P19. Reference is made to contours $C_1$ and $C_2$ on figure 60. $\boxed{0<a<1}$

a) $f_1(z) = \dfrac{z^{-a}}{z+1}$ ; $z = re^{i\theta}, |z|>0, -\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$

$$f_1(z) = \frac{\exp[-a\log(z)]}{z+1} = \frac{r^{-a} e^{-ia\theta}}{re^{i\theta}+1}$$

on $\Gamma_R \Rightarrow \quad f_1(z) = \dfrac{R^{-a} e^{-ia\theta}}{Re^{i\theta}+1}$ ; $0 \le \theta \le \theta_0$ ; $\pi < \theta_0 < \dfrac{3\pi}{2}$

on $\Gamma_\rho \Rightarrow f_1(z) = \dfrac{\rho^{-a} e^{-ia\theta}}{\rho e^{i\theta}+1}$, $0 \le \theta \le \theta_0$ ; $\pi < \theta_0 < \dfrac{3\pi}{2}$

on $L \Rightarrow f_1(z) = \dfrac{r^{-a} e^{-ia\theta_0}}{re^{i\theta_0}+1}$ ; $\theta = \theta_0$ ; $\rho \le r \le R$

On $z = x \Rightarrow f_1(z) = \dfrac{r^{-a}}{r+1}$, $\theta = 0$ ; $\rho \le r \le R$, $dz = dr$

Besides, we have: $\displaystyle\int_{C_1} f(z)\,dz = 2\pi i \operatorname{Res}[f_1(z), -1] = 2\pi i e^{-ia\pi}$

$$\int_{C_1} f_1(z)\,dz = \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz + \int_\rho^R \frac{r^{-a}}{r+1}\,dr = 2\pi i e^{-ia\pi}$$

from the previous example on pages 193-195, we have:

$$\left|\int_{\Gamma_R} f_1(z)\,dz\right| \le \frac{R^{-a}}{R-1}\theta_0 R = \frac{\theta_0 R}{(R-1)R^a} \Rightarrow \lim_{R\to\infty} |\cdots| = 0$$

$$\left|\int_{\Gamma_\rho} f_1(z)\,dz\right| \le \frac{\rho^{-a}}{1-\rho}\theta_0\rho = \frac{\theta_0\rho^{1-a}}{1-\rho} \Rightarrow \lim_{\rho\to 0} |\cdots| = 0$$

additional considerations $\to 1$

$$\left|r^{-a} e^{-ia\theta_0}\right| = |r^{-a}|\,\cancel{|e^{-ia\theta_0}|} = |r^{-a}| = r^{-a}$$

$$\left||re^{i\theta}| - |1|\right| \le |re^{i\theta}+1| \le |re^{i\theta}| + |1| = r+1$$

$$|r-1| \le |re^{i\theta}+1| \le r+1 \Rightarrow \frac{1}{|r-1|} \ge \frac{1}{|re^{i\theta}+1|} \ge \frac{1}{r+1}$$

or alternatively, we can write:

$$\left||1| - |re^{i\theta}|\right| \le |re^{i\theta}+1| \le |re^{i\theta}| + |1| = r+1$$

$$\Rightarrow \frac{1}{|1-r|} \ge \frac{1}{|re^{i\theta}+1|} \ge \frac{1}{r+1}$$

Therefore, we have two inequalities

$$\frac{r^{-a}}{|r-1|} \ge |f_1(z)| \ge \frac{r^{-a}}{r+1} \quad \text{and} \quad \frac{r^{-a}}{|1-r|} \ge |f_1(z)| \ge \frac{r^{-a}}{r+1}$$

and these two forms justify the above limits ($\rho\to 0$ and $R\to\infty$). In fact, they've been used on page 195. Neither of them is helpful to tackle the integral on L, though. For over L the arc length $l \to \infty$. The only way to rid the problem of that integral is to sum the two contours $C_1$ and $C_2$ up. [ $\rho < 1$ and $R > 1$ ]

b) $f_2(z) = \dfrac{z^{-a}}{z+1}$, $z = re^{i\theta}$, $|z|>0$; $\dfrac{\pi}{2} < \theta < \dfrac{5\pi}{2}$

$f_2(z) = \dfrac{r^{-a} e^{-ia\theta}}{re^{i\theta}+1}$

The solution continues on the back of pages 200-201 $\to \to$