Equação do Biocalor

Problemas

Problema #10.13

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Studies have concluded that the plates on the back of the dinosaur Stegosaurus served a thermoregulatory function as heat dissipating fins*. There are indications that the network of channels within the plates may be blood vessels. Model the plate as a rectangular fin of width W, length L and thickness t. Use the Pennes model to formulate the heat equation for this blood perfused plate. The plate exchanges heat with the ambient air by convection. The heat transfer coefficient is h and the ambient temperature is T. Assume that blood reaches each part of the plate at temperature T_{a0} and that it equilibrates at the local temperature T. Assume further that (1) blood perfusion is uniform, (2) negligible metabolic heat production, (3) negligible temperature variation along plate thickness t (fin approximation is valid, $Bi << 1$), (4) steady state and (5) constant properties. Show that the heat equation for this model is given by

$$\frac{d^2\theta}{d\xi^2} - \left[m' + \beta\right]\theta + m' = 0 \qquad m' = \frac{2h\left(W+t\right)L^2}{kWt} \qquad \beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}} \qquad \xi = \frac{x}{L}$$

*Farlow, J.O., Thompson, C.V., and Rosner, D.E., “Plates of the Dinosaur Stegosaurus: Forced Convection Heat Loss Fins?”, Science, vol. 192, pp.1123-1125, 1976.

https://blogs.scientificamerican.com/tetrapod-zoology/the-stegosaurus-plate-controversy/

Problema #10.13

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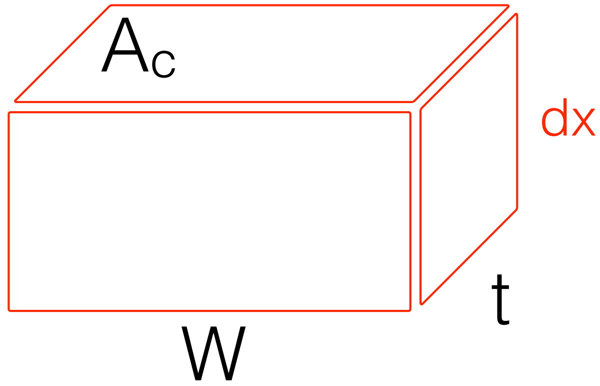
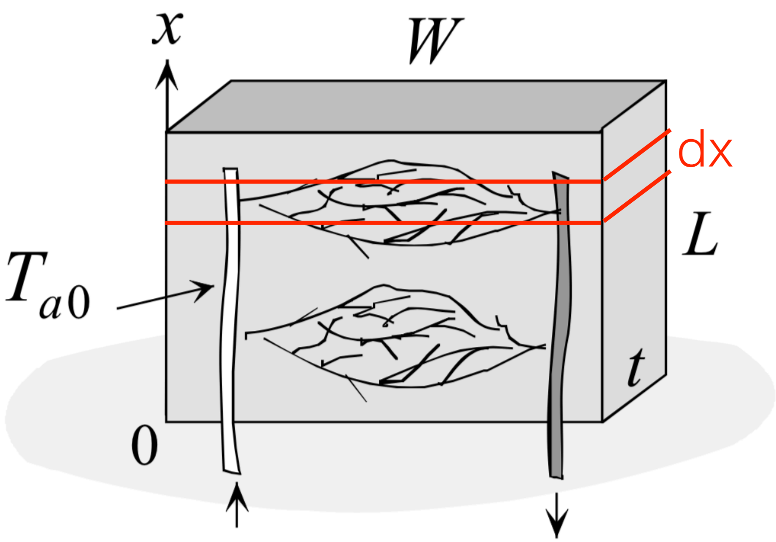


$A_C = Wt$

$dA_S = Pdx$

$P = 2(W+t)$

$d\forall = Wt.dx$



Balanço de energia:

$$q_x - q_{x+dx} - dq_{conv} + dq_{blood} = 0$$

$$dq_{conv} = h dA_s \left(T - T_\infty\right)$$

$$dq_{blood} = \dot{w}_b d\forall \rho_b c_b \left(T_{a0} - T\right)$$

Dedução

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Balanço de energia: $q_x - q_{x+dx} - dq_{conv} + dq_{blood} = 0$

