Hoje (05/10/20): Mudança de variáveis em distribuição
+ Distribuição de velocidades de Maxwell-Boltzmann

Lista de exercícios de Distribuição: Ex 2 e Ex 6

Ex 2.

$p(v) = A v^2 e^{-\alpha v^2}$

$E = \frac{1}{2} m v^2$

Achar $P(E)$

Ex 6.

$f(d) = A e^{-\alpha (d - \langle d \rangle)^2}$

$V = \frac{4}{3}\pi r^3$ e $d = 2r$

Achar $P(V)$

Exemplo de mudança de variável numa função

A) $p = mv$ $\quad E = \frac{1}{2} m v^2$ $\Rightarrow E(v) = \frac{1}{2} m v^2$

$\hookrightarrow v = \frac{p}{m}$ $\quad \Rightarrow E(p) = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{1}{2} \frac{p^2}{m}$

$E(p) = \frac{1}{2} \frac{p^2}{m}$ $\qquad E(v) = E(p)$

B) $d = 2r$ $\quad V = \frac{4}{3}\pi r^3$ $\Rightarrow V(d) = \frac{4}{3}\pi \left(\frac{d}{2}\right)^3 = \frac{\cancel{4}}{3} \pi \frac{d^3}{\cancel{8}\,2}$

$r = \frac{d}{2}$

$V(d) = \frac{1}{6}\pi d^3$ $\qquad V(r) = V(d)$

Para fazer a mudança de variável na distribuição de probabilidade temos que nos preocupar com o elemento ou variável de integração.

$$\boxed{dp = f(x)dx = f(s)ds} \Rightarrow f(x) \neq f(s)$$

Exercicio 2) lista

$$dp = p(v)\,dv = p(E)\,dE$$

$$p(v) = A v^2 e^{-\alpha v^2}$$

$$p(E) = A \frac{2E}{m} e^{-\alpha 2E/m}$$

$$p(v)\,dv = A \frac{2E}{m} e^{-\alpha 2E/m} \sqrt{\frac{1}{2mE}}\, dE$$

$$dp = \frac{2A}{m} E \left(\frac{1}{2mE}\right)^{1/2} e^{-2\alpha E/m}\, dE$$

$$dp = A \left(\frac{2^{\cancel{2}} E^{\cancel{2}}}{m^2} \frac{1}{\cancel{2}m\cancel{E}}\right)^{1/2} e^{-2\alpha E/m}\, dE = A \left(\frac{2E}{m^3}\right)^{1/2} e^{-2\alpha E/m}\, dE$$

$$p(v) = A v^2 e^{-\alpha v^2}$$

$$E = \frac{1}{2} m v^2 \Rightarrow v^2 = \frac{2E}{m} \therefore v = \sqrt{\frac{2E}{m}}$$

$$dE = \frac{1}{\cancel{2}} m \cancel{2} v\, dv$$

$$dE = m v\, dv$$

$$dv = \frac{1}{mv} dE = \frac{1}{m}\sqrt{\frac{m}{2E}}\, dE$$

$$dv = \sqrt{\frac{1}{2mE}}\, dE$$

$$\boxed{dp(E) = \frac{2^{1/2} A}{m^{3/2}} E^{1/2} e^{-2\alpha E/m}\, dE}$$

Exercicio 6) $f(d) = A e^{-\alpha (d - \langle d \rangle)^2}$

$$dp = f(d)\,dd = \underline{f(V)\,dV}$$

$$f(d) = A e^{-\alpha (d - \langle d \rangle)^2} \qquad \langle d \rangle = \mu$$

$$f(d) = A e^{-\alpha \left[\left(\frac{6V}{\pi}\right)^{1/3} - \mu\right]^2}$$

$$V = \frac{1}{6}\pi d^3$$

$$dV = \frac{\pi}{\cancel{6}} \cancel{3} d^2\, dd$$

$$dV = \frac{\pi}{2} d^2\, dd$$

$$dd = \frac{2}{\pi} d^{-2}\, dV$$

$$dd = \frac{2}{\pi}\left(\frac{6V}{\pi}\right)^{-2/3} dV$$

$$d^3 = \frac{6V}{\pi}$$

$$d = \left(\frac{6V}{\pi}\right)^{1/3}$$

$$dp = f(d)\,dd$$

$$dp = A e^{-\alpha \left[\left(\frac{6V}{\pi}\right)^{1/3} - \mu\right]^2} \frac{2}{\pi}\left(\frac{6V}{\pi}\right)^{-2/3} dV$$

$$dp = A \frac{2}{\pi} \frac{\pi^{2/3}}{6^{2/3}} V^{-2/3} e^{-\left(\frac{6}{\pi}\right)^{2/3} \alpha \left[V^{1/3} - \underbrace{\left(\frac{\pi}{6}\right)^{1/3} \mu}_{\nu}\right]^2} dV$$

$$\boxed{dp = \frac{A}{\pi^{1/3}\, 6^{2/3}}\, V^{-2/3}\, e^{-\left(\frac{6}{\pi}\right)^{2/3}\alpha\left[V^{1/3}-\nu\right]^2}\, dV}$$

onde $\nu = \left(\frac{\pi}{6}\right)^{2/3}\mu^2$

$$V_{mp} \Rightarrow \left.\frac{dp}{dV}\right|_{V_{mp}} = 0 \neq d_{mp} \Rightarrow \left.\frac{dp}{dd}\right|_{d_{mp}} = 0 \Rightarrow V_{mp} \neq \frac{\pi}{6} d_{mp}^3$$

