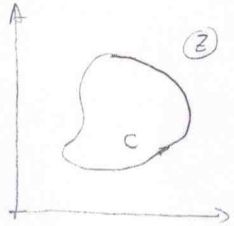
An alternative ~~approach~~ approach to the analytic functions definition, is given on page 128 of Milne-Thomson's book "Theoretical hydrodynamics" 5th ed. 1968 MacMillan Press Ltd.

There, a function $f(z) = \phi(x,y) + i\psi(x,y)$ of $z = x + iy$ is said to be holomorphic within a simple contour C if it satisfies the following conditions:



a)- $f(z)$ is finite and one-valued in C.

b) for each $z \in C$, $f(z)$ has a one-valued finite differential coefficient with respect to z, that is:

Since $x = \frac{1}{2}(z+\bar{z})$, $y = -\frac{i}{2}(z+\bar{z})$ any function of x and y is a function of z and $\bar{z} = x - iy$.

Thus, we can writes

$$f(z,\bar{z}) = \phi(x,y) + i\psi(x,y) \Rightarrow df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \lim_{|dz|\to 0}\frac{d\bar{z}}{dz} \; ; \; \lim_{|dz|\to 0}\frac{d\bar{z}}{dz} = \lim_{\substack{\delta x\to 0\\ \delta y\to 0}} \frac{\delta x - i\delta y}{\delta x + i\delta y} =$$

$$= \lim_{\substack{\delta x\to 0\\ \delta y\to 0}} \frac{1 - i\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}} = \underline{\text{indeterminate}}$$

the indeterminacy implies that the only way for $\frac{df}{dz}$ to exist is to impose the condition $\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$ $\Rightarrow$ which means that an holomorphic function is necessarily independent of $\bar{z}$ ∎

Now, for $f(z) = \phi(x,y) + i\psi(x,y)$, the above condition implies:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = 0 \left\{ \begin{array}{l} \text{where } x = \frac{1}{2}(z+\bar{z}) \text{ and } y = -\frac{i}{2}(z-\bar{z}) \\ \frac{\partial x}{\partial z} = \frac{1}{2} \; ; \; \frac{\partial y}{\partial z} = -\frac{i}{2} \\ \frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \; ; \; \frac{\partial y}{\partial \bar{z}} = +\frac{i}{2} \end{array} \right.$$

$$\frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} = 0$$

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) + \frac{i}{2}\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = 0$$

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\right) = 0 \Rightarrow \left\{ \begin{array}{l} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{array} \right\} \Rightarrow CR$$

↳ the sufficient conditions for $f(z)$ to be holomorphic require that all partial derivatives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$ be continuous as well.

Side results:

$$\frac{\partial \bar{f}}{\partial z} = \frac{\partial \bar{f}}{\partial x}\overbrace{\frac{\partial x}{\partial z}}^{1/2} + \frac{\partial \bar{f}}{\partial y}\overbrace{\frac{\partial y}{\partial z}}^{-i/2} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x}\right) - \frac{i}{2}\left(\frac{\partial \phi}{\partial y} - i\frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial \bar{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) - \frac{i}{2}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\right) = 0$$ <u>provided that f(z) is analytic</u> ∎

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \frac{\partial \bar{f}}{\partial x}\overbrace{\frac{\partial x}{\partial \bar{z}}}^{1/2} + \frac{\partial \bar{f}}{\partial y}\overbrace{\frac{\partial y}{\partial \bar{z}}}^{i/2} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x}\right) + \frac{i}{2}\left(\frac{\partial \phi}{\partial y} - i\frac{\partial \psi}{\partial y}\right)$$

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{2}{2}\frac{\partial \phi}{\partial x} - i\frac{2}{2}\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x} = \overline{\left[\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right]} = \overline{\left(\frac{df}{dz}\right)}$$

→ see next back page for further comments → →

An interesting result of the previous discussion is that $\bar{f}(\bar{z})$ will be analytic if $f(z)$ is analytic. However, we must be careful here.