Usando uma série de Taylor truncada:

$$q_x - q_x + \frac{dq_x}{dx}dx - hP\left(T - T_\infty\right)dx + \dot{w}_b W t \rho_b c_b \left(T_{a0} - T\right)dx = 0$$

Usando a lei de Fourier:

$$-\frac{d}{dx}\left(-kA_c \frac{dT}{dx}\right)dx - hP\left(T - T_\infty\right)dx + \dot{w}_b W t \rho_b c_b \left(T_{a0} - T\right)dx = 0$$

Dedução

$$-\frac{d}{dx}\left(-kA_c\frac{dT}{dx}\right)dx - hP\left(T-T_\infty\right)dx + \dot{w}_b W t \rho_b c_b \left(T_{a0}-T\right)dx = 0$$

$$kWt\frac{d^2T}{dx^2} - 2h\left(W+t\right)\left(T-T_\infty\right) + \dot{w}_b W t \rho_b c_b \left(T_{a0}-T\right) = 0$$

$$\frac{d^2T}{dx^2} - \frac{2h\left(W+t\right)}{kWt}\left(T-T_\infty\right) + \frac{\dot{w}_b \rho_b c_b}{k}\left(T_{a0}-T\right) = 0$$

Dedução

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$$\frac{d^2T}{dx^2} - \frac{2h\left(W+t\right)}{kWt}\left(T - T_\infty\right) + \frac{\dot{w}_b \rho_b c_b}{k}\left(T_{a0} - T\right) = 0$$

$$\frac{d^2T}{dx^2} - \left[\frac{2h\left(W+t\right)}{kWt} + \frac{\dot{w}_b \rho_b c_b}{k}\right]T + \frac{2h\left(W+t\right)}{kWt}T_\infty + \frac{\dot{w}_b \rho_b c_b}{k}T_{a0} = 0$$

$$\frac{d^2T}{dx^2} - \left[\frac{2h\left(W+t\right)}{kWt} + \frac{\dot{w}_b \rho_b c_b}{k}\right]\left(T - T_{a0}\right) + \frac{2h\left(W+t\right)}{kWt}T_\infty - \frac{2h\left(W+t\right)}{kWt}T_{a0} = 0$$

$$\frac{d^2T}{dx^2} - \left[\frac{2h\left(W+t\right)}{kWt} + \frac{\dot{w}_b \rho_b c_b}{k}\right]\left(T - T_{a0}\right) + \frac{2h\left(W+t\right)}{kWt}\left(T_\infty - T_{a0}\right) = 0$$

Adimensionlização

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$$\frac{d^2T}{dx^2}-\left[\frac{2h\left(W+t\right)}{kWt}+\frac{\dot{w}_b\rho_b c_b}{k}\right]\left(T-T_{a0}\right)+\frac{2h\left(W+t\right)}{kWt}\left(T_\infty-T_{a0}\right)=0$$

$$\theta=\frac{T-T_{a0}}{T_\infty-T_{a0}} \qquad \Rightarrow d^2\theta=\frac{d^2T}{T_\infty-T_{a0}} \qquad \Rightarrow d^2T=\left(T_\infty-T_{a0}\right)d^2\theta$$

$$\xi=\frac{x}{L} \qquad \Rightarrow d\xi=\frac{dx}{L} \qquad \Rightarrow dx^2=L^2d\xi^2$$

$$\frac{T_\infty-T_{a0}}{L^2}\frac{d^2\theta}{d\xi^2}-\left[\frac{2h\left(W+t\right)}{kWt}+\frac{\dot{w}_b\rho_b c_b}{k}\right]\theta\left(T_\infty-T_{a0}\right)+\frac{2h\left(W+t\right)}{kWt}\left(T_\infty-T_{a0}\right)=0$$