Sugestão: Voltar para lista de distribuição e resolver os exercícios 2 e 6

$$\langle V\rangle \neq \frac{\pi}{6}\langle d\rangle^3$$

$$\left\langle \frac{\pi}{6} d^3\right\rangle \neq \frac{\pi}{6}\langle d\rangle^3$$

$$\frac{\pi}{6}\langle d^3\rangle \neq \frac{\pi}{6}\langle d\rangle^3$$

Distribuição de velocidades de Maxwell (1859)

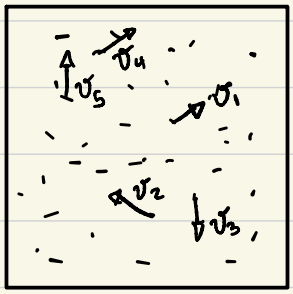
Com $\langle v^2\rangle \Rightarrow \underbrace{\langle E_c\rangle}_{\text{energia cinética}} = \left\langle \frac{1}{2} m v^2\right\rangle = \frac{1}{2} m \langle v^2\rangle$

$\Rightarrow \underbrace{\langle P\rangle}_{\text{pressão}} = \frac{1}{3}\rho\langle v^2\rangle$

m = massa

ρ = densidade

Gás



$$dp(\vec{v}_1, \vec{v}_2, \dots \vec{v}_3) = \underbrace{dp(\vec{v}_1)} \times dp(\vec{v}_2) \cdots dp(\vec{v}_N)$$

$v_i = (v_{xi}, v_{yi}, v_{zi})$

$dp(v_x, v_y, v_z) \Leftarrow$ 1 partícula.

Hipótese de Maxwell sobre os sistemas físicos:

1) O espaço é isotrópico
2) As componentes (x, y e z) são independentes
3) Validade do Teorema de Equipartição da Energia (T.E.E)

1) O espaço é isotrópico = não existe direção preferencial.

Do ponto de vista probabilístico

$$dp(v_x) = dp(v_y) = dp(v_z)$$

$$\boxed{dp(v_x) = dp(-v_x)} \Rightarrow dp(v_x^2)\ ;\ dp(v_y^2)\ ;\ dp(v_z^2)$$

2) As componentes (x, y, z) são independentes

$$dp(v_x, v_y, v_z) = dp(v_x)\,dp(v_y)\,dp(v_z)$$

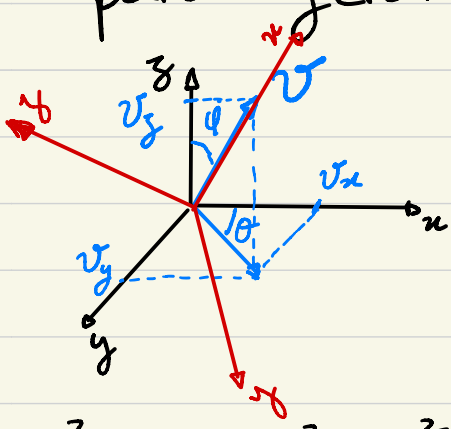
3) T.E.E. = a energia do sistema é distribuida igualmente para cada grau de liberdade com o valor de $\frac{1}{2}kT$ para da partícula.

Ex. Gas monatômica (N partículas) $\Rightarrow$ 3 graus de liberdade

$$\langle E \rangle = \frac{3}{2} NkT$$

translação x, y e z

Vamos ver como estas 3 hipóteses se juntam para gerar a distribuição de velocidades.



$$0 \le \varphi \le 2\pi \quad e \quad 0 \le \theta \le \pi$$

$$\underbrace{dp(v_x, v_y, v_z)}_{1\ \text{partícula}} = f(v_x^2)\,f(v_y^2)\,f(v_z^2)\,dv_x\,dv_y\,dv_z$$

$$dp(v, 0, 0) = \underline{f(v^2)\,f(0)\,f(0)}\,dv_x\,dv_y\,dv_z$$

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$dv_x\,dv_y\,dv_z = v^2 \operatorname{sen}\theta\, dv\, d\theta\, d\varphi$$

$$\underline{dv_x\,dv_y\,dv_z = 4\pi v^2\, dv}$$

onde integramos $\displaystyle\int_0^{2\pi}\int_0^{\pi} \operatorname{sen}\theta\, d\theta\, d\varphi = 4\pi$

$$\overbrace{\int_0^{2\pi} d\varphi \underbrace{\int_0^{\pi} \operatorname{sen}\theta\, d\theta}_{2}}^{} = 2\pi \times 2 = 4\pi$$

$$dp(v,0,0) = dp(v) = A f(v^2)\, 4\pi v^2 dv$$

$$dp(v) = 4\pi A\, v^2 f(v^2)\, dv \quad *$$

Qual função tem a seguinte propriedade:

