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N-S. equations for ideal Gases, Newtonian fluids

$$\frac{\partial U}{\partial t} + \nabla \cdot (\vec{F} - \vec{F}_v) = Q$$

$$U \equiv \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_T \end{pmatrix} ; \quad \vec{F} = \left(\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ \rho u h_T \end{pmatrix}, \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ \rho v h_t \end{pmatrix}, \begin{pmatrix} \rho w \\ \rho w u \\ \rho w v \\ \rho w^2 + p \\ \rho w h_t \end{pmatrix} \right)$$

where $e_t \equiv e_i + \frac{(\vec{u} \cdot \vec{u})}{2}$, $h_t = h + \frac{(\vec{u} \cdot \vec{u})}{2}$

$$\vec{F}_v = \left(\begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ \phi_x + k\frac{\partial T}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ \phi_y + k\frac{\partial T}{\partial y} \end{pmatrix}; \begin{pmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ \phi_z + k\frac{\partial T}{\partial z} \end{pmatrix} \right)$$

where: $\tau_{ij} \equiv \mu \left[(\partial_i u_j + \partial_j u_i) - \frac{2}{3}(\partial_k u_k)\delta_{ij} \right]$; $\phi_i \equiv \tau_{ij} u_j$

and $Q = 0 \Rightarrow \boxed{\frac{\partial U}{\partial t} + \nabla \cdot \vec{F} = \nabla \cdot \vec{F}_v}$

$F \Rightarrow$ { Euler fluxes : convective term plus pressure grad
Hyperbolic in time-space domain

$F_v \Rightarrow$ viscous, dissipative, fluxes $\Rightarrow$ elliptic terms.

Euler $\Rightarrow$ { physics of convection
wave, directional, character

$$\frac{\partial \vec{U}}{\partial t} + \left(\frac{\partial \vec{F}}{\partial \vec{U}}\right) \cdot \nabla \vec{U} = 0$$

$A \equiv \left(\frac{\partial \vec{F}}{\partial \vec{U}}\right)$ Flux Jacobian matrix

Homogeneous Property } $\Rightarrow$ $\vec{F}$ is homogeneous of order one on $\vec{U}$. Therefore, one can show that

$$\vec{F} = A . \vec{U} \Rightarrow \text{exact result !!}$$
$\hookrightarrow$ (3rd order tensor)

Furthermore, one can make:

$$\vec{V} = \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix} \Rightarrow \begin{aligned} d\vec{U} &= \frac{\partial \vec{U}}{\partial \vec{V}} d\vec{V} \Rightarrow M \equiv \left(\frac{\partial \vec{U}}{\partial \vec{V}}\right) \\ d\vec{V} &= \frac{\partial \vec{V}}{\partial \vec{U}} d\vec{U} \Rightarrow M^{-1} \equiv \left(\frac{\partial \vec{V}}{\partial \vec{U}}\right) \end{aligned}$$

But one must beware of the fact that:

$$\vec{U} \neq M\vec{V} \quad \text{and} \quad \vec{V} \neq M^{-1}\vec{U}$$

For the above relations only hold in differential form.

Well, in any case, we can make:

$$M^{-1}\frac{\partial \vec{U}}{\partial t} + M^{-1} A\, M\, M^{-1}\frac{\partial \vec{U}}{\partial \vec{x}} = 0$$

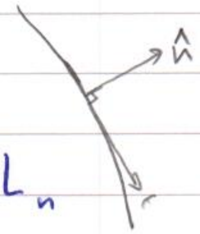
$$\frac{\partial \vec{V}}{\partial t} + \tilde{A}\frac{\partial \vec{V}}{\partial \vec{x}} = 0$$

where we define: $\tilde{A} \equiv M^{-1} A M$

$\hookrightarrow$ 3rd order tensor

Besides, for a flux Jacobian that's associated with a given direction, say $A.\hat{n}$, where $\|\hat{n}\|=1$, one can make:

$$A_n \equiv A.\hat{n}$$



$$\Lambda_n = L_n^{-1}\tilde{A}_n L_n = L_n^{-1} M^{-1} A_n M L_n$$

where Λ is a diagonal matrix of the form:

$$\Lambda_n = \begin{pmatrix} \vec{u}\cdot\hat{n} & 0 & 0 & 0 & 0 \\ 0 & \vec{u}\cdot\hat{n} & 0 & 0 & 0 \\ 0 & 0 & \vec{u}\cdot\hat{n} & 0 & 0 \\ 0 & 0 & 0 & (\vec{u}\cdot\hat{n}+c) & 0 \\ 0 & 0 & 0 & 0 & (\vec{u}\cdot\hat{n}-c) \end{pmatrix}$$

