2.3 Matrix Norms

The analysis of matrix algorithms requires use of matrix norms. For example, the quality of a linear system solution may be poor if the matrix of coefficients is "nearly singular." To quantify the notion of near-singularity, we need a measure of distance on the space of matrices. Matrix norms can be used to provide that measure.

2.3.1 Definitions

Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, the definition of a matrix norm should be equivalent to the definition of a vector norm. In particular, $f:\mathbb{R}^{m\times n} \to \mathbb{R}$ is a matrix norm if the following three properties hold:

$$\begin{aligned} f(A) &\geq 0, & & A \in \mathbb{R}^{m\times n}, \quad (f(A) = 0 \text{ iff } A = 0)\\ f(A+B) &\leq f(A) + f(B), & & A, B \in \mathbb{R}^{m\times n},\\ f(\alpha A) &= |\alpha| f(A), & & \alpha \in \mathbb{R}, A \in \mathbb{R}^{m\times n}. \end{aligned}$$

As with vector norms, we use a double bar notation with subscripts to designate matrix norms, i.e., $\| A \| = f(A)$.

The most frequently used matrix norms in numerical linear algebra are the Frobenius norm

$$\| A \|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} \tag{2.3.1}$$

and the p-norms

$$\| A \|_p = \sup_{x\neq 0} \frac{\| Ax \|_p}{\| x \|_p}. \tag{2.3.2}$$

Note that the matrix p-norms are defined in terms of the vector p-norms discussed in the previous section. The verification that (2.3.1) and (2.3.2) are matrix norms is left as an exercise. It is clear that $\| A \|_p$ is the p-norm of the largest vector obtained by applying A to a unit p-norm vector:

$$\| A \|_p = \sup_{x\neq 0} \left\| A\left(\frac{x}{\| x \|_p}\right)\right\|_p = \max_{\|x\|_p = 1} \| Ax \|_p .$$

It is important to understand that (2.3.2) defines a family of norms—the 2-norm on $\mathbb{R}^{3\times 2}$ is a different function from the 2-norm on $\mathbb{R}^{5\times 6}$. Thus, the easily verified inequality

$$\| AB \|_p \leq \| A \|_p \| B \|_p, \qquad A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times q} \tag{2.3.3}$$

is really an observation about the relationship between three different norms. Formally, we say that norms f_1, f_2, and f_3 on $\mathbb{R}^{m\times q}$, $\mathbb{R}^{m\times n}$, and $\mathbb{R}^{n\times q}$ are *mutually consistent* if for all matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$ we have $f_1(AB) \leq f_2(A)f_3(B)$, or, in subscript-free norm notation:

$$\| AB \| \leq \| A \| \| B \|. \tag{2.3.4}$$

Not all matrix norms satisfy this property. For example, if $\| A \|_\Delta = \max |a_{ij}|$ and

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

then $\| AB \|_\Delta > \| A \|_\Delta \| B \|_\Delta$. For the most part, we work with norms that satisfy (2.3.4).

The p-norms have the important property that for every $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$ we have

$$\| Ax \|_p \le \| A \|_p \| x \|_p.$$

More generally, for any vector norm $\| \cdot \|_\alpha$ on $\mathbb{R}^n$ and $\| \cdot \|_\beta$ on $\mathbb{R}^m$ we have $\| Ax \|_\beta \le \| A \|_{\alpha,\beta} \| x \|_\alpha$ where $\| A \|_{\alpha,\beta}$ is a matrix norm defined by

$$\| A \|_{\alpha,\beta} = \sup_{x\neq 0} \frac{\| Ax \|_\beta}{\| x \|_\alpha}. \tag{2.3.5}$$

We say that $\| \cdot \|_{\alpha,\beta}$ is *subordinate* to the vector norms $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$. Since the set $\{x \in \mathbb{R}^n : \| x \|_\alpha = 1\}$ is compact and $\| \cdot \|_\beta$ is continuous, it follows that

$$\| A \|_{\alpha,\beta} = \max_{\|x\|_\alpha = 1} \| Ax \|_\beta = \| Ax_* \|_\beta \tag{2.3.6}$$

for some $x_* \in \mathbb{R}^n$ having unit α-norm.

2.3.2 Some Matrix Norm Properties

The Frobenius and p-norms (especially $p = 1, 2, \infty$) satisfy certain inequalities that are frequently used in the analysis of a matrix computation. If $A \in \mathbb{R}^{m\times n}$ we have

$$\| A \|_2 \le \| A \|_F \le \sqrt{\min\{m,n\}}\, \| A \|_2, \tag{2.3.7}$$

$$\max_{i,j} |a_{ij}| \le \| A \|_2 \le \sqrt{mn} \max_{i,j} |a_{ij}|, \tag{2.3.8}$$

$$\| A \|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|, \tag{2.3.9}$$

$$\| A \|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|, \tag{2.3.10}$$

$$\frac{1}{\sqrt{n}} \| A \|_\infty \le \| A \|_2 \le \sqrt{m}\, \| A \|_\infty, \tag{2.3.11}$$

$$\frac{1}{\sqrt{m}} \| A \|_1 \le \| A \|_2 \le \sqrt{n}\, \| A \|_1. \tag{2.3.12}$$

If $A \in \mathbb{R}^{m\times n}$, $1 \le i_1 \le i_2 \le m$, and $1 \le j_1 \le j_2 \le n$, then

$$\| A(i_1{:}i_2, j_1{:}j_2) \|_p \le \| A \|_p. \tag{2.3.13}$$

The proofs of these relationships are left as exercises. We mention that a sequence $\{A^{(k)}\} \in \mathbb{R}^{m\times n}$ *converges* if there exists a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$\lim_{k\to\infty} \| A^{(k)} - A \| = 0.$$

The choice of norm is immaterial since all norms on $\mathbb{R}^{m\times n}$ are equivalent.