Maxwell propos:

$$f(v^2) = f(v_x^2) f(v_y^2) f(v_z^2) \Rightarrow \boxed{f(v_i^2) = e^{-\alpha v_i^2}}$$

$$f(v^2) = e^{-\alpha v_x^2} e^{-\alpha v_y^2} e^{-\alpha v_z^2} = e^{-\alpha(v_x^2+v_y^2+v_z^2)}$$

$$\underline{f(v^2) = e^{-\alpha v^2}} \quad *$$

$$dp(v) = 4\pi A v^2 e^{-\alpha v^2} dv$$

Para achar A e α vamos normalizar dp e usar a hipótese 3: $\langle E \rangle = \frac{3}{2} NkT \Rightarrow$ gás monoatômico.

1 partícula $\Rightarrow \langle E \rangle = \frac{3}{2} kT = \langle \frac{1}{2} m v^2 \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$

$$\int_0^\infty dp(v) = 1 \qquad \int_0^\infty 4\pi A v^2 e^{-\alpha v^2} dv = 1 \qquad \text{normalização}$$

$0 \leq v \leq \infty$
módulo

$$4\pi A \underbrace{\int_0^\infty v^2 e^{-\alpha v^2} dv}_{G_2} = 1 \Rightarrow 4\pi A G_2 = 1$$

$$\cancel{4}\pi A \frac{1}{\cancel{4}} \sqrt{\frac{\pi}{\alpha^3}} = 1$$

$$A \sqrt{\left(\frac{\pi}{\alpha}\right)^3} = 1 \quad \therefore \quad A = \left(\frac{\alpha}{\pi}\right)^{3/2}$$

Usando T.E.E. $\langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT \Rightarrow \langle v^2 \rangle = 3 \frac{kT}{m}$

$$\langle v^2 \rangle = \int_0^\infty v^2 \, dp(v) = 3\frac{kT}{m} \Rightarrow 4\pi A G_4 = 3\frac{kT}{m}$$

$$\langle v^2 \rangle = \int_0^\infty v^2 \, 4\pi A v^2 e^{-\alpha v^2} dv = 4\pi A \underbrace{\int_0^\infty v^4 e^{-\alpha v^2} dv}_{= G_4}$$

$$i = 2 \Rightarrow G_4 = \frac{3}{2^3}\sqrt{\frac{\pi}{\alpha^5}} \Rightarrow \cancel{4}\pi A \frac{\cancel{3}}{\cancel{8}}\sqrt{\frac{\pi}{\alpha^5}} = \cancel{3}\frac{kT}{m}$$

$$\frac{\pi A}{2}\sqrt{\frac{\pi}{\alpha^5}} = \frac{kT}{m} \therefore \frac{A}{2}\sqrt{\frac{\pi^3}{\alpha^5}} = \frac{kT}{m} \therefore \frac{1}{2}\left(\frac{\alpha}{\cancel{\pi}}\right)^{3/2} \frac{\cancel{\pi^{3/2}}}{\alpha^{5/2}} = \frac{kT}{m}$$

$$\frac{1}{2}\frac{\alpha^{3/2}}{\alpha^{5/2}} = \frac{kT}{m} \therefore \frac{1}{\alpha} = 2\frac{kT}{m} \therefore \boxed{\alpha = \frac{m}{2kT}}$$

$$A = \left(\frac{\alpha}{\pi}\right)^{3/2} = \frac{\alpha^{3/2}}{\pi^{3/2}} = \frac{1}{\pi^{3/2}}\left(\frac{m}{2kT}\right)^{3/2} \qquad \boxed{A = \left(\frac{m}{2\pi kT}\right)^{3/2}}$$

$$dp(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\alpha v^2} dv$$

$$dp(v) = \left(\frac{4^{2/3} m \pi^{2/3}}{2\pi kT}\right)^{3/2} v^2 e^{-\alpha v^2} dv = \left(\frac{2^{\overbrace{4/3-\frac{3}{3}}^{1/3}} m \pi^{\overbrace{2/3-\frac{3}{3}}^{-1/3}}}{kT}\right)^{3/2} \cdots$$

$$\boxed{dp(v) = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{m}{kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv}$$

Distribuição de velocidades de Maxwell.

ou $$\boxed{dp(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv}$$