Even if both $f(z)$ and $\bar{f}(\bar{z})$ are indeed analytic, the sum: $f(z) + \bar{f}(\bar{z})$ won't be analytic. For each of them is analytic on a different complex variable, that is:

a) $f(z)$ is analytic on z, but it's not analytic on $\bar{z}$.

b) $\bar{f}(\bar{z})$ is analytic on $\bar{z}$, but it isn't analytic on z.

The same idea holds for the difference $f(z) - \bar{f}(\bar{z})$.

To put it another way, the functions

$$\left.\begin{aligned} h(z,\bar{z}) &= f(z) + \bar{f}(\bar{z}) \\ g(z,\bar{z}) &= f(z) - \bar{f}(\bar{z}) \end{aligned}\right\} \text{cannot be analytic because they depend on both } z \text{ and } \bar{z}$$

The circle Theorem. Milne-Thomson "Theoretical hydrodynamics, page 157.

Let there be irrotational 2-D potential incompressible flow in the (z) plane. There are no rigid boundaries and the complex potential is given by $f(z)$ where all the singularities of $f(z)$ are at a distance greater than a from the origin.

If a circular cylinder $C \Rightarrow |z| = a$ be introduced into the flow field, the complex potential becomes:

$$W = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$$

Proof: on the contour C, we have $\bar{z} = a^2/z$

$z = ae^{i\theta} \Rightarrow \bar{z} = ae^{-i\theta} = a^2/z$

It implies that W is purely real on C and, thus, $\psi\big|_C = 0 \Rightarrow C$ is a streamline.

if $z = re^{i\theta} \Rightarrow \zeta = \frac{a^2}{r}e^{-i\theta}$

	z	ζ	
$r > a$	out	in	position relative to C
$r < a$	in	out	

Since all the singularities of $f(z)$ are, by hypothesis outside C, all the singularities of $\bar{f}\left(\frac{a^2}{z}\right)$ must be inside C. In particular $\bar{f}(a^2/z)$ has no singularity at infinity, because $f(z)$ has none at $z=0$.

Thus W has the same singularities as $f(z)$ outside the cylinder.

Example: Put a cylinder at the origin in a uniform flow: $f(z) = Uze^{-i\alpha} \Rightarrow \bar{f}(\bar{z}) = U\bar{z}e^{i\alpha} \Rightarrow \bar{f}(z) = Ue^{i\alpha}z$; $\bar{f}\left(\frac{a^2}{z}\right) = \frac{Ue^{i\alpha}a^2}{z}$

$$W = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = Ue^{-i\alpha}z + \frac{Ue^{i\alpha}a^2}{z}$$

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Cauchy Riemann Conditions: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (0)

$f(z) = u(x,y) + i v(x,y)$

coordinate transformation: $\begin{cases} x = r c\theta \\ y = r s\theta \end{cases}$

$$\begin{cases} dx = c\theta\, dr - r s\theta\, d\theta \\ dy = s\theta\, dr + r c\theta\, d\theta \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} c\theta & -rs\theta \\ s\theta & rc\theta \end{vmatrix} = r$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = c\theta \frac{\partial u}{\partial x} + s\theta \frac{\partial u}{\partial y} \quad (1)$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{1}{r}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -s\theta \frac{\partial u}{\partial x} + c\theta \frac{\partial u}{\partial y} \quad (2)$$

Similarly

$$\begin{cases} \frac{\partial v}{\partial r} = c\theta \frac{\partial v}{\partial x} + s\theta \frac{\partial v}{\partial y} & (3) \\ \frac{1}{r}\frac{\partial v}{\partial \theta} = -s\theta \frac{\partial v}{\partial x} + c\theta \frac{\partial v}{\partial y} & (4) \end{cases}$$

$$\left.\begin{aligned} \frac{\partial v}{\partial r} &= -c\theta \frac{\partial u}{\partial y} + s\theta \frac{\partial u}{\partial x} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \\ \frac{1}{r}\frac{\partial v}{\partial \theta} &= +s\theta \frac{\partial u}{\partial y} + c\theta \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} &= -\frac{\partial v}{\partial r} \end{aligned}}$$