And, althoug the eigenvalue $\vec{u}.\hat{n}$ has algebraic multiplicity 3, it also has a geometric multiplicity three. That is, the matrix has a complete set of linearly independent eigenvectors that form a complete eigensystem.

The main result of this idea is that the 3rd order tensor A, can be diagonilized in one direction, say the normal direction to a given surface. But its components along directions tangent to that surface will <u>not</u> be diagonilized.

And what's the use of such diagonalization, if it's not complete?

Well, for starters, it allows us to study the fluxes of mass momentum and energy that cross any given surface. And that's why its so useful, eventhough it may not be explicitly used for numerical algorithms.

$$P_n \equiv M L_n ; \; P_n^{-1} \equiv L_n^{-1} M^{-1} \Rightarrow \Lambda_n = P_n^{-1} A_n P_n$$

$$L^{-1} M^{-1} \frac{\partial \vec{U}}{\partial t} + L^{-1} M^{-1} A M L L^{-1} M^{-1} \frac{\partial \vec{U}}{\partial \bar{x}} = 0$$

$$L_n^{-1} \frac{\partial \vec{V}}{\partial t} + L_n^{-1} \tilde{A} L_n L_n^{-1} \frac{\partial \vec{V}}{\partial \bar{x}} = 0$$

$$\frac{\partial \vec{W}}{\partial t} + L_n^{-1} \tilde{A} L_n \frac{\partial \vec{W}}{\partial \bar{x}} = 0$$

$$P^{-1} \frac{\partial \vec{U}}{\partial t} + P_{\cdot}^{-1} A . P . P^{-1} \frac{\partial \vec{U}}{\partial x} = 0$$

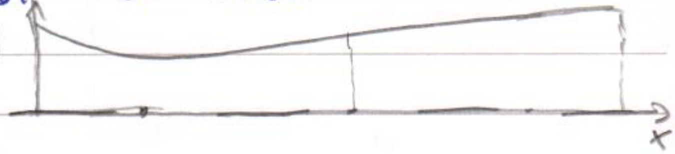
Rieman Variables $\delta \vec{W}$

Rieman Variables are only defined in differential form.

$$\delta\vec{W} \equiv L_n^{-1}\,\delta\vec{V} = P_n^{-1}\,\delta\vec{U} = \begin{pmatrix} \delta\rho - \frac{1}{c^2}\delta p \\ k_x\,\delta u^z - k_z\,\delta u^x \\ k_y\,\delta u^x - k_x\,\delta u^y \\ \frac{1}{\rho c}\delta p + k_{i'}\,\delta u^{i'} \\ \frac{1}{\rho c}\delta p - k_{i'}\,\delta u^{i'} \end{pmatrix} \begin{matrix} \text{(waves)} \\ \text{entropy} \\ \text{vorticity } y \\ \text{vorticity } z \\ \text{acoustic} \\ \text{acoustic} \end{matrix}$$

These variables, even in differential form, will tell us which part of the waves go in which direction. Therefore, they are responsible for what kinds of boundary conditions we can impose on the flow, as we shall see in the simple quasi-1D case.

Quasi-1D flow



$S(x) \sim$ cross-sectional area; $\frac{dS}{dx} \ll 1$

$$\frac{\partial U}{\partial t} + \frac{1}{S}\frac{\partial(FS)}{\partial x} = Q$$

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho e_t \end{pmatrix} ; \quad \vec{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho h_t u \end{pmatrix} ; \quad Q = \begin{pmatrix} 0 \\ \frac{p}{S}\frac{dS}{dx} \\ 0 \end{pmatrix}$$

