# Equação da continuidade e Teorema de Ehrenfest

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Problema 1: Verifique a equação de continuidade para um estado estacionário $\psi(x,t) = \phi(x)\, e^{-\frac{i\,E\,t}{\hbar}}$ , onde $\phi$ satisfaz $\mathrm{H}\phi = E\phi$.

# Equação da continuidade

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \qquad (1)$$

$$-i\hbar\frac{\partial\psi^*(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t) \qquad (2)$$

$Eq.Continuidade \rightarrow \psi^*$ (1) - $\psi$ (2)

$$i\hbar\left[\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right] = -\frac{\hbar^2}{2m}\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right] + \cancel{V\psi\psi^*} - \cancel{V\psi^*\psi}$$

$$i\hbar\,\partial_t[\underbrace{\psi\psi^*}_{|\psi|^2=\rho(x,t)}] = -\frac{\hbar^2}{2m}\partial_x\left[\psi^*\partial_x\psi - \psi\partial_x\psi^*\right]$$

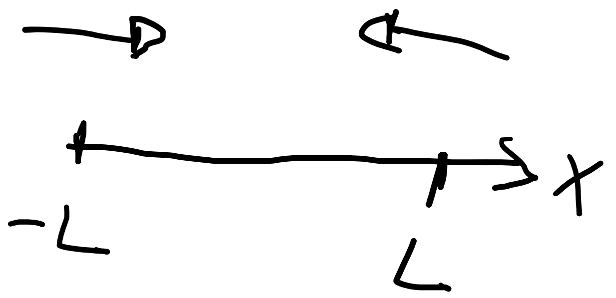
$$\partial_t\rho(x,t) = -\partial_x\left[\underbrace{\frac{\hbar}{2mi}\left(\psi^*\partial_x\psi - \psi\partial_x\psi^*\right)}_{J(x,t)}\right]$$

# Equação da continuidade – versão integrada

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \qquad \rho(x,t) = |\psi(x,t)|^2 \qquad J(x,t) = \frac{\hbar}{2mi}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

Considerando intervalo: $x \in [-L, L]$ $\qquad P(t) = \int_{-L}^{L} \rho(x,t)\, dx = \int_{-L}^{L} |\psi(x,t)|^2\, dx$

$$\int_{-L}^{L} \frac{\partial \rho(x,t)}{\partial t}\, dx = -\int_{-L}^{L} \frac{\partial J(x,t)}{\partial x}\, dx$$




$$\frac{d}{dt}\underbrace{\int_{-L}^{L} \rho(x,t)\, dx} = -J(L,t) + J(-L,t)$$

$$\frac{d}{dt} P(t) = -J(L,t) + J(-L,t)$$

$$\frac{\partial \rho(t,\vec{r})}{\partial t} + \nabla \cdot \vec{J} = 0$$

Problema 1: Verifique a equação de continuidade para um estado estacionário $\psi(x,t) = \phi(x)\, e^{-\frac{iEt}{\hbar}}$, onde $\phi$ satisfaz $\mathrm{H}\phi = E\phi$.

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \qquad \rho(x,t) = |\psi(x,t)|^2 \qquad J(x,t) = \frac{\hbar}{2mi}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

$$\rho(x,t) = \psi\psi^* = \phi e^{\frac{iEt}{\hbar}} \phi^* e^{\frac{-iEt}{\hbar}} = \phi\phi^* = |\phi(x)|^2 \rightarrow \frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial t}|\phi(x)|^2 = 0$$

$$\frac{\partial J}{\partial x} = \frac{\hbar}{2mi}\left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right] = \frac{\hbar}{2mi}\left[\phi^* \frac{\partial^2 \phi}{\partial x^2} - \phi \frac{\partial^2 \phi^*}{\partial x^2}\right]$$

$$\rightarrow H\phi = E\phi$$

$$-\frac{\hbar^2}{2m}\partial_x^2 \phi + V\phi = E\phi$$

$$\partial_x^2 \phi = (V-E)\frac{2m}{\hbar^2}\phi$$

$$\partial_x^2 \phi^* = (V-E)\frac{2m}{\hbar^2}\phi^*$$

$$\frac{\partial J}{\partial x} = \frac{\hbar}{2mi}\left[\phi^*(V-E)\frac{2m}{\hbar^2}\phi - \phi(V-E)\frac{2m}{\hbar^2}\phi^*\right] = 0$$

Problema 2: Ehrenfest para oscilador – um oscilador harmônico, massa m e frequência $\omega$, é descrito em $t = 0$ pelo pacote $\varphi_0(x)$. Quais os valores médios de $x$ e $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$ em um instante $t$ qualquer?

Eq .Ehrenfest

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{dV(x)}{dx} \right\rangle$$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{m}\langle p\rangle$$

$\Longleftrightarrow$

Eq. variáveis canônicas x e p (mecânica clássica)

$$\frac{dp}{dt} = -\frac{dV(x)}{dx}$$

$$\frac{dx}{dt} = \frac{1}{m}p$$

$$\left\langle \frac{dV(x)}{dx} \right\rangle = \frac{dV(\langle x\rangle)}{dx}$$

$V(x) = x^n$

$n = 3 \rightarrow \partial_x V \propto x^2 \quad \langle x^2\rangle \neq \langle x\rangle^2$

Potencial harmônico

$$V(x) = \frac{1}{2}m\omega^2x^2 \rightarrow \frac{dV}{dx} = m\omega^2 x$$

$\Downarrow$

$$\begin{cases} \dfrac{d\langle p\rangle}{dt} = -m\omega^2\langle x\rangle \\[2ex] \dfrac{d\langle x\rangle}{dt} = \dfrac{1}{m}\langle p\rangle \end{cases}$$

$$\underline{\frac{d^2\langle x\rangle}{dt^2} = -\omega^2\langle x\rangle}$$

$$\frac{d^2\langle x\rangle}{dt^2} = -\omega^2\langle x\rangle$$

$$\frac{d\langle p\rangle}{dt} = -m\omega^2\langle x\rangle$$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{m}\langle p\rangle$$

Solução Geral:

$$\bar{x}(t) = A\cos(wt) + B\sin(wt)$$

$t = 0$

$$\bar{x}(0) = x_0 = \int dx\, \varphi_0^* \, x \, \varphi_0$$

$$\bar{p}(0) = p_0 = \int dx\, \varphi_0^* \frac{\hbar}{i}\frac{\partial}{\partial x}\varphi_0$$

$$\bar{p}(t) = m\frac{d\bar{x}}{dt} = -mAw\,\text{sen}(wt) + Bmw\cos(wt)$$

$$\bar{p}(0) = Bmw$$

$A = x_0$

$$B = \frac{p_0}{mw}$$

$$\therefore \quad \bar{x}(t) = x_0\cos(wt) + \frac{p_0}{mw}\text{sen}(wt)$$

$$\bar{p}(t) = -m x_0 w\,\text{sen}(wt) + p_0\cos(wt)$$