# PEF-5916
# Dinâmica e Estabilidade das Estruturas

Prof. Dr. Carlos Eduardo Nigro Mazzilli

# Vibração forçada com carregamento harmônico

$$m\ddot{u} + c\dot{u} + ku = p_0 \operatorname{sen} \bar{\omega} t$$

Para amortecimento subcrítico $\xi < 1$

$$u(t) = e^{-\xi\omega t}\rho\cos(\omega_D t - \theta) + \bar{\rho}\operatorname{sen}(\bar{\omega}t - \bar{\theta})$$

Condições iniciais

$$u_0 = \rho\cos\theta - \bar{\rho}\operatorname{sen}\bar{\theta},$$

$$\dot{u}_0 = -\xi\omega\rho\cos\theta + \omega_D\rho\operatorname{sen}\theta + \bar{\omega}\bar{\rho}\cos\bar{\theta}$$

⇓

$$\rho = \sqrt{\left(u_0 + \bar{\rho}\operatorname{sen}\bar{\theta}\right)^2 + \left[\frac{\dot{u}_0 - \bar{\omega}\bar{\rho}\cos\bar{\theta} + \xi\omega\left(u_0 + \bar{\rho}\operatorname{sen}\bar{\theta}\right)}{\omega_D}\right]^2},$$

$$\theta = \arctan\frac{\dot{u}_0 - \bar{\omega}\bar{\rho}\cos\bar{\theta} + \xi\omega\left(u_0 + \bar{\rho}\operatorname{sen}\bar{\theta}\right)}{\omega_D\left(u_0 + \bar{\rho}\operatorname{sen}\bar{\theta}\right)}.$$

# Vibração forçada com carregamento harmônico

$$m\ddot{u} + c\dot{u} + ku = p_0 \,\mathrm{sen}\, \bar{\omega} t$$

Para amortecimento subcrítico $\xi < 1$

$$u(t) = e^{-\xi\omega t}\rho\cos(\omega_D t - \theta) + \bar{\rho}\,\mathrm{sen}\,(\bar{\omega}t - \bar{\theta})$$

desprezível após poucos ciclos

$$\bar{\rho} = D\, u_e$$

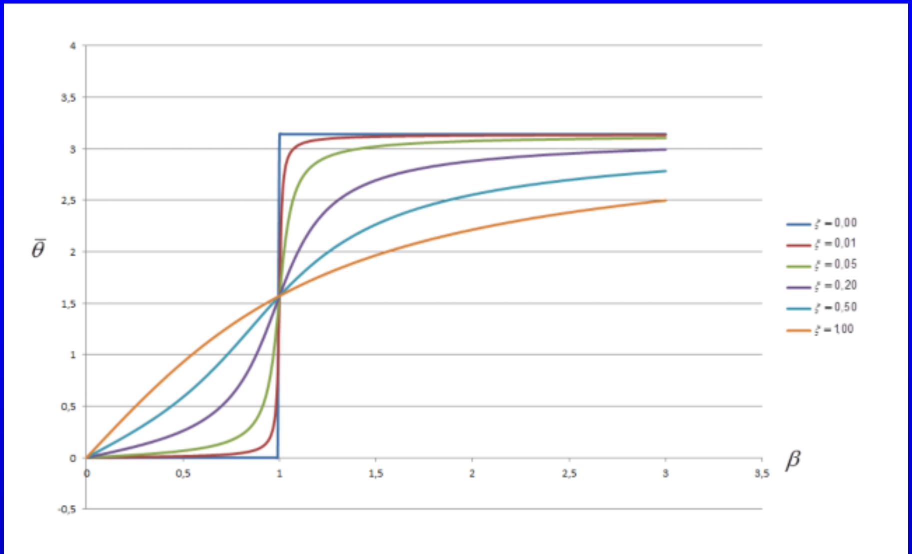
$$\bar{\theta} = \arctan\left(\frac{2\xi\beta}{1-\beta^2}\right), \qquad 0 \leqslant \bar{\theta} \leqslant \pi$$