$p = (\gamma - 1)e_i$; $c = \sqrt{\gamma R T}$; $\gamma \equiv \frac{c_p}{c_v}$

$p v = RT$

$$\frac{\partial \vec{U}}{\partial t} + \frac{1}{S} A \frac{\partial (US)}{\partial x} = Q$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)u & (\gamma-1) \\ (\gamma-1)u^3 - \gamma e_t u & \gamma e_t - 3(\gamma-1)\frac{u^2}{2} & \gamma u \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \alpha & \rho u & 1/\beta \end{pmatrix} \quad ; \quad L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ \alpha\beta & -u\beta & \beta \end{pmatrix}$$

$$\beta \equiv (\gamma - 1) \quad ; \quad \alpha \equiv u^2/2$$

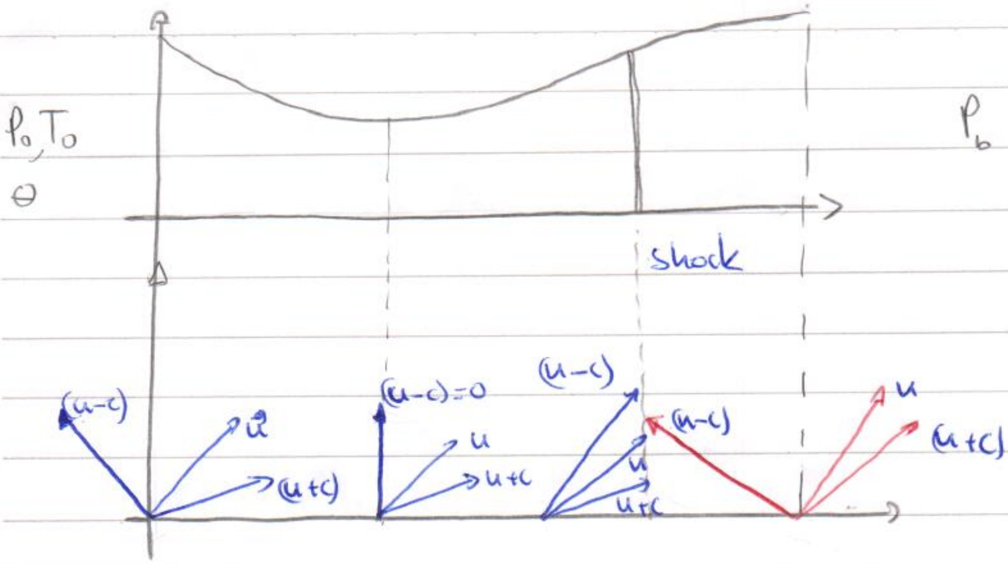
$$M = \begin{pmatrix} 1 & 1/2c^2 & 1/2c^2 \\ 0 & 1/2\rho c & -1/2\rho c \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad ; \quad M^{-1} = \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & \rho c & 1 \\ 0 & -\rho c & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} u & 0 & 0 \\ 0 & (u+c) & 0 \\ 0 & 0 & (u-c) \end{pmatrix}$$

$F_+ = A_+ U$ and $F_- = A_- U$

$A_+ = P\Lambda_+ P^{-1}$; $A_- = P\Lambda_-$

$$\frac{\partial U}{\partial t} + \frac{1}{S}\frac{\partial (F_+ S)}{\partial x} + \frac{1}{S}\frac{\partial (F_- S)}{\partial x} = Q$$



$$T_0 = T\left[1 + \frac{(\gamma-1)}{2} M^2\right]$$

$$P_0 = P\left[1 + \frac{(\gamma-1)}{2} M^2\right]^{\gamma/(\gamma-1)}$$

$$\frac{P_{02}}{P_{01}} = \exp\left(\frac{s_1 - s_2}{R}\right)$$

$$\frac{D\vec{\omega}}{Dt} = \underbrace{\vec{\omega}\cdot\nabla\vec{u} - \vec{\omega}\,\nabla\cdot\vec{u}}_{\text{distribution}} + \underbrace{\nabla T \times \nabla s - \frac{\nabla\rho}{\rho^2} \times \nabla\cdot\tau - \frac{1}{\rho}\nabla\times(\nabla\cdot\tau)}_{\text{sources}}$$

$$\nabla T \times \nabla s = -\nabla\times(v\,\nabla p) = -\nabla\times\left(\frac{\nabla p}{\rho}\right) = 0 \quad \text{whenever } \rho = \text{constant}$$