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$$E(\theta) = e^{i\theta} \; ; \; z = re^{i\theta}$$

$$\Delta z = (r_0 + \Delta r)e^{i(\theta + \Delta\theta)} - r_0 e^{i\theta}$$

$$= (r_0 + \Delta r)e^{i\theta} e^{i\Delta\theta} - r_0 e^{i\theta}$$

$$= e^{i\theta}\left[(r_0 + \Delta r)e^{i\Delta\theta} - r_0\right]$$

$$= e^{i(\theta + \Delta\theta)}\left[r_0 + \Delta r - r_0 e^{-i\Delta\theta}\right]$$

$$= e^{i(\theta + \Delta\theta)}\left[\Delta r + r_0\left(1 - e^{-i\Delta\theta}\right)\right]$$

$$\boxed{\Delta z = e^{i(\theta + \Delta\theta)}\left\{\Delta r + r_0\left[1 - \cos(\Delta\theta) + i\sin(\Delta\theta)\right]\right\}}$$

$$\Delta z = e^{i(\theta + \Delta\theta)}\left\{\Delta r + i r_0 \Delta\theta + r_0\Delta\theta\underbrace{\left[\frac{1 - \cos(\Delta\theta)}{\Delta\theta} - i\left(\frac{\Delta\theta - \sin\Delta\theta}{\Delta\theta}\right)\right]}_{h(\Delta\theta)}\right\}$$

$$\boxed{\Delta z = e^{i(\theta+\Delta\theta)}\left\{\Delta r + i r_0 \Delta\theta + r_0 \Delta\theta\, h(\Delta\theta)\right\}}$$

goes to zero

$$\lim_{\Delta\theta \to 0} h(\Delta\theta) = \lim_{\Delta\theta \to 0}\left\{\frac{1 - \cos(\Delta\theta)}{\Delta\theta}\right\} - i \lim_{\Delta\theta \to 0}\left\{\frac{\Delta\theta - \sin\Delta\theta}{\Delta\theta}\right\}$$

goes to zero

L'Hospital

$$= \lim_{\Delta\theta \to 0}\left\{\sin(\Delta\theta)\right\} - i \lim_{\Delta\theta \to 0}\left\{1 - \frac{\sin\Delta\theta}{\Delta\theta}\right\} \quad ? \lim_{\Delta\theta \to 0}\frac{\sin\Delta\theta}{\Delta\theta} = 1$$

$$= 0 - 0i = 0$$

$$\boxed{\lim_{\Delta\theta \to 0} h(\Delta\theta) = 0}$$

from P8 we have: for $\underline{r_0 > 0}$

$$\Delta z = e^{i(\theta+\Delta\theta)}\left[\Delta r + i r_0 \Delta\theta + r_0 \Delta\theta\, h(\Delta\theta)\right]$$

where $\lim\limits_{\Delta\theta \to 0} h(\Delta\theta) = 0$

for $r_0 > 0$, show that $\left|\dfrac{\Delta\theta}{\Delta z}\right|$ is bounded for all Δr and $\Delta\theta$ when $|\Delta\theta|$ is small

$$|\Delta z| = \left|\cancelto{1}{e^{i(\theta+\Delta\theta)}}\right| \left|\left[\Delta r + i r_0 \sin(\Delta\theta) + r_0\left(1-\cos(\Delta\theta)\right)\right]\right|$$

$$|\Delta z|^2 = \left[\Delta r + r_0(1-\cos(\Delta\theta))\right]^2 + r_0^2 \sin^2(\Delta\theta)$$

$$= \Delta r^2 + 2r_0\Delta r\left(1-\cos(\Delta\theta)\right) + r_0^2\left(1-\cos(\Delta\theta)\right)^2 + r_0^2 \sin^2(\Delta\theta)$$

$$= \Delta r^2 + 2r_0\Delta r\left(1-\cos(\Delta\theta)\right) + r_0^2 - 2r_0^2\cos(\Delta\theta) + \overbrace{r_0^2\cos^2(\Delta\theta) + r_0^2\sin^2(\Delta\theta)}^{r_0^2}$$