Adimensionlização

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$$\frac{T_\infty - T_{a0}}{L^2}\frac{d^2\theta}{d\xi^2} - \left[\frac{2h(W+t)}{kWt} + \frac{\dot{w}_b \rho_b c_b}{k}\right]\theta\left(T_\infty - T_{a0}\right) + \frac{2h(W+t)}{kWt}\left(T_\infty - T_{a0}\right) = 0$$

$$\frac{d^2\theta}{d\xi^2} - \left[\frac{2h(W+t)L^2}{kWt} + \frac{\dot{w}_b \rho_b c_b L^2}{k}\right]\theta + \frac{2h(W+t)L^2}{kWt} = 0$$

$$\frac{d^2\theta}{d\xi^2} - \left[m' + \beta\right]\theta + m' = 0$$

$$m' = \frac{2h(W+t)L^2}{kWt}$$

$$\beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}$$

Condições de contorno

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$$\frac{d^2\theta}{d\xi^2} - \left[m' + \beta\right]\theta + m' = 0$$

Condições de contorno:

$$T\left(x = 0\right) = T_{a0} \Rightarrow \theta_b\left(\xi = 0\right) = 0$$

$$-kA_c \left.\frac{dT}{dx}\right|_{x=L} = 0 \Rightarrow \left.\frac{d\theta}{d\xi}\right|_{\xi=1} = 0 \qquad A_c << A_s$$

Solução geral

$$\frac{d^2\theta}{d\xi^2} - \left[m' + \beta\right]\theta + m' = 0$$

Definindo: $m^2 = m' + \beta$ $\quad\rightarrow\quad$ $\dfrac{d^2\theta}{d\xi^2} - m^2\theta + m' = 0$

$$\theta' = \theta - \frac{m'}{m^2} \quad\rightarrow\quad \theta = \theta' + \frac{m'}{m^2} \quad\rightarrow\quad \frac{d^2\theta'}{d\xi^2} - m^2\theta' = 0$$

Equação homogênea:

$$\frac{d^2\theta'}{d\xi^2} - m^2\theta' = 0 \quad\rightarrow\quad \theta' = C_1 \operatorname{senh}(m\xi) + C_2 \cosh(m\xi)$$

ou

$$\theta' = C_3 e^{m\xi} + C_4 e^{-m\xi}$$

Solução geral

$$\frac{d^2\theta'}{dx^2} - m^2\theta' = 0 \qquad\qquad \theta = \theta' + \frac{m'}{m^2}$$

Condições de contorno:

$$\theta_b\left(\xi = 0\right) = 0 \Rightarrow \theta_b'\left(\xi = 0\right) = -\frac{m'}{m^2}$$

$$\left.\frac{d\theta}{dx}\right|_{\xi=1} = 0 \Rightarrow \left.\frac{d\theta'}{dx}\right|_{\xi=1} = 0$$

Solução particular

Partimos da solução:

$$\theta' = C_1 \operatorname{senh}(m\xi) + C_2 \cosh(m\xi) \qquad (1)$$

Apliquemos as condições de contorno:

$$\theta_b'(\xi = 0) = -\frac{m'}{m^2} \qquad \Rightarrow C_2 = \theta_b'$$

$$\left.\frac{d\theta'}{dx}\right|_{\xi=1} = 0 \qquad \Rightarrow C_1 m \cosh(m) + C_2 m \operatorname{senh}(m) = 0$$

Combinando as duas condições de contorno:

$$\Rightarrow C_1 \cosh(m) + \theta_b' \operatorname{senh}(m) = 0$$

$$\Rightarrow C_1 = -\theta_b' \tanh(m)$$

Solução particular

Substituindo em (1): $\theta' = -\theta_b' \tanh(m)\operatorname{senh}(m\xi) + \theta_b' \cosh(m\xi)$

$$\frac{\theta'}{\theta_b'} = -\frac{\operatorname{senh}(m)}{\cosh(m)}\operatorname{senh}(m\xi) + \cosh(m\xi)$$

$$\frac{\theta'}{\theta_b'} = \frac{\cosh(m)\cosh(m\xi) - \operatorname{senh}(m)\operatorname{senh}(m\xi)}{\cosh(m)}$$

Nossa solução fica: $$\boxed{\frac{\theta'}{\theta_b'} = \frac{\cosh\left[m(1-\xi)\right]}{\cosh(m)}}$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \operatorname{senh}(x)\operatorname{senh}(y)$$