$$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$u_e = \frac{p_0}{k}$$

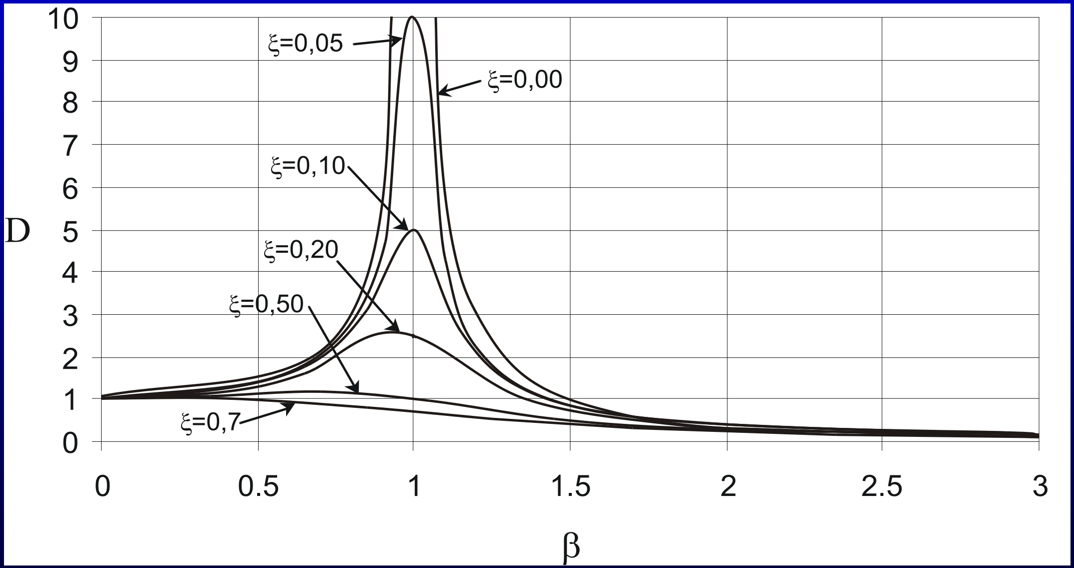
$$\beta = \frac{\bar{\omega}}{\omega}$$

# Vibração forçada com carregamento harmônico



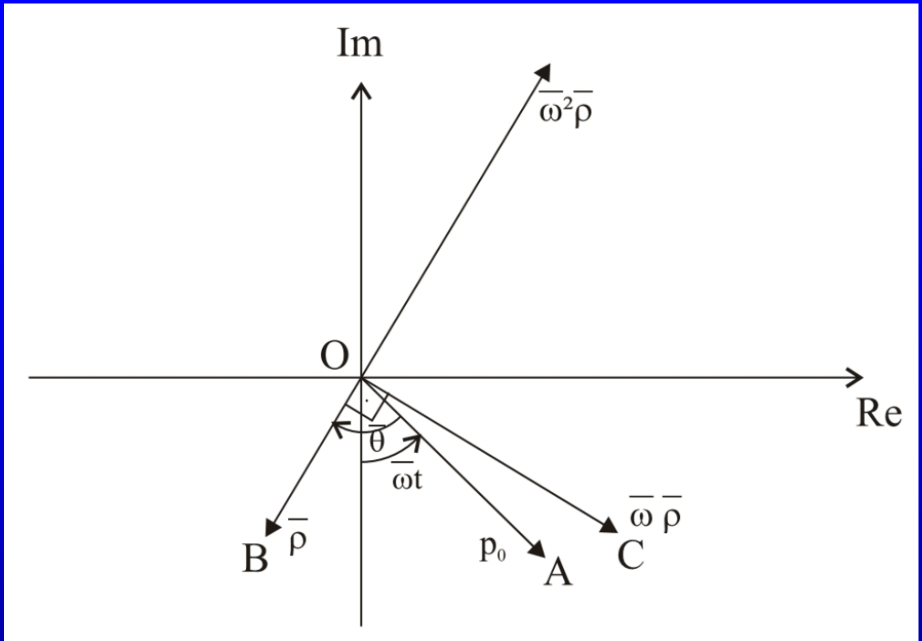

⟵ Ângulo de fase

Coeficiente de amplificação dinâmica

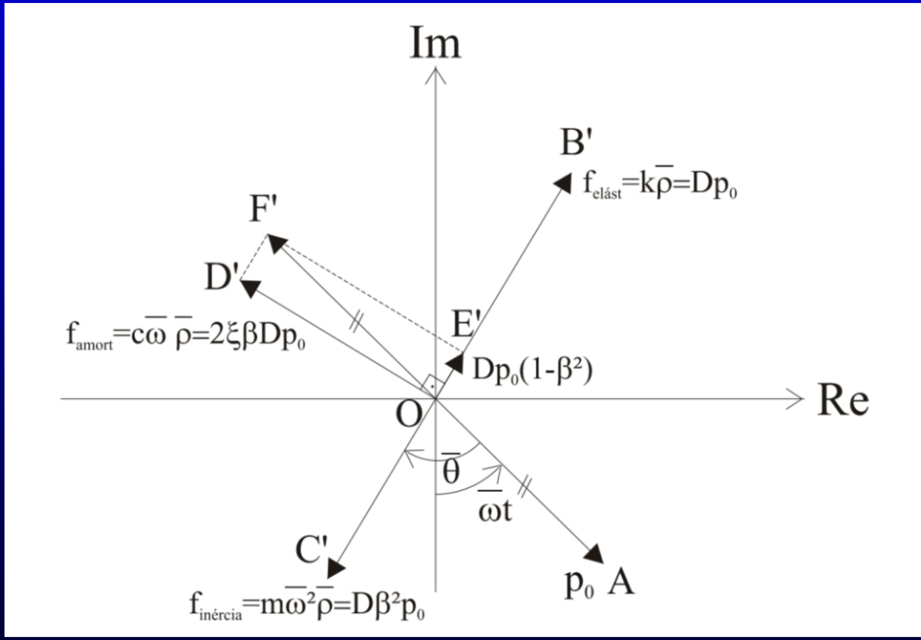



$$D_{max} \cong \frac{1}{2\xi}$$

# Vibração forçada com carregamento harmônico

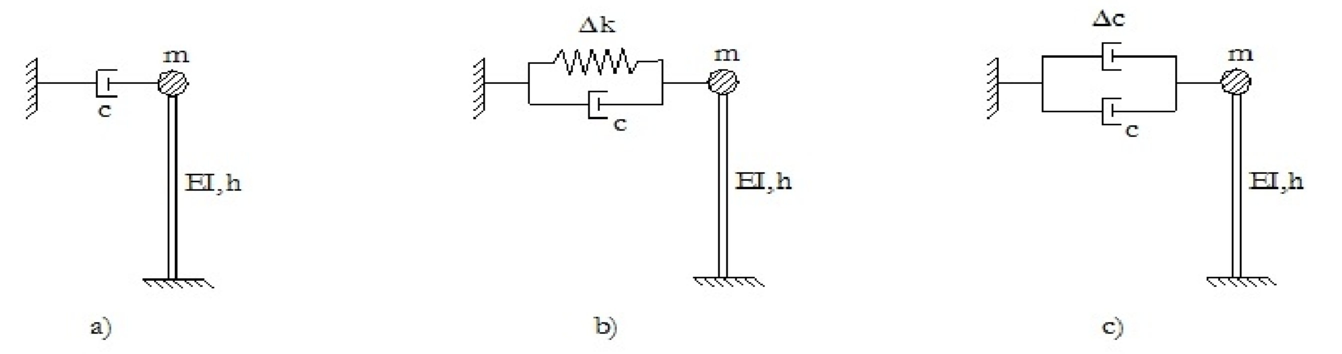



Interpretação geométrica no plano de Argand



# Vibração forçada com carregamento harmônico

## Exemplo: carregamento harmônico ressonante $\beta = 1$




$$m = 10^6\,kg,\ h = 15\,m,\ E = 3 \times 10^{10}\,Nm^{-2},\ I = 2,25\,m^4$$

Caso a: sabendo-se que $D_a = 4$, calcular $\xi$ e $c$

$$D_a = \frac{1}{2\xi_a} \Rightarrow \xi_a = 0,125$$

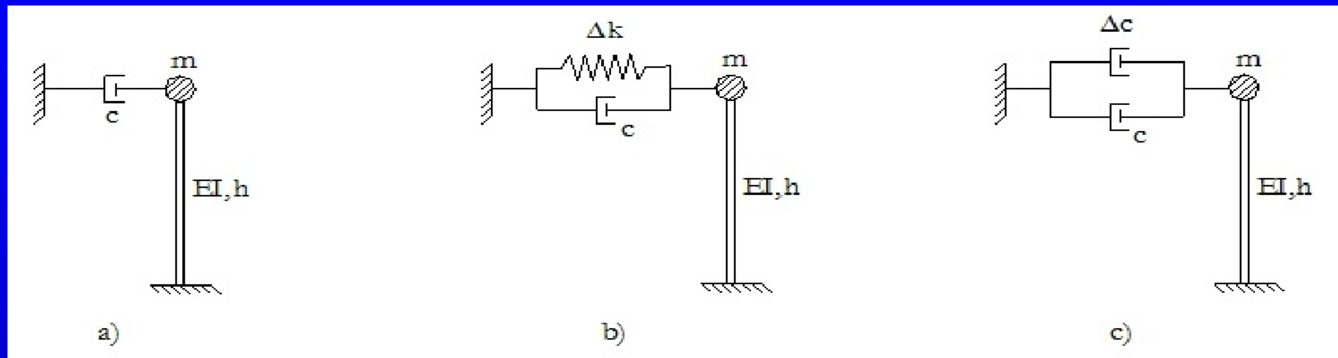
