AULAS ANTERIORES 09/10/2020

$\langle \cdot , \cdot \rangle$ define um produto interno em um EV V se satisfizer 4 propriedades ($\forall \vec{u}, \vec{v}, \vec{w} \in V$ e $\forall \alpha \in \mathbb{R}$):

P1. Simetria: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

P2. Positividade: $\langle \vec{u}, \vec{u} \rangle \geq 0$ e $\langle \vec{u}, \vec{u} \rangle = 0 \Leftrightarrow \vec{u} = \vec{0}$

P3. Distributividade: $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

P4. Homogeneidade: $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

Norma:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

$$\longrightarrow \quad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \quad \therefore \|\vec{u}\| = 1$$

VETOR NORMALIZADO

IMPORTANTE:

* Norma depende do produto interno definido.
* Vetor será normalizado em relação a um dado produto interno.

Slide 12 - Exemplos : Ortogonalidade

$\vec{v_i} = (x_i, y_i, z_i) \in \mathbb{R}^3$

1) $B = \{ \underset{\vec{v_1}}{(1,1,1)}, \underset{\vec{v_2}}{(-1,1,0)}, \underset{\vec{v_3}}{(-1,-1,2)} \}$

B será ortogonal se $< \vec{v_i}, \vec{v_j} > = 0$ para $i \neq j$ $\therefore \begin{cases} < \vec{v_1}, \vec{v_2} > = 0 \\ < \vec{v_1}, \vec{v_3} > = 0 \\ < \vec{v_2}, \vec{v_3} > = 0 \end{cases}$

PI Usual : $< \vec{v_i}, \vec{v_j} > = x_i x_j + y_i y_j + z_i z_j$

$< \vec{v_1}, \vec{v_2} > = x_1 x_2 + y_1 y_2 + z_1 z_2 = 1(-1) + 1(1) + 1(0) = 0$

$< \vec{v_1}, \vec{v_3} > = x_1 x_3 + y_1 y_3 + z_1 z_3 = 1(-1) + 1(-1) + 1(2) = 0$

$< \vec{v_2}, \vec{v_3} > = x_2 x_3 + y_2 y_3 + z_2 z_3 = (-1)(-1) + 1(-1) + 0(2) = 0$

$\therefore$ B é um conjunto ortogonal

2. a) PI Usual :

$$< f, g > = \int_a^b f(x)\, g(x)\, dx , \quad \forall\, f, g \in \mathcal{C}([a,b])$$

$f(x) = 1$; $g(x) = x$; $[a,b] = [-1, 1]$.

$$\therefore < f, g > = \int_{-1}^{1} (1)(x)\, dx = \left[\frac{x^2}{2} \right]_{-1}^{1} = 0$$

$\therefore$ $f(x)$ e $g(x)$ são ortogonais

b) Para B_1, deve-se lembrar da relação trigonométrica:

$$\text{sen}(nx)\,\text{sen}(mx) = \frac{\cos((n-m)x) - \cos((n+m)x)}{2}$$

Portanto:

$$\langle \text{sen}(nx), \text{sen}(mx) \rangle = \int_{-\pi}^{\pi} \text{sen}(nx)\,\text{sen}(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)\,dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\,dx$$

$$= \frac{1}{2(n-m)}\Big[\text{sen}((n-m)x)\Big]_{-\pi}^{\pi} - \frac{1}{2(n+m)}\Big[\text{sen}((n+m)x)\Big]_{-\pi}^{\pi} = \underline{\underline{0}}$$

$\therefore$ $\langle \text{sen}(nx), \text{sen}(mx) \rangle = 0$ se $n \neq m$ e B_1 é um conjunto ortogonal.

Para B_2, deve-se lembrar da relação trigonométrica:

$$\cos(nx)\cos(mx) = \frac{\cos((n-m)x) + \cos((n+m)x)}{2}$$

Portanto:

$$\langle \cos(nx), \cos(mx) \rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\,dx$$

$$= \frac{1}{2(n-m)}\Big[\text{sen}((n-m)x)\Big]_{-\pi}^{\pi} + \frac{1}{2(n+m)}\Big[\text{sen}((n+m)x)\Big]_{-\pi}^{\pi} = \underline{\underline{0}}$$

$\therefore$ $\langle \cos(nx), \cos(mx) \rangle = 0$ se $n \neq m$ e B_2 é um conjunto ortogonal.

