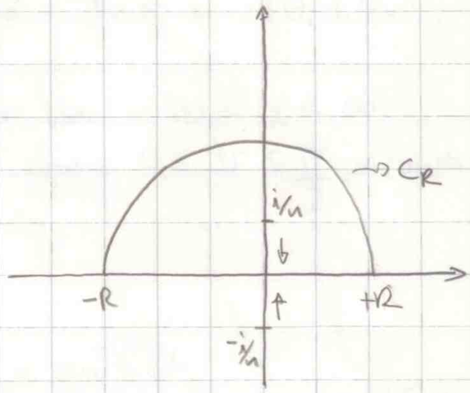
Problem 1 P258. Butkov. (Distributions theory)

$f(z)$ is an arbitrary function, which is analytic on the upper half-plane

Show that: $\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x) f(x)\,dx = f(0)$

where $\phi_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2} = \frac{\cancel{n}}{\pi n^2(x^2+1/n^2)} = \frac{1}{\pi n(x^2+1/n^2)}$



$C = C_R \cup [-R, R]$

$$g(z) \equiv \frac{f(z)}{\pi n(z^2+1/n^2)} = \frac{f(z)}{\pi n(z-i/n)(z+i/n)}$$

$$\int_C g(z)\,dz = \int_C \frac{\{f(z)/[\pi n(z+i/n)]\}}{(z-i/n)}dz = \frac{2\pi i\, f(i/n)}{\pi n\left(2\frac{i}{n}\right)}$$

$$\int_C g(z)\,dz = f(i/n)$$

On assuming that $f(z)$ [$f(x)$, for that matter] is a test function, it should go to zero as $x\to\infty$. Therefore, let's assume that $f(z)\to 0$ as $z\to\infty$ or, to put it another way, $f(z)$ is a bounded function on C_R, regardless of the value of R.

$0 \leq |f(z)| \leq M$ on C_R

$$\left|\int_{C_R} g(z)\,dz\right| \leq \int \frac{|f(z)|}{n\pi|z^2+1/n^2|}|dz| \leq \frac{M}{n\pi}\cdot\frac{\pi R}{|\frac{1}{n^2}-R^2|} \Rightarrow \lim_{R\to\infty} \frac{RM}{R^2\left|\frac{1}{n^2R^2}-1\right|} = 0$$

$$\left||z^2| - |1/n^2|\right| \leq |z^2+1/n^2| \leq |z^2| + |1/n^2|$$

$$\left|R^2-\frac{1}{n^2}\right| \leq \left|z^2+\frac{1}{n^2}\right| \leq R^2+\frac{1}{n^2}$$

$$\frac{1}{|R^2-1/n^2|} \geq \frac{1}{|z^2+1/n^2|} \geq \frac{1}{R^2+\frac{1}{n^2}}$$

Hence, we have: $\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n(x)f(x)\,dx = \lim_{\substack{R\to\infty\\ n\to\infty}} \int_C g(z)\,dz = \lim_{n\to\infty} f(i/n) = f(0)$