$$= \Delta r^2 + 2r_0\Delta r(1-\cos(\Delta\theta)) + 2r_0^2 - 2r_0^2\cos(\Delta\theta) = \Delta r_0^2 + 2r_0\Delta r(1-\cos(\Delta\theta)) + 2r_0^2(1-\cos(\Delta\theta))$$

$$\boxed{|\Delta z|^2 = \Delta r^2 + 2r_0(r_0+\Delta r)\left[1-\cos(\Delta\theta)\right]}$$

On defining $f(\Delta\theta) = 1-\cos(\Delta\theta) \Rightarrow \boxed{0 \le f(\Delta\theta) \le 2}$, $f(0) = 0$

$$|\Delta z| = \left[\Delta r^2 + 2r_0(r_0+\Delta r) f(\Delta\theta)\right]^{1/2}$$

$$\frac{|\Delta\theta|}{|\Delta z|} = \left\{\frac{\Delta\theta^2}{\Delta r^2 + 2r_0(r_0+\Delta r) f(\Delta\theta)}\right\}^{1/2}, \quad \frac{|\Delta r|}{|\Delta z|} = \left\{\frac{\Delta r^2}{\Delta r^2 + 2r_0(r_0+\Delta r) f(\Delta\theta)}\right\}^{1/2}$$

$$\lim_{\Delta\theta\to 0}\frac{|\Delta\theta|}{|\Delta z|} = 0$$

$$\lim_{\Delta\theta\to\infty}\frac{|\Delta\theta|}{|\Delta z|} = \infty$$

$$\lim_{\Delta r\to 0}\frac{|\Delta\theta|}{|\Delta z|} = \frac{\Delta\theta}{r_0\sqrt{2 f(\Delta\theta)}}$$

$$\lim_{\Delta r\to\infty}\frac{|\Delta\theta|}{|\Delta z|} = 0$$

$$\lim_{\Delta\theta\to 0}\frac{|\Delta r|}{|\Delta z|} = \sqrt{\frac{\Delta r^2}{\Delta r^2}} = 1$$

$$\lim_{\Delta\theta\to\infty}\frac{|\Delta r|}{|\Delta z|} = \sqrt{\frac{\Delta r^2}{\Delta r^2 + 2r_0(r+\Delta r) f(\Delta\theta)}}$$

$$\lim_{\Delta r\to 0}\frac{|\Delta r|}{|\Delta z|} = 0$$

$$\lim_{\Delta r\to\infty}\frac{|\Delta r|}{|\Delta z|} = \left\{\frac{1}{1 + \left[\cancelto{0}{\frac{2r_0^2}{\Delta r^2}} + \cancelto{0}{\frac{2r_0}{\Delta r}}\right] f(\Delta\theta)}\right\}^{1/2} = 1$$

Therefore, the two quantities are bounded for $|\Delta\theta|$ finite and $|\Delta r| < r_0$

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$$\Delta f = f(z_0 + \Delta z) - f(z_0) \qquad \text{where} \quad f(z) = u(r,\theta) + i\,v(r,\theta),\ z_0 = r_0 e^{i\theta_0}$$

$$\Delta f = \Big[u(r_0+\Delta r, \theta_0+\Delta\theta) - u(r_0,\theta_0)\Big] + i\Big[v(r_0+\Delta r,\theta_0+\Delta\theta) - v(r_0,\theta_0)\Big]$$

$$\Delta f = \left\{\frac{\partial u}{\partial r}\Delta r + \underbrace{\frac{1}{r}\frac{\partial u}{\partial \theta} r\Delta\theta}_{\left(-\frac{\partial v}{\partial r} r\Delta\theta\right)_{r=r_0}} + \varepsilon_1\Delta r + \varepsilon_2\Delta\theta\right\}_{z=z_0} + i\left\{\frac{\partial v}{\partial r}\Delta r + \underbrace{\frac{1}{r}\frac{\partial v}{\partial\theta} r\Delta\theta}_{\left(\frac{\partial u}{\partial r} r\Delta\theta\right)_{r=r_0}} + \varepsilon_3\Delta r + \varepsilon_4\Delta\theta\right\}_{z=z_0}$$

