Lista 8: Função degrau (extraídos do livro de Boyce e DiPrima)

# Problems

In each of Problems 1 through 4, sketch the graph of the given function on the interval $t \geq 0$.

**1.** $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

**2.** $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$

**3.** $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$

**4.** $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 5 through 8:

**a.** Sketch the graph of the given function.

**b.** Express $f(t)$ in terms of the unit step function $u_c(t)$.

**5.** $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$

**6.** $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$

**7.** $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$

**8.** $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 9 through 12, find the Laplace transform of the given function.

**9.** $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$

**10.** $f(t) = \begin{cases} 0, & t < \pi \\ t-\pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

**11.** $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

**12.** $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

In each of Problems 13 through 16, find the inverse Laplace transform of the given function.

**13.** $F(s) = \dfrac{3!}{(s-2)^4}$

**14.** $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

**15.** $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$

**16.** $F(s) = \dfrac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$

**17.** Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

**a.** Show that if $c$ is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \quad s > ca.$$

**b.** Show that if $k$ is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

**c.** Show that if $a$ and $b$ are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

In each of Problems 18 through 20, use the results of Problem 17 find the inverse Laplace transform of the given function.

**18.** $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

**19.** $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

**20.** $F(s) = \dfrac{1}{9s^2-12s+3}$

In each of Problems 21 through 23, find the Laplace transform of given function. In Problem 23, assume that term-by-term integrat of the infinite series is permissible.

**21.** $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

**22.** $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

**23.** $f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$. See Figure 6.3.8.

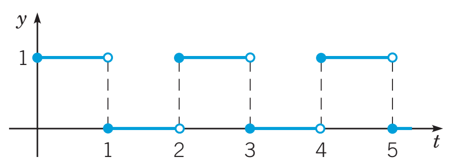


**FIGURE 6.3.8** The function $f(t)$ in Problem 23; a square wave.

**24.** Let $f$ satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for so fixed positive number $T$; $f$ is said to be **periodic with period** $T$ $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}.$$

In each of Problems 25 through 28, use the result of Problem 24 find the Laplace transform of the given function.

**25.** $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & 1 \leq t < 2; \end{cases} \quad f(t+2) = f(t).$

Compare with Problem 23.

**26.** $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2; \end{cases} \quad f(t+2) = f(t).$

See Figure 6.3.9.

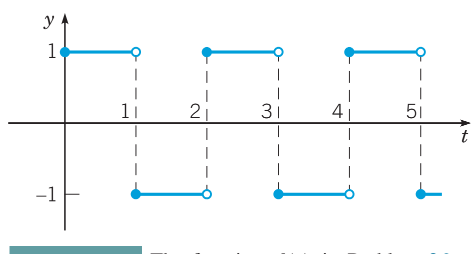


**FIGURE 6.3.9** The function $f(t)$ in Problem 26; a square wave.

**27.** $f(t) = t, \quad 0 \le t < 1; \quad f(t+1) = f(t).$
See Figure 6.3.10.

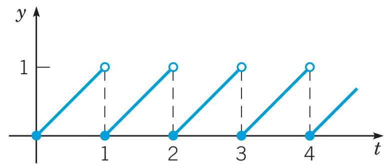


**FIGURE 6.3.10** The function $f(t)$ in Problem 27; a sawtooth wave.

**28.** $f(t) = \sin t, \quad 0 \le t < \pi; \quad f(t+\pi) = f(t).$
See Figure 6.3.11.

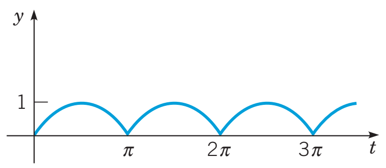


**FIGURE 6.3.11** The function $f(t)$ in Problem 28; a rectified sine wave.

**29.** **a.** If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$. Sketch the graph of $y = f(t)$. Compare with Problem 21.
**b.** Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function $f$ is defined in part a. Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$. Use your expression for $\mathcal{L}\{g(t)\}$ to find an explicit formula for $g(t)$. *Hint:* See Problem 28 in Section 6.2.
**c.** Let $h(t) = g(t) - u_1(t)g(t-1)$, where $g$ is defined in part b. Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$. Use your expression for $\mathcal{L}\{h(t)\}$ to find an explicit formula for $h(t)$.

**30.** Consider the function $p$ defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

**a.** Sketch the graph of $y = p(t)$.
**b.** Find $\mathcal{L}\{p(t)\}$ by noting that $p$ is the periodic extension of the function $h$ in Problem 29c; then use the result of Problem 24.
**c.** Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)dt,$$

where $f$ is the function in Problem 26; then use Theorem 6.2.1.