Slide 15 - Exercícios

1. O ângulo entre dois vetores não-nulos decorre da Desigualdade de Cauchy-Schwarz:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\| \longrightarrow \cos\theta = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\| \, \|\vec{v}\|}$$

a) $V = \mathbb{R}^4$: $\vec{u} = (1,1,1,0)$; $\vec{v} = (2,1,-2,1)$

PI Usual : $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i$

$$\langle \vec{u}, \vec{v} \rangle = 1 \cdot 2 + 1 \cdot 1 + 1 \cdot (-2) + 0 = 1$$

$$\langle \vec{u}, \vec{u} \rangle = 1 + 1 + 1 = 3 \longrightarrow \|\vec{u}\| = \sqrt{3}$$

$$\langle \vec{v}, \vec{v} \rangle = 4 + 1 + 4 + 1 = 10 \longrightarrow \|\vec{v}\| = \sqrt{10}$$

$$\therefore \cos\theta = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{1}{\sqrt{3}\sqrt{10}} = \frac{\sqrt{30}}{30} \longrightarrow \theta = \arccos\left(\frac{\sqrt{30}}{30}\right)$$

b) $V = P_2(x)$: $p = -1 + 5x + 2x^2$; $q = 2 + 4x - 4x^2$

Produto interno : $\langle p, q \rangle = \sum_{i=0}^{n} a_i b_i$

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 = (-1) \cdot 2 + 5 \cdot 4 + 2(-4) = 10$$

$$\langle p, p \rangle = a_0^2 + a_1^2 + a_2^2 = 30 \longrightarrow \|p\| = \sqrt{30}$$

$$\langle q, q \rangle = b_0^2 + b_1^2 + b_2^2 = 36 \longrightarrow \|q\| = \sqrt{36} = 6$$

$$\therefore \cos\theta = \frac{|\langle p, q \rangle|}{\|p\| \, \|q\|} = \frac{10}{6\sqrt{30}} = \frac{5\sqrt{30}}{3 \cdot 30} = \frac{\sqrt{30}}{18}$$

$$\theta = \arccos\left(\frac{\sqrt{30}}{18}\right)$$

c) $V = M(2,2)$: $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$; $B = \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix}$.

PI Usual : $\langle A, B \rangle = tr(B^T A)$

$B^T = \begin{bmatrix} -3 & 4 \\ 0 & 2 \end{bmatrix}$ ⟶ $B^T A = \begin{bmatrix} -3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ -2 & 6 \end{bmatrix}$

$B^T B = \begin{bmatrix} -3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 8 \\ 8 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ ⟶ $A^T A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 25 \end{bmatrix}$

$$\langle A, B \rangle = tr \langle B^T A \rangle = \sum_{i=1}^{2} a_{ii} = -10 + 6 = 4$$

$$\langle A, A \rangle = tr \langle A^T A \rangle = \sum_{i=1}^{2} a_{ii} = 5 + 25 = 30 \longrightarrow \|A\| = \sqrt{30}$$

$$\langle B, B \rangle = tr \langle B^T B \rangle = \sum_{i=1}^{2} a_{ii} = 25 + 4 = 29 \longrightarrow \|B\| = \sqrt{29}$$

$$\therefore \quad \cos\theta = \frac{|\langle A, B \rangle|}{\|A\| \, \|B\|} = \frac{4}{\sqrt{30}\sqrt{29}} = \frac{4\sqrt{30}\sqrt{29}}{30 \cdot 29} = \frac{\sqrt{870}}{174}$$

$$\theta = \arccos\left(\frac{2\sqrt{870}}{435}\right)$$

2. $\vec{u} \perp \vec{v}$ se $\langle \vec{u}, \vec{v} \rangle = 0$, $\vec{u}$ e $\vec{v}$ não nulos

a) $V = \mathbb{R}^4$: $\vec{u} = (-4, 6, -10, 1)$; $\vec{v} = (2, 1, -2, 9)$

PI Usual : $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i$

$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = -8 + 35 = -27$

$\langle \vec{u}, \vec{v} \rangle \neq 0 \therefore \vec{u}$ e $\vec{v}$ não são ortogonais

b) $V = P_2(x)$: $p = 1 - x + 2x^2$; $q = 2x + x^2$

Produto interno : $\langle p, q \rangle = \sum_{i=0}^{n} a_i b_i$

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 = 1 \cdot 0 + (-1) \cdot 2 + 2 \cdot 1 = 0$

$\langle p, q \rangle = 0 \therefore p$ e q são ortogonais

c) $V = M(2,2)$: $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

PI Usual : $\langle A, B \rangle = tr(B^T A)$ ou $\langle B, A \rangle = tr(A^T B)$

$B^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \longrightarrow B^T A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$

$\langle A, B \rangle = tr \langle B^T A \rangle = \sum_{i=1}^{2} a_{ii} = 2 - 2 = 0$

$\langle A, B \rangle = 0 \therefore A$ e B são ortogonais

3. $\langle \vec{u}, \vec{v} \rangle = af + be + cd \quad \begin{cases} \vec{u} = (a, b, c) \\ \vec{v} = (d, e, f) \end{cases}, \quad V = \mathbb{R}^3.$

$$B = \{ \underset{\vec{v}_1}{(1,1,1)}, \underset{\vec{v}_2}{(\alpha, 0, -1)}, \underset{\vec{v}_3}{(1, \beta, 1)} \}$$

$\alpha, \beta = ?$ para que B seja uma base ortogonal.

