathcal{C}$$

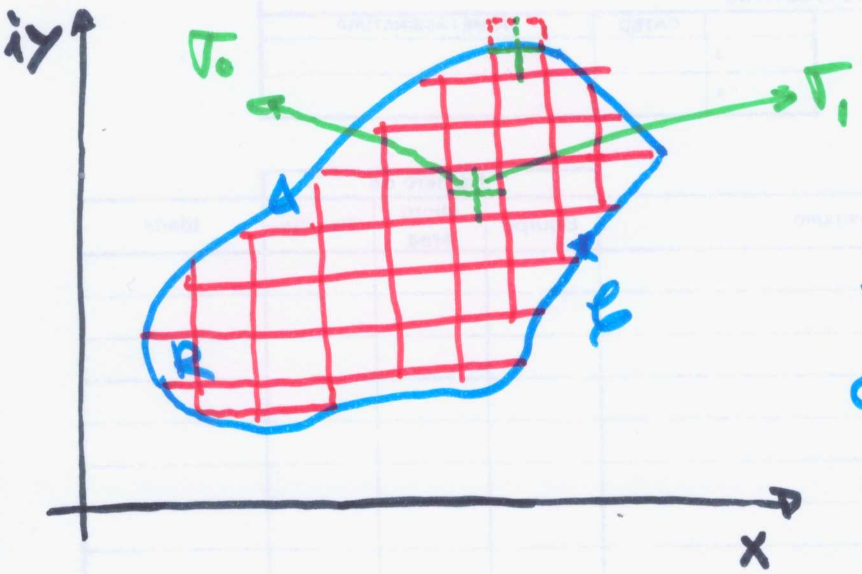
Thus far, we have:

given $f(z)$ analytic in $R \cup \mathcal{C}$ with $u(x,y)$ and $v(x,y)$ that are C^1 there, then:

$$\oint_{\mathcal{C}} f(z)\,dz = 0$$

Read Churchill's book page 109 _ 113

Sections 36 and 37



(Lemma)

$f(z)$ analytic in R and on $\mathcal{C}$ ($R \cup \mathcal{C}$)

$\forall\, \varepsilon > 0,\ \varepsilon \in \mathbb{R}$, the region R can be covered with a Finite number of squares and partial squares, indexed by $j = 1, 2, \ldots n$, such that in each one there is a fixed point z_j for which

$$\left| \frac{f(z) - f(z_j)}{(z - z_j)} - f'(z_j) \right| < \varepsilon \quad \forall\, z \in \sigma_j$$

where σ_j is a square (entire of partial)

This inequality shall be proved by contradiction

Let's assume for a moment that $\exists\, \sigma_j$ where the inequality does not hold: ie, $\nexists\, z_j$ that has that property

On assuming that there is a square or a partial square where we can not find such a point z_j.

we'll then further divide that square (or partial square) into 4 smaller units by splitting each side into halves — do the same for partial squares and then disregard the portion that lies outside C

⋮

Now, we know that $f(z)$ is analytic in $R \cup C$ (by hypothesis). Hence $f'(z)$ exists and it is well defined there.

Therefore, for a given nested (finite) partition of "halving" the squares sides we eventually reach a point where the diagonal of the square "d" is such that $d < \delta$ where $\delta > 0$, $\delta \in \mathbb{R}$ and $\exists$ a z_0 in that square such that

$$|z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{(z - z_0)} - f'(z_0) \right| < \varepsilon$$

Since this is a nested finite partition, that is

$$\sigma_K \subset \sigma_{K-1} \subset \sigma_{K-2} \ldots\ldots \subset \sigma_J$$