2.3.3 The Matrix 2-Norm

A nice feature of the matrix 1-norm and the matrix ∞-norm is that they are easy, $O(n^2)$ computations. (See (2.3.9) and (2.3.10).) The calculation of the 2-norm is considerably more complicated.

Theorem 2.3.1. *If $A \in \mathbb{R}^{m\times n}$, then there exists a unit 2-norm n-vector z such that $A^TAz = \mu^2 z$ where $\mu = \| A \|_2$.*

Proof. Suppose $z \in \mathbb{R}^n$ is a unit vector such that $\| Az \|_2 = \| A \|_2$. Since z maximizes the function

$$g(x) = \frac{1}{2}\frac{\| Ax \|_2^2}{\| x \|_2^2} = \frac{1}{2}\frac{x^TA^TAx}{x^Tx}$$

it follows that it satisfies $\nabla g(z) = 0$ where ∇g is the gradient of g. A tedious differentiation shows that for $i = 1{:}n$

$$\frac{\partial g(z)}{\partial z_i} = \left[(z^Tz)\sum_{j=1}^{n}(A^TA)_{ij}z_j - (z^TA^TAz)z_i\right] \Big/ (z^Tz)^2 .$$

In vector notation this says that $A^TAz = (z^TA^TAz)z$. The theorem follows by setting $\mu = \| Az \|_2$. $\square$

The theorem implies that $\| A \|_2^2$ is a zero of $p(\lambda) = \det(A^TA - \lambda I)$. In particular,

$$\| A \|_2 = \sqrt{\lambda_{\max}(A^TA)}$$

We have much more to say about eigenvalues in Chapters 7 and 8. For now, we merely observe that 2-norm computation is iterative and a more involved calculation than those of the matrix 1-norm or ∞-norm. Fortunately, if the object is to obtain an order-of-magnitude estimate of $\| A \|_2$, then (2.3.7), (2.3.8), (2.3.11), or (2.3.12) can be used.

As another example of norm analysis, here is a handy result for 2-norm estimation.

Corollary 2.3.2. *If $A \in \mathbb{R}^{m\times n}$, then $\| A \|_2 \le \sqrt{\| A \|_1 \| A \|_\infty}$.*

Proof. If $z \neq 0$ is such that $A^TAz = \mu^2 z$ with $\mu = \| A \|_2$, then $\mu^2 \| z \|_1 = \| A^TAz \|_1 \le \| A^T \|_1 \| A \|_1 \| z \|_1 = \| A \|_\infty \| A \|_1 \| z \|_1$. $\square$

2.3.4 Perturbations and the Inverse

We frequently use norms to quantify the effect of perturbations or to prove that a sequence of matrices converges to a specified limit. As an illustration of these norm applications, let us quantify the change in A^{-1} as a function of change in A.

Lemma 2.3.3. *If $F \in \mathbb{R}^{n\times n}$ and $\| F \|_p < 1$, then $I - F$ is nonsingular and*

$$(I - F)^{-1} = \sum_{k=0}^{\infty} F^k$$

with

$$\| (I - F)^{-1} \|_p \le \frac{1}{1 - \| F \|_p}.$$

Proof. Suppose $I - F$ is singular. It follows that $(I - F)x = 0$ for some nonzero x. But then $\| x \|_p = \| Fx \|_p$ implies $\| F \|_p \ge 1$, a contradiction. Thus, $I - F$ is nonsingular. To obtain an expression for its inverse consider the identity

$$\left(\sum_{k=0}^{N} F^k\right)(I - F) = I - F^{N+1}.$$

Since $\| F \|_p < 1$ it follows that $\lim_{k\to\infty} F^k = 0$ because $\| F^k \|_p \le \| F \|_p^k$. Thus,

$$\left(\lim_{N\to\infty} \sum_{k=0}^{N} F^k\right)(I - F) = I.$$

It follows that $(I - F)^{-1} = \lim_{N\to\infty} \sum_{k=0}^{N} F^k$. From this it is easy to show that

$$\| (I - F)^{-1} \|_p \le \sum_{k=0}^{\infty} \| F \|_p^k = \frac{1}{1 - \| F \|_p}$$

completing the proof of the theorem. □

Note that $\| (I - F)^{-1} - I \|_p \le \| F \|_p / (1 - \| F \|_p)$ is a consequence of the lemma. Thus, if $\epsilon \ll 1$, then $O(\epsilon)$ perturbations to the identity matrix induce $O(\epsilon)$ perturbations in the inverse. In general, we have

Theorem 2.3.4. *If A is nonsingular and $r \equiv \| A^{-1}E \|_p < 1$, then $A+E$ is nonsingular and*

$$\| (A + E)^{-1} - A^{-1} \|_p \le \frac{\| E \|_p \| A^{-1} \|_p^2}{1 - r}.$$

Proof. Note that $A + E = (I + F)A$ where $F = -EA^{-1}$. Since $\| F \|_p = r < 1$, it follows from Lemma 2.3.3 that $I + F$ is nonsingular and $\| (I + F)^{-1} \|_p \le 1/(1 - r)$.

Thus, $(A+E)^{-1} = A^{-1}(I+F)^{-1}$ is nonsingular and

$$(A+E)^{-1} - A^{-1} = A^{-1}(A-(A+E))(A+E)^{-1} = -A^{-1}EA^{-1}(I+F)^{-1}.$$

The theorem follows by taking norms. $\square$