Perfil de temperatura adimensional

$$\frac{\theta'}{\theta'_b} = \frac{\cosh\left[m\left(1-\xi\right)\right]}{\cosh\left(m\right)} \qquad \theta = \theta' + \frac{m'}{m^2} \qquad \theta'_b = -\frac{m'}{m^2}$$

$$\frac{\theta - {m'}/{m^2}}{-{m'}/{m^2}} = \frac{\cosh\left[m\left(1-\xi\right)\right]}{\cosh\left(m\right)} \Rightarrow \theta = \frac{m'}{m^2} - \frac{m'}{m^2}\frac{\cosh\left[m\left(1-\xi\right)\right]}{\cosh\left(m\right)}$$

$$m^2 = m' + \beta$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}$$

$$\Rightarrow \theta = \frac{m'}{m'+\beta}\left\{1 - \frac{\cosh\left[\sqrt{m'+\beta}\left(1-\xi\right)\right]}{\cosh\left(\sqrt{m'+\beta}\right)}\right\}$$

Perfil de temperatura adimensional

$$\boxed{\theta = \frac{m'}{m'+\beta}\left\{1-\frac{\cosh\left[\sqrt{m'+\beta}\left(1-\xi\right)\right]}{\cosh\left(\sqrt{m'+\beta}\right)}\right\}}$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}$$

$$m' = \frac{2h\left(W+t\right)L^2}{kWt}$$

$$\xi = \frac{x}{L}$$

$$\beta = \frac{\dot{w}_b \rho_b c_b L^2}{k}$$

Calor total transferido

Vamos calcular o calor total transferido a partir da aleta:

$$\boxed{\frac{\theta'}{\theta'_b} = \frac{\cosh m(1-\xi)}{\cosh(m)}}$$

$$q_a = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = -\frac{kWt}{L}\left(T_\infty - T_{a0}\right)\left.\frac{d\theta'}{d\xi}\right|_{\xi=0}$$

$$q_a = -\frac{kWt}{L}\left(T_\infty - T_{a0}\right)\left[-\theta'_b m \frac{\operatorname{senh} m(1-\xi)}{\cosh(m)}\right]_{\xi=0}$$

$$q_a = \frac{kWt}{L}\left(T_\infty - T_{a0}\right)\theta'_b m \frac{\operatorname{senh}(m)}{\cosh(m)}$$

$$q_a = \frac{kWt}{L}\left(T_\infty - T_{a0}\right)\theta'_b m \tanh(m)$$

Calor total transferido

$$q_a = \frac{kWt}{L}\left(T_\infty - T_{a0}\right)\theta_b' m \tanh\left(m\right)$$

$$\theta_b' = -\frac{m'}{m^2} \rightarrow q_a = -\frac{kWt}{L}\left(T_\infty - T_{a0}\right)\frac{m'}{m^2}\sqrt{m'+\beta}\tanh\left(\sqrt{m'+\beta}\right)$$

$$m^2 = m' + \beta \rightarrow q_a = \frac{kWt}{L}\left(T_{a0} - T_\infty\right)\frac{m'}{m^2}\sqrt{m'+\beta}\tanh\left(\sqrt{m'+\beta}\right)$$

$$\boxed{q_a = \frac{kWt}{L}\left(T_{a0} - T_\infty\right)\frac{m'}{\sqrt{m'+\beta}}\tanh\left(\sqrt{m'+\beta}\right)}$$

Problema #10.22

Prolonged exposure to cold environment of elephants can result in frost bite on their ears. Model the elephant ear as a sheet of total surface area (two sides) A and uniform thickness δ. Assume uniform blood perfusion $\dot{w}_b$ and uniform metabolic heat q'''_m. The ear loses heat by convection. The ambient temperature is T_∞ and the heat transfer coefficient is h. Using lumped capacity approximation and the Pennes model, show that the dimensionless transient heat equation is given by

