rive the integration formula

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2},$$

which is important in the theory of the *Fourier integral.**

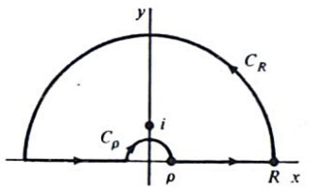

Figure 58

14. Derive the integration formula

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

Suggestion: Note that $2\sin^2 x = \text{Re}(1 - e^{i2x})$, and integrate the function $f(z) = (1 - e^{i2z})/z^2$ around the closed contour shown in Fig. 58. Also, refer to Exercise 12.

15. Show that

$$(a)\ \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0; \quad (b)\ \int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx = -\frac{\pi}{4}.$$

Suggestion: Using the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the logarithmic function, integrate the functions

$$\frac{\log z}{z^2+1} \quad \text{and} \quad \frac{\log z}{(z^2+1)^2}$$

around the closed contour in Fig. 58. Also, use the integration formulas derived in Exercise 1, Sec. 59.

16. The integral of the function

$$f(x) = \frac{1}{x(x^2 - 4x + 5)}$$

over an interval that includes the origin does not exist. Show that the *principal value* of the integral of that function along the entire x axis,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to 0}\left[\int_{-\infty}^{-\rho} f(x)\,dx + \int_\rho^\infty f(x)\,dx\right] \qquad (\rho > 0),$$

does exist by finding that value with the aid of the contour in Fig. 58 and the result found in Exercise 12.

Ans. $2\pi/5$.

* See the authors' "Fourier Series and Boundary Value Problems," 4th ed., pp. 181–183, 1987.

17. Given that (see the footnote to Exercise 11)

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

derive the integration formula

$$\int_0^\infty \exp(-x^2)\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}\exp(-b^2) \qquad (b > 0)$$

by integrating the function $\exp(-z^2)$ around the rectangular path shown in Fig. 59 and then letting a tend to infinity.

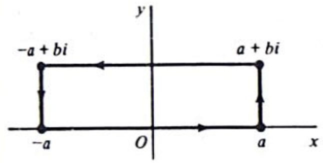

Figure 59

18. The *beta function* is this function of two real variables:

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}\,dt \qquad (p > 0, q > 0).$$

Make the substitution $t = 1/(x + 1)$, and use the result obtained in Sec. 61 to show that

$$B(p, 1 - p) = \frac{\pi}{\sin p\pi} \qquad (0 < p < 1).$$

19. Consider the two simple closed contours shown in Fig. 60 and obtained by dividing into two pieces the annulus bounded by the circles C_ρ and C_R in Fig. 56 (Sec. 61). The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.

(a) Show how it follows from the residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{-a}/(z + 1)$ is integrated around the closed contour on the left in Fig. 60,

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z).$$

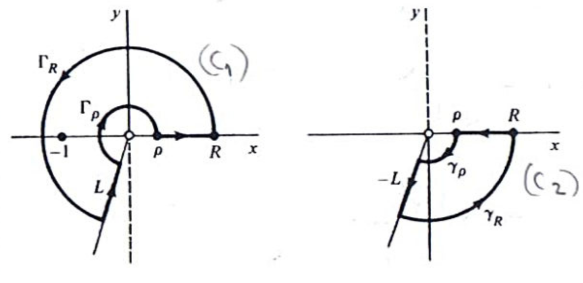

Figure 60