B será ortogonal se $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ para $i \neq j$ $\therefore \begin{cases} \langle \vec{v}_1, \vec{v}_2 \rangle = 0 \\ \langle \vec{v}_1, \vec{v}_3 \rangle = 0 \\ \langle \vec{v}_2, \vec{v}_3 \rangle = 0 \end{cases}$

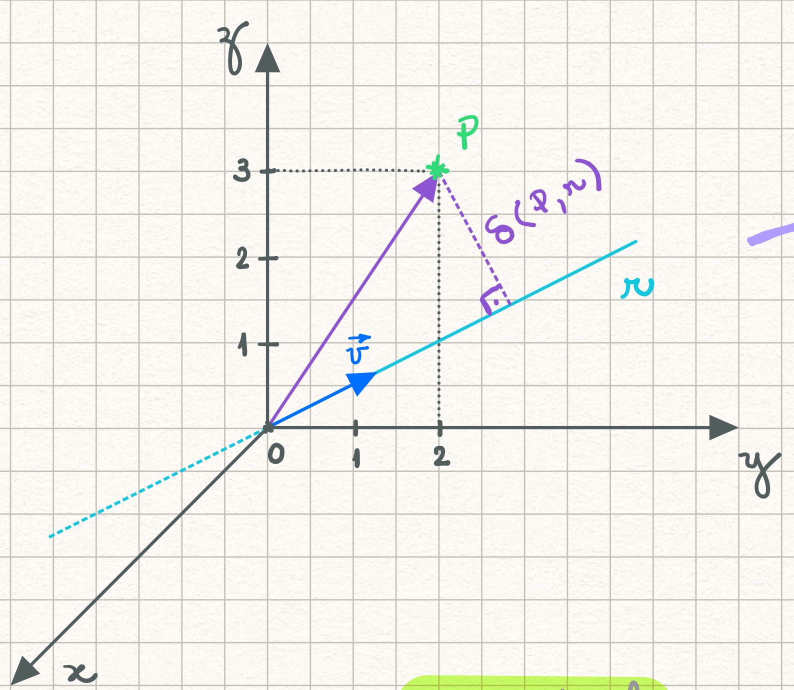
Aplicando o produto interno aqui definido:

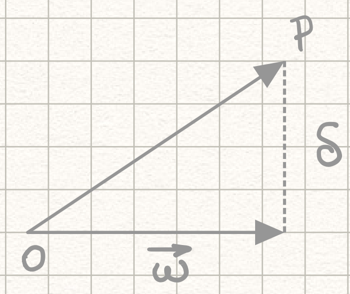
$$\langle \vec{v}_1, \vec{v}_2 \rangle = \langle (1,1,1), (\alpha, 0, -1) \rangle = -1 + \alpha = 0 \quad \therefore \underline{\underline{\alpha = 1}}$$

$$\langle \vec{v}_1, \vec{v}_3 \rangle = \langle (1,1,1), (1, \beta, 1) \rangle = 1 + \beta + 1 = 0 \quad \therefore \underline{\underline{\beta = -2}}$$

$\therefore \alpha = 1 ; \beta = -2$

4.





$\delta(P, r) = \| \overrightarrow{OP} - \vec{w} \|,$

$\vec{w} = \text{proj}_{\vec{v}} \overrightarrow{OP}$

* P1 Usual

$$\text{proj}_{\vec{v}}\,\overrightarrow{OP} = \left(\frac{\langle \overrightarrow{OP}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\right)\vec{v} \quad , \quad \overrightarrow{OP} = P - O = (0,2,3)$$

$$\langle \overrightarrow{OP}, \vec{v} \rangle = \langle (0,2,3), (1,2,1) \rangle = 0\cdot 1 + 2\cdot 2 + 3\cdot 1 = 7$$

$$\langle \vec{v}, \vec{v} \rangle = \langle (1,2,1), (1,2,1) \rangle = 1\cdot 1 + 2\cdot 2 + 1\cdot 1 = 6$$

$$\longrightarrow \vec{\omega} = \text{proj}_{\vec{v}}\,\overrightarrow{OP} = \frac{7}{6}\,\vec{v} = \frac{7}{6}(1,2,1) = \left(\frac{7}{6}, \frac{14}{6}, \frac{7}{6}\right)$$

$$\longrightarrow \overrightarrow{OP} - \vec{\omega} = (0,2,3) - \left(\frac{7}{6}, \frac{14}{6}, \frac{7}{6}\right) = \left(-\frac{7}{6}, -\frac{2}{6}, \frac{11}{6}\right)$$

Portanto:

$$\delta(P, r) = \| \overrightarrow{OP} - \vec{\omega} \| \quad , \quad \vec{d} = \overrightarrow{OP} - \vec{\omega}$$

$$\therefore \| \vec{d} \| = \sqrt{\langle \vec{d}, \vec{d} \rangle}$$

$$\langle \vec{d}, \vec{d} \rangle = \left(-\frac{7}{6}\right)^2 + \left(-\frac{2}{6}\right)^2 + \left(\frac{11}{6}\right)^2 = \frac{174}{36} = \frac{29}{6}$$

$$\longrightarrow \| \vec{d} \| = \sqrt{\frac{29}{6}} = \frac{\sqrt{174}}{6} = \delta(P, r)$$

Assim, $\delta(P, r) = \boxed{\dfrac{\sqrt{174}}{6}}$