and $d\Big|_{\sigma_K} < \delta$ so that

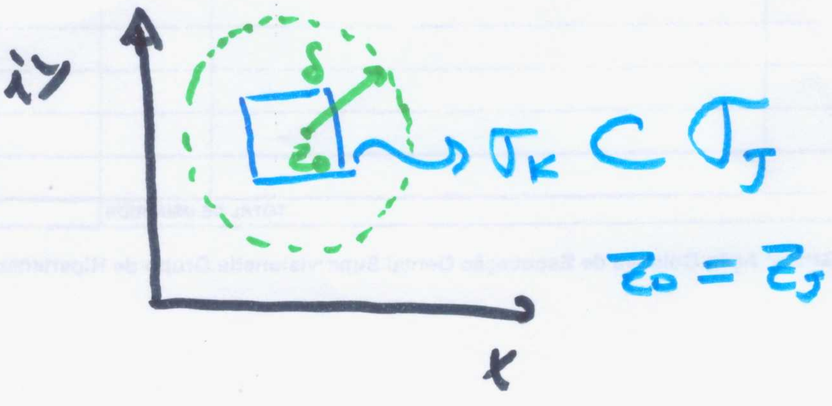
$$|z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

Since $f'(z)\Big|_{z=z_0}$ exists by hypothesis

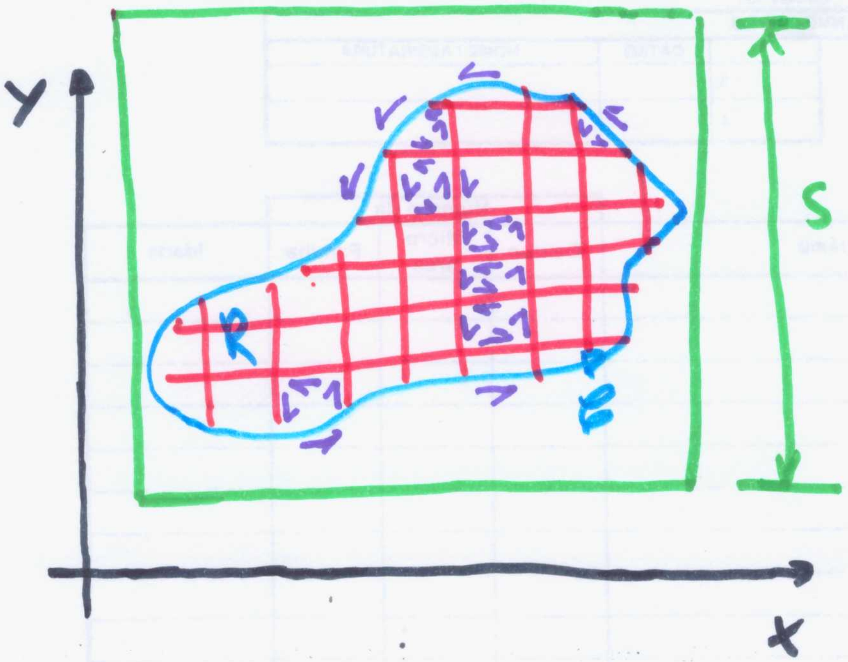
Hence $z_0 \in \sigma_K \Rightarrow z_0 \in \sigma_J$

Therefore, we can take $z_J\Big|_{\sigma_J} = z_0$

As a result of this, we get to a contradition, and the Lemma is proved.



Proof of Cauchy-Goursatz Theorem



$f(z)$ is analytic in $R \cup C$

We want to prove that $\oint_C f(z)\,dz = 0$

Then, on making use of the previous Lemma, let us define

$$\delta_j(z) = \begin{cases} \dfrac{f(z)-f(z_j)}{(z-z_j)} - f'(z) & \text{for } z \neq z_j \\ 0 & \text{for } z = z_j \end{cases}$$

Now, what our Lemma tells us is that

$$|\delta_j(z)| < \varepsilon$$

Moreover, we get that $\delta_j(z)$ is a continuous function in $R \cup C$, since $f(z)$ is continuous and, hence, we can write:

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

Let's now have C_J for $J = 1 \ldots n$ denote the positively oriented contours of the squares and partial squares $\mathcal{T}_J$.

