3. Espaços Vetoriais com Produto Interno

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Slide 4

$\langle \vec{u}+\vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

* $\langle \vec{u}, \vec{v}+\vec{w} \rangle$

$\xrightarrow{P1} \langle \vec{v}+\vec{w}, \vec{u} \rangle \xrightarrow{P3} \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle$

$\xrightarrow{P1} \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

* $\langle \vec{u}, \alpha \vec{v} \rangle$ $\quad \langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

$\xrightarrow{P1} \langle \alpha \vec{v}, \vec{u} \rangle \xrightarrow{P4} \alpha \langle \vec{v}, \vec{u} \rangle \xrightarrow{P1} \alpha \langle \vec{u}, \vec{v} \rangle$

Slide 5 - Exemplos

1) $\vec{u}, \vec{v} \in \mathbb{R}^n$ tal que $\begin{cases} \vec{u} = (u_1, u_2, \dots, u_n) \Rightarrow U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ \vec{v} = (v_1, v_2, \dots, v_n) \Rightarrow V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \end{cases}$

PI usual: $\langle \vec{u}, \vec{v} \rangle = V^T U = V^T I U$

$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{bmatrix}_{n \times n} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}_{1 \times n} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_{n \times 1} = v_1 u_1 + \dots + v_n u_n$

$= u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$

2) Se $\begin{cases} f(x) = x \\ g(x) = e^x \\ f, g \in \mathcal{C}([0,1]) \end{cases}$, então:

$\langle f, g \rangle = \int_0^1 x e^x \, dx = \left[e^x (x-1) \right]_0^1 = 0 - e^0(-1)$

$= 1$

3) Traço de uma matriz A :

$tr(A)$ ... soma dos elementos da diagonal principal de A.

4) Seja o EV real $V = \mathbb{R}^n$ com produto interno usual. Se A for uma matriz simétrica ($A = A^T$), então:

$$\boxed{T_A(\vec{v}) = A\vec{v}}$$

$$\langle T_A(\vec{x}), \vec{y} \rangle = \langle \vec{x}, T_A(\vec{y}) \rangle$$

Sejam:

$$\vec{x} = (x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = X \; ;$$

$$\vec{y} = (y_1, \dots, y_n) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = Y .$$

Assim:

$$\langle T_A(\vec{x}), \vec{y} \rangle = \langle \overset{\vec{w}}{A\vec{x}}, \vec{y} \rangle \xrightarrow{\text{Exemplo 1}} = \boxed{Y^T \overset{W}{A X}} = \boxed{(A^T Y)^T} X$$

$$\boxed{(AB)^T = B^T A^T}$$

$$= (\underset{A^T = A}{A} Y)^T X = \langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, T_A(\vec{y}) \rangle$$

$$\boxed{\begin{aligned} (AB)^T &= B^T A^T \\ \therefore \; (A^T Y)^T &= Y^T A \end{aligned}}$$

$B = Y$

$A = A^T$

Slide 8 - Exemplo: Norma

a) $\|\vec{w}\| = \sqrt{<\vec{w}, \vec{w}>}$, $\vec{w} = (\overset{x}{-2}, \overset{y}{1}, \overset{z}{2})$

$<\vec{w}, \vec{w}> = 3x^2 + 2y^2 + z^2 = 3(-2)^2 + 2(1)^2 + (2)^2 = 18$

$\longrightarrow \|\vec{w}\| = \sqrt{18} \neq 1 \quad \therefore$ NORMALIZAR: $\vec{u}_1 = \dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{1}{\sqrt{18}}(-2, 1, 2)$

b) PI Usual: $<\vec{u}, \vec{v}> = x_1 x_2 + y_1 y_2 + z_1 z_2$

Logo: $\|\vec{w}\| = \sqrt{<\vec{w}, \vec{w}>}$, $\vec{w} = (\overset{x}{-2}, \overset{y}{1}, \overset{z}{2})$

$<\vec{w}, \vec{w}> = x^2 + y^2 + z^2 = (-2)^2 + (1)^2 + (2)^2 = 9$

$\longrightarrow \|\vec{w}\| = \sqrt{9} \neq 1 \quad \therefore$ NORMALIZAR: $\vec{u}_2 = \dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{1}{3}(-2, 1, 2)$

c) 1) A norma depende do produto interno estabelecido.