after all, they are assumed to satisfy C.R. conditions

$$\Delta f = \left(\frac{\partial u}{\partial r}\Delta r - \frac{\partial v}{\partial r} r_0\Delta\theta\right)_{z=z_0} + i\left(\frac{\partial v}{\partial r}\Delta r + \frac{\partial u}{\partial r} r_0\Delta\theta\right)_{z=z_0} + \underbrace{(\varepsilon_1 + i\varepsilon_3)}_{\sigma_1}\Delta r + \underbrace{(\varepsilon_2 + i\varepsilon_4)}_{\sigma_2}\Delta\theta$$

$$\boxed{\Delta f = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\Delta r + i r_0\Delta\theta) + \sigma_1\Delta r + \sigma_2\Delta\theta}$$

$$= \frac{\partial u}{\partial r}\Delta r + i\frac{\partial u}{\partial r} r_0\Delta\theta + i\frac{\partial v}{\partial r}\Delta r - \frac{\partial v}{\partial r} r_0\Delta\theta + \sigma_1\Delta r + \sigma_2\Delta\theta$$

From P8, we have: $(\Delta r + i r_0\Delta\theta) = \Delta z\, e^{-i(\theta+\Delta\theta)} - r_0\Delta\theta\, h(\Delta\theta)$

Then we can write:

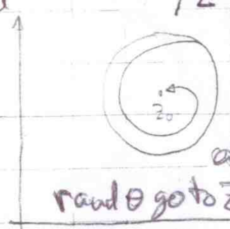
$$\Delta f = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\Delta z\, e^{-i(\theta+\Delta\theta)} - \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) r_0\Delta\theta\, h(\Delta\theta) + \sigma_1\Delta r + \sigma_2\Delta\theta$$

$$\frac{\Delta f}{\Delta z} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) e^{-i(\theta+\Delta\theta)} - \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial v}\right) r_0 \underbrace{\frac{\Delta\theta}{\Delta z}}_{\text{bounded}} \overset{\to 0}{h(\Delta\theta)} + \cancel{\sigma_1}^{\,0} \underbrace{\frac{\Delta r}{\Delta z}}_{\text{bounded}} + \cancel{\sigma_2}^{\,0} \underbrace{\frac{\Delta\theta}{\Delta z}}_{\text{bounded}}$$

and from P8 and P9 results we should have

$$\boxed{\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)e^{-i\theta}}$$

$|z - z_0| < \delta$



depending on the rates at which r and θ go to zero, the path to the limit is Arbitrary!

$$\Delta z \to 0 \Rightarrow \begin{cases}\Delta r \to 0\\ \Delta\theta \to 0\end{cases}$$

L'Hospitals' rule

$$\lim_{\Delta\theta\to 0}\left\{\lim_{\Delta r\to 0}\frac{|\Delta\theta|}{|\Delta z|}\right\} = \lim_{\Delta\theta\to 0}\frac{\Delta\theta}{r_0\sqrt{2}\sqrt{1-\cos(\Delta\theta)}} = \frac{1}{r_0\sqrt{2}}\lim_{\Delta\theta\to 0}\frac{2\sqrt{1-\cos(\Delta\theta)}}{\sin(\Delta\theta)} =$$

$$= \frac{2}{r_0\sqrt{2}}\lim_{\Delta\theta\to 0}\sqrt{\frac{1-\cos(\Delta\theta)}{1-\cos^2(\Delta\theta)}} = \frac{2}{r_0\sqrt{2}}\lim_{\Delta\theta\to 0}\frac{1}{\sqrt{1+\cos(\Delta\theta)}} = \frac{2}{r_0\sqrt{2}\sqrt{2}} = \boxed{\frac{1}{r_0}}$$

~ bounded