(*b*) Apply the Cauchy-Goursat theorem to the branch

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$$

of $z^{-a}/(z+1)$, integrated around the closed contour on the right in Fig. 60, to show that

$$-\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0.$$

(*c*) Point out why, in the last three integrals in parts (*a*) and (*b*), the branches $f_1(z)$ and $f_2(z)$ of $z^{-a}/(z+1)$ can be replaced by the branch

$$f(z) = \frac{z^{-a}}{z+1} \qquad (|z| > 0, 0 < \arg z < 2\pi).$$

Then, by adding corresponding sides of those two equations, derive equation (4), Sec. 61, which was obtained only formally there.

62. INVERSE LAPLACE TRANSFORMS

Suppose that a function F of the complex variable s is analytic throughout the finite s plane except for a finite number of poles. Then let L_R denote a vertical line segment $s = \gamma + it\ (-R \leqq t \leqq R)$, where the constant γ is positive and large enough that the segment lies to the right of all of those singularities. A new function f of the real variable t is defined for positive values of t by means of the equation s ∈ ℂ, t ∈ ℝ

$$f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)\,ds \qquad (t > 0), \tag{1}$$

provided this limit exists. Expression (1) is usually written

$$f(t) = \frac{1}{2\pi i}\,\text{P.V.}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)\,ds \qquad (t > 0) \tag{2}$$

[compare equation (4), Sec. 58], and such an integral is called a *Bromwich integral*.

It can be shown that, when fairly general conditions are imposed on the functions involved, $f(t)$ is the *inverse* Laplace transform of $F(s)$. That is, if $F(s)$ is the *Laplace transform* of $f(t)$, defined by the equation

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{3}$$

then $f(t)$ is retrieved by means of equation (1), where the choice of the positive number γ is immaterial as long as L_R lies to the right of the poles of F.* Laplace transforms and their inverses are important in solving both ordinary and partial differential equations.

* For an extensive treatment of such details regarding Laplace transforms, see R. V. Churchill, "Operational Mathematics," 3d ed., 1972, where transforms $F(s)$ with an infinite number of isolated singular points, or with branch cuts, are also discussed.

Residues can often be used to evaluate the limit in expression (1) when the function $F(s)$ is specified. Using the variable z instead of s, we let $z_1, z_2, \ldots, z_n$ denote the poles of $F(z)$. We then let R_0 denote the largest of the moduli of those poles, and we consider a semicircle C_R with parametric representation

$$z = \gamma + Re^{i\theta} \qquad \left(\frac{\pi}{2} \leqq \theta \leqq \frac{3\pi}{2}\right), \tag{4}$$

where $R > R_0 + \gamma$. We note that, for each pole z_k,

$$|z_k - \gamma| \leqq |z_k| + \gamma \leqq R_0 + \gamma < R.$$

Hence the poles all lie in the interior of the semicircular region bounded by C_R and L_R (Fig. 61), and the residue theorem tells us that

$$\int_{L_R} e^{zt}F(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}[e^{zt}F(z)] - \int_{C_R} e^{zt}F(z)\,dz. \tag{5}$$

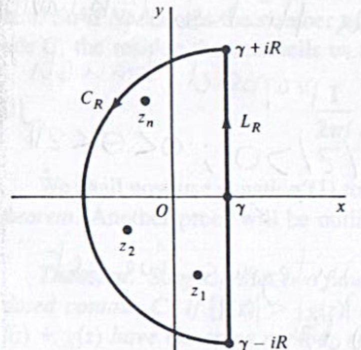

Figure 61

Suppose now that, *for all points z on C_R, there is a positive constant M_R such that $|F(z)| \leqq M_R$, where M_R tends to zero as R tends to infinity.* We may use the parametric representation (4) for C_R to write

$$\int_{C_R} e^{zt}F(z)\,dz = \int_{\pi/2}^{3\pi/2} \exp(\gamma t + Rte^{i\theta})F(\gamma + Re^{i\theta})Rie^{i\theta}\,d\theta.$$

Then, since

$$|\exp(\gamma t + Rte^{i\theta})| = e^{\gamma t}e^{Rt\cos\theta} \quad \text{and} \quad |F(\gamma + Re^{i\theta})| \leqq M_R,$$

we find that

$$\left|\int_{C_R} e^{zt}F(z)\,dz\right| \leqq e^{\gamma t}M_R R\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\,d\theta. \tag{6}$$