2) Os vetores normalizados são unitários em relação ao produto interno definido.

Slide 10 - Exercícios

1. Para que $<\cdot, \cdot>$ seja um produto interno, deve satisfazer 4 propriedades ($\forall \vec{u}, \vec{v}, \vec{w} \in V$ e $\forall \alpha \in \mathbb{R}$):

P1) Simetria: $<\vec{u}, \vec{v}> = <\vec{v}, \vec{u}>$

P2) Positividade: $<\vec{u}, \vec{u}> \geq 0$ e $<\vec{u}, \vec{u}> = 0 \iff \vec{u} = \vec{0}$

P3) Distributividade: $<\vec{u} + \vec{w}, \vec{v}> = <\vec{u}, \vec{v}> + <\vec{w}, \vec{v}>$

P4) Homogeneidade: $<\alpha\vec{u}, \vec{v}> = \alpha <\vec{u}, \vec{v}>$

a) $\langle \vec{u}, \vec{v} \rangle = 2x_1x_2 - x_1y_2 - x_2y_1 + 2y_1y_2$ $\begin{cases} \vec{u} = (x_1, y_1) \\ \vec{v} = (x_2, y_2) \end{cases}$

P1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

$$2x_1x_2 - x_1y_2 - x_2y_1 + 2y_1y_2 = 2x_2x_1 - x_2y_1 - x_1y_2 + 2y_2y_1$$
$$= 2x_1x_2 - x_1y_2 - x_2y_1 + 2y_1y_2$$

CONFERE!

P2) $\langle \vec{u}, \vec{u} \rangle \geq 0$ e $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

$$\langle \vec{u}, \vec{u} \rangle = 2x_1x_1 - x_1y_1 - x_1y_1 + 2y_1y_1$$

$$\langle \vec{u}, \vec{u} \rangle = 2x_1^2 - 2x_1y_1 + 2y_1^2 \begin{cases} > 0 & \forall\ x_1, y_1 \neq 0 \\ = 0 & \iff x_1 = y_1 = 0 \therefore \vec{u} = \vec{0} \end{cases}$$

CONFERE!

P3) $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

$$\vec{w} = (x_3, y_3) \quad ; \quad \vec{u} + \vec{w} = (x_1 + x_3, y_1 + y_3)$$

$$2(x_1+x_3)x_2 - (x_1+x_3)y_2 - x_2(y_1+y_3) + 2(y_1+y_3)y_2 =$$

$$2x_1x_2 - x_1y_2 - x_2y_1 + 2y_1y_2 + 2x_3x_2 - x_3y_2 - x_2y_3 + 2y_2y_3$$

$$2(x_1+x_3)x_2 - (x_1+x_3)y_2 - x_2(y_1+y_3) + 2(y_1+y_3)y_2 =$$
$$2(x_1+x_3)x_2 - (x_1+x_3)y_2 - x_2(y_1+y_3) + 2(y_1+y_3)y_2$$

CONFERE!

P4) $\langle \alpha\vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

$$\alpha\vec{u} = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$$

$$\langle \alpha\vec{u}, \vec{v} \rangle = 2\alpha x_1x_2 - \alpha x_1y_2 - x_2\alpha y_1 + 2\alpha y_1y_2 =$$
$$\alpha(2x_1x_2 - x_1y_2 - x_2y_1 + 2y_1y_2) = \alpha \langle \vec{u}, \vec{v} \rangle$$

CONFERE!