Problema #10.22

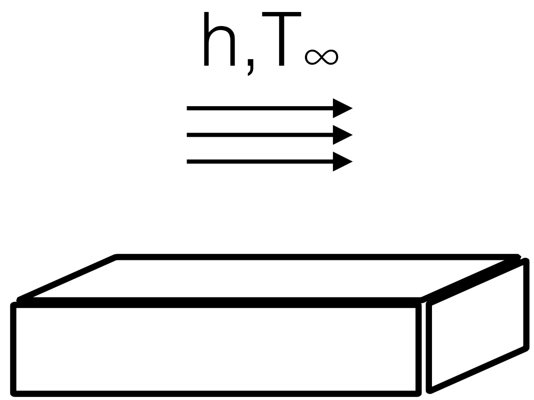
$$\frac{d\theta}{d\tau} = (1+\gamma) - (1+\beta)\theta \,,$$

where

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}, \quad \tau = \frac{2h}{\delta \rho c} t, \quad \beta = \frac{\dot{w}_b \rho_b c_b \delta}{2h}, \quad \gamma = \frac{q_m''' \delta}{2h(T_\infty - T_{a0})}.$$

Here ρ is tissue density and c tissue specific heat. The subscript b refers to blood. Determine the maximum time a zoo elephant can remain outdoors on a cold winter day without resulting in frost bite when the ambient temperature is lower than freezing temperature T_f. Assume that initially the ears are at uniform temperature T_i.

Balanço de energia



Balanço de energia: $$M + q_{blood} - q_{conv} = mc\frac{dT}{dt}$$

$$q_{conv} = hA\left(T - T_{\infty}\right)$$

$$q_{blood} = \dot{w}_b \forall \rho_b c_b \left(T_{a0} - T\right)$$

$$\forall = (A/2)\,\delta$$

$$M = q'''_m \forall$$

$$m = \rho \forall$$

Equação diferencial

Balanço de energia: $M + q_{blood} - q_{conv} = mc\frac{dT}{dt}$

$$\rho \forall c \frac{dT}{dt} = q_m''' \forall + \dot{w}_b \forall \rho_b c_b \left(T_{a0} - T\right) - hA\left(T - T_\infty\right)$$

$$\rho c \frac{dT}{dt} = q_m''' + \dot{w}_b \rho_b c_b \left(T_{a0} - T\right) - \frac{2h}{\delta}\left(T - T_\infty\right)$$

Adimensionalização

$$\rho c \frac{dT}{dt} = q'''_m + \dot{w}_b \rho_b c_b \left(T_{a0} - T\right) - \frac{2h}{\delta}\left(T - T_\infty\right) = 0$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}} \Rightarrow d\theta = \frac{dT}{T_\infty - T_{a0}} \Rightarrow dT = \left(T_\infty - T_{a0}\right) d\theta$$

$$\tau = \frac{2h}{\delta \rho c} t \Rightarrow d\tau = \frac{2h}{\delta \rho c} dt \Rightarrow dt = \frac{\delta \rho c}{2h} d\tau$$

$$\rho c \left(T_\infty - T_{a0}\right) \frac{2h}{\delta \rho c} \frac{d\theta}{d\tau} = q'''_m + \dot{w}_b \rho_b c_b \left(T_{a0} - T\right) - \frac{2h}{\delta}\left(T - T_\infty\right) = 0$$

Adimensionalização

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$$\left(T_\infty - T_{a0}\right)\frac{2h}{\delta}\frac{d\theta}{d\tau} = q_m''' + \dot{w}_b\rho_b c_b\left(T_{a0} - T\right) - \frac{2h}{\delta}\left(T - T_\infty\right)$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}}$$

$$\frac{d\theta}{d\tau} = \frac{q_m'''\delta}{2h\left(T_\infty - T_{a0}\right)} + \frac{\dot{w}_b\rho_b c_b\delta}{2h}\frac{\left(T_{a0} - T\right)}{\left(T_\infty - T_{a0}\right)} - \frac{\left(T - T_\infty\right)}{\left(T_\infty - T_{a0}\right)}$$

$$\gamma = \frac{q_m'''\delta}{2h\left(T_\infty - T_{a0}\right)}$$

$$\beta = \frac{\dot{w}_b\rho_b c_b\delta}{2h}$$

$$\Rightarrow \quad \frac{d\theta}{d\tau} = \gamma - \beta\theta - \frac{\left(T - T_\infty\right)}{\left(T_\infty - T_{a0}\right)}$$