From the definition of $\delta_J(z)$ we have:

$$f(z) = f(z_J) - z_J f'(z_J) + f'(z_J) z + (z - z_J)\delta_J(z)$$

Whence we can make:

$$\oint_{C_J} f(z)\,dz = [f(z_J) - z_J f'(z_J)] \oint_{C_J} 1\,dz + f'(z_J) \oint_{C_J} z\,dz + \oint_{C_J} (z - z_J)\,dz$$

Now, functions 1 and z are defined and have anti-derivatives in $R \cup \mathcal{C}$:

$$[z]' = 1 \quad \text{and} \quad \left[\frac{z^2}{2}\right]' = z$$

Hence these integrals, when computed along any closed contour will be <u>Zero</u>

$$\oint_{C_J} 1\,dz = [z]_{z_K}^{z_K} = 0 \qquad \oint_{C_J} z\,dz = \left[\frac{z^2}{2}\right]_{z_K}^{z_K} = 0$$

Therefore we get:

$$\oint_{C_J} f(z)\,dz = \oint_{C_J} (z-z_J)\,\delta_J(z)\,dz$$

And the integral of $f(z)$ over C as a whole corresponds to:

$$\oint_{C} f(z)\,dz = \underbrace{\sum_{J=1}^{n}}_{\text{finite summation}} \oint_{C_J} f(z)\,dz = \sum_{J=1}^{n} \oint_{C_J} (z-z_J)\,\delta_J(z)\,dz$$

Now, we can make use of the 1st triangular inequality to write:

$$\left|\oint_{C} f(z)\,dz\right| \leq \sum_{J=1}^{n} \left|\oint_{C_J} (z-z_J)\,\delta_J(z)\,dz\right|$$

However, we know that:

$$|z-z_J| \leq \sqrt{2}\,S_J \qquad \text{and} \qquad |\delta_J(z)| < \varepsilon$$

where S_J is the side of the square C_J,

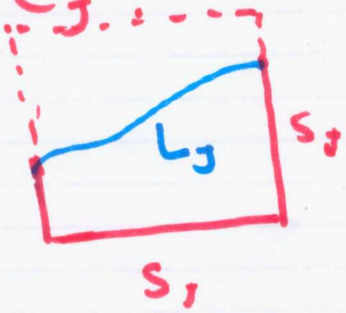
which implies that: $|(z-z_J)\,\delta_J(z)| = |z-z_J|\,|\delta_J(z)| < \sqrt{2}\,S_J\,\varepsilon$

Let's now denote the area of a complete square
by: $A_J = s_J^2$. Then we write:

$$\left|\oint_{C_J}(z-z_J)\,\delta_J(z)\,dz\right| < \sqrt{2}\,s_J\,\varepsilon\,(4s_J) = \sqrt{2}\,\varepsilon\,4A_J$$

where $4s_J$ is the perimeter of $\Gamma_J \Rightarrow \oint_{\Gamma_J} dz = 4s_J$

On the other hand, for the so-called partial squares, we can claim that their contour length does <u>not</u> exceed $(4s_J + L_J)$ where L_J is the length of the portion of Γ that is part of that C_J:



Hence, for a partial square, we have:

$$\left|\oint_{C_J}(z-z_J)\,\delta_J(z)\,dz\right| < \sqrt{2}\,s_J\,\varepsilon\,(4s_J+L_J) < 4\sqrt{2}\,A_J\,\varepsilon + \sqrt{2}\,s_J L_J$$

And, to further ensure that the inequality holds, and that it does so for any C_J, we make

$$\left|\oint_{C_J}(z-z_J)\,\delta_J(z)\,dz\right| < 4\sqrt{2}\,A_J\,\varepsilon + \sqrt{2}\,S L_J\,\varepsilon$$

Finally, we make

$$\left| \oint_C f(z)\, dz \right| < \left(4\sqrt{2}\, S^2 + \sqrt{2}\, S L \right) \varepsilon$$

where S is the length of the side of the biggest square and L is the whole length of $\mathcal{C}$.

Now, since ε is arbitrary, it can be made as small as we want. Then, the LHS must be zero, for it is independent of ε and that is the only way of having the inequality hold for any ε.

Therefore we get that:

$$\boxed{\oint_{\mathcal{C}} f(z)\, dz = 0}$$

for f(z) analytic in $R \cup \mathcal{C}$