Logo, $\langle \cdot, \cdot \rangle$ define um produto interno em $V = \mathbb{R}^2$.

b) $\langle \vec{u}, \vec{v} \rangle = \dfrac{x_1 x_2}{a^2} + \dfrac{y_1 y_2}{b^2} \quad \begin{cases} \vec{u} = (x_1, y_1) \\ \vec{v} = (x_2, y_2) \end{cases}, \quad a, b \neq 0$

P1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

$$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} = \frac{x_2 x_1}{a^2} + \frac{y_2 y_1}{b^2}$$

$$= \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2}$$

CONFERE!

P2) $\langle \vec{u}, \vec{u} \rangle \geqslant 0$ e $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

$$\langle \vec{u}, \vec{u} \rangle = \frac{x_1 x_1}{a^2} + \frac{y_1 y_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \begin{cases} > 0 \ \forall x_1, y_1 \neq 0 \\ = 0 \iff x_1 = y_1 = 0 \therefore \vec{u} = \vec{0} \end{cases}$$

CONFERE!

P3) $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

$\vec{w} = (x_3, y_3)$; $\vec{u} + \vec{w} = (x_1 + x_3, y_1 + y_3)$

$$\frac{(x_1 + x_3) x_2}{a^2} + \frac{(y_1 + y_3) y_2}{b^2} = \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{x_3 x_2}{a^2} + \frac{y_3 y_2}{b^2}$$

$$= \frac{(x_1 + x_3) x_2}{a^2} + \frac{(y_1 + y_3) y_2}{b^2}$$

CONFERE!

P4) $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle$

$\alpha \vec{u} = \alpha (x_1, y_1) = (\alpha x_1, \alpha y_1)$

$$\langle \alpha \vec{u}, \vec{v} \rangle = \frac{\alpha x_1 x_2}{a^2} + \frac{\alpha y_1 y_2}{b^2} = \alpha \left(\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} \right)$$

$$= \alpha \langle \vec{u}, \vec{v} \rangle$$

CONFERE!

Logo, $\langle \cdot, \cdot \rangle$ define um produto interno em $V = \mathbb{R}^2$.

c) $\langle f, g \rangle = \sum_{i=0}^{n} a_i b_i$ $\begin{cases} f(t) = a_0 + a_1 t + \dots + a_n t^n \\ g(t) = b_0 + b_1 t + \dots + b_n t^n \end{cases}$

P1) $\langle f, g \rangle = \langle g, f \rangle$

$$\sum_{i=0}^{n} a_i b_i = \sum_{i=0}^{n} b_i a_i$$

$$a_0 b_0 + a_1 b_1 + \dots + a_n b_n = b_0 a_0 + b_1 a_1 + \dots + b_n a_n$$

$$= a_0 b_0 + a_1 b_1 + \dots + a_n b_n$$

CONFERE!

P2) $\langle f, f \rangle \geqslant 0$ e $\langle f, f \rangle = 0 \iff f = 0$

$$\langle f, f \rangle = \sum_{i=0}^{n} a_i a_i = a_0^2 + a_1^2 + \dots + a_n^2$$

$$\langle f, f \rangle \begin{cases} > 0 \text{ se } a_i \neq 0, & i = 0, \dots, n \\ = 0 \text{ se } a_i = 0, & i = 0, \dots, n \quad \therefore \quad f = 0 \end{cases}$$

CONFERE!

P3) $\langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$

$h(t) = c_0 + c_1 t + \dots + c_n t^n$;

$f + h = (a_0 + c_0) + (a_1 + c_1) t + \dots + (a_n + c_n) t^n$

$$\sum_{i=0}^{n} (a_i + c_i) b_i = \sum_{i=0}^{n} a_i b_i + \sum_{i=0}^{n} c_i b_i$$

$$(a_0 + c_0) b_0 + (a_1 + c_1) b_1 + \dots + (a_n + c_n) b_n =$$

$$a_0 b_0 + a_1 b_1 + \dots + a_n b_n + c_0 b_0 + c_1 b_1 + \dots + c_n b_n$$

$$(a_0 + c_0) b_0 + (a_1 + c_1) b_1 + \dots + (a_n + c_n) b_n =$$

$$(a_0 + c_0) b_0 + (a_1 + c_1) b_1 + \dots + (a_n + c_n) b_n$$

CONFERE!