Adimensionalização

$$\frac{d\theta}{d\tau}=\gamma-\beta\theta-\frac{\left(T-T_{\infty}\right)}{\left(T_{\infty}-T_{a0}\right)}$$

$$\theta=\frac{T-T_{a0}}{T_{\infty}-T_{a0}} \qquad \tau=\frac{2h}{\delta\rho c}t$$

$$\frac{d\theta}{d\tau}=\gamma-\beta\theta-\frac{\left(T-T_{\infty}\right)-T_{a0}+T_{a0}}{\left(T_{\infty}-T_{a0}\right)}$$

$$\frac{d\theta}{d\tau}=\gamma-\beta\theta-\frac{\left(T-T_{a0}\right)}{\left(T_{\infty}-T_{a0}\right)}-\frac{T_{a0}-T_{\infty}}{\left(T_{\infty}-T_{a0}\right)}$$

$$\frac{d\theta}{d\tau}=\gamma-\beta\theta-\theta+1$$

$$\boxed{\frac{d\theta}{d\tau}=\left(1+\gamma\right)-\left(1+\beta\right)\theta}$$

Equação diferencial

$$\boxed{\frac{d\theta}{d\tau} = \left(1+\gamma\right) - \left(1+\beta\right)\theta}$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}} \qquad \tau = \frac{2h}{\delta\rho c}t$$

Condição inicial: $T(t=0) = T_i$ $\Rightarrow$ $\theta(\tau=0) = \theta_i = \dfrac{T_i - T_{a0}}{T_\infty - T_{a0}}$

Solução (mudança de variável)

$$\frac{d\theta}{d\tau} = (1+\gamma) - (1+\beta)\theta$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}} \qquad \tau = \frac{2h}{\delta\rho c} t$$

$$\theta(\tau = 0) = \theta_i = \frac{T_i - T_{a0}}{T_\infty - T_{a0\ i}}$$

$$\theta' = \theta - \frac{1+\gamma}{1+\beta} \quad \Rightarrow \quad d\theta' = d\theta$$

$$\theta = \frac{1+\gamma}{1+\beta} + \theta'$$

$$\frac{d\theta'}{d\tau} = -(1+\beta)\theta'$$

$$\theta'_i = \theta_i - \frac{1+\gamma}{1+\beta}$$

Solução (mudança de variável)

$$\frac{d\theta'}{d\tau} = -\left(1+\beta\right)\theta' \quad \Rightarrow \quad \frac{\theta'}{\theta'_i} = e^{-\left(1+\beta\right)\tau}$$

$$\theta'_i = \theta_i - \frac{1+\gamma}{1+\beta} \quad \Rightarrow \quad \frac{\theta - \dfrac{1+\gamma}{1+\beta}}{\theta_i - \dfrac{1+\gamma}{1+\beta}} = e^{-\left(1+\beta\right)\tau}$$

$$\theta = \frac{1+\gamma}{1+\beta} + \left[\theta_i - \frac{1+\gamma}{1+\beta}\right] e^{-\left(1+\beta\right)t} = \theta_i + \frac{1+\gamma}{1+\beta}\left[1 - e^{-\left(1+\beta\right)\tau}\right]$$

$$\theta = \frac{T - T_{a0}}{T_\infty - T_{a0}} \quad \Rightarrow \quad \boxed{\frac{T - T_{a0}}{T_\infty - T_{a0}} = \frac{T_i - T_{a0}}{T_\infty - T_{a0}} + \frac{1+\gamma}{1+\beta}\left[1 - e^{-\left(1+\beta\right)\tau}\right]}$$

Solução (mudança de variável)

$$\boxed{\frac{T-T_{a0}}{T_\infty - T_{a0}} = \frac{T_i - T_{a0}}{T_\infty - T_{a0}} + \frac{1+\gamma}{1+\beta}\left[1 - e^{-(1+\beta)\frac{2h}{\delta\rho c}t}\right]}$$

$$\gamma = \frac{q_m'''\delta}{2h\left(T_\infty - T_{a0}\right)}$$

$$\beta = \frac{\dot{w}_b \rho_b c_b \delta}{2h}$$