P4) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

$\alpha f = \alpha a_0 + \alpha a_1 t + \dots + \alpha a_n t^n$

$$\langle \alpha f, g \rangle = \sum_{i=0}^{n} \alpha a_i b_i = \alpha a_0 b_0 + \alpha a_1 b_1 + \dots + \alpha a_n b_n$$

$$= \alpha \, (a_0 b_0 + a_1 b_1 + \dots + a_n b_n) = \alpha \sum_{i=0}^{n} a_i b_i$$

CONFERE!

$$= \alpha \langle f, g \rangle$$

Logo, $\langle \cdot, \cdot \rangle$ define um produto interno em $V = P_n(t)$.

2. $\vec{u} = (\overset{x_1}{6}, \overset{y_1}{a}, \overset{z_1}{-1})$; $\|\vec{u}\| = \sqrt{41} \longrightarrow a = ?$ PI Usual

PI Usual: $\langle \vec{u}, \vec{v} \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2 \quad \begin{cases} \vec{u} = (x_1, y_1, z_1) \\ \vec{v} = (x_2, y_2, z_2) \end{cases}$

$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{x_1^2 + y_1^2 + z_1^2} = \sqrt{a^2 + 37}$

Mas:

$$\|\vec{u}\| = \sqrt{41}$$

$$\therefore \sqrt{a^2 + 37} = \sqrt{41} \qquad (\ )^2$$

$$a^2 + 37 = 41$$

$$a^2 = 4$$

$$a = \pm 2$$

3. $\overset{\quad v_1\ \ v_2\ v_3\ v_4}{\vec{v} = (-2,3,0,6)}$; $\|k\vec{v}\| = 5 \longrightarrow k = ?$ PI Usual

PI Usual: $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{4} u_i v_i$ $\begin{cases} \vec{u} = (u_1, u_2, u_3, u_4) \\ \vec{v} = (v_1, v_2, v_3, v_4) \end{cases}$

$$k\vec{v} = k(v_1, v_2, v_3, v_4) = (kv_1, kv_2, kv_3, kv_4)$$

$$\|k\vec{v}\| = \sqrt{\langle k\vec{v}, k\vec{v} \rangle} = \sqrt{k^2(v_1^2 + v_2^2 + v_3^2 + v_4^2)} = \sqrt{49k^2}$$

Mas:

$$\|k\vec{v}\| = 5$$

$$\therefore \sqrt{49k^2} = 5 \qquad (\ )^2$$

$$49k^2 = 25 \longrightarrow k^2 = \frac{25}{49} \longrightarrow k = \pm \frac{5}{7}$$

4. $\|\vec{u}\| = 1$; $\|\vec{v}\| = 1$; $\|\vec{u} - \vec{v}\| = 2 \longrightarrow \langle \vec{u}, \vec{v} \rangle = ?$

$$\langle \vec{u} - \vec{v}, \underbrace{\boxed{\vec{u} - \vec{v}}}_{\vec{w}} \rangle = (\|\vec{u} - \vec{v}\|)^2 = 4$$

$$\langle \vec{u} - \vec{v}, \vec{w} \rangle \xrightarrow{P3} \langle \vec{u}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle = 4$$

$$\langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle = 4$$

P3

$$\langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle - (-\langle \vec{v}, \vec{v} \rangle) = 4$$

P1

$$\langle \vec{u}, \vec{u} \rangle - 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle = 4$$

$$\|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 = 4$$

$$-2\langle \vec{u}, \vec{v} \rangle = 4 - (1)^2 - (1)^2$$

$$-2\langle \vec{u}, \vec{v} \rangle = 2$$

$$\langle \vec{u}, \vec{v} \rangle = -1$$