$$\xi_a = \frac{c}{2m\omega_a}$$

$$k_a = \frac{3EI}{h^3} = 60 \times 10^6\ N/m \Rightarrow \omega_a = \sqrt{\frac{k_a}{m}} = 7,746\,rad/s$$

$$\Rightarrow c = 1,937 \times 10^6\ Ns/m$$

# Vibração forçada com carregamento harmônico

## Exemplo: carregamento harmônico ressonante $\beta_a = 1$




$$m = 10^6\,kg,\ h = 15\,m,\ E = 3 \times 10^{10}\,Nm^{-2},\ I = 2,25\,m^4$$

## Caso b: quanto vale $\Delta k$ para $D_b = 2$ ?

$$k_b = k_a + \Delta k \Rightarrow \omega_b = \sqrt{\frac{k_b}{m}} > \omega_a \Rightarrow \beta_b = \frac{\bar{\omega}}{\omega_b} = \frac{\omega_a}{\omega_b} < \beta_a = 1$$

$$\Downarrow$$

$$\xi_b = \frac{c}{2m\omega_b} = \frac{c}{2m\omega_a}\left(\frac{\omega_a}{\omega_b}\right) = \xi_a\beta_b = 0,125\beta_b < \xi_a$$

$$\Leftarrow D_b = 2 = \frac{1}{\sqrt{\left[1-(\beta_b)^2\right]^2 + \left[0,25\,(\beta_b)^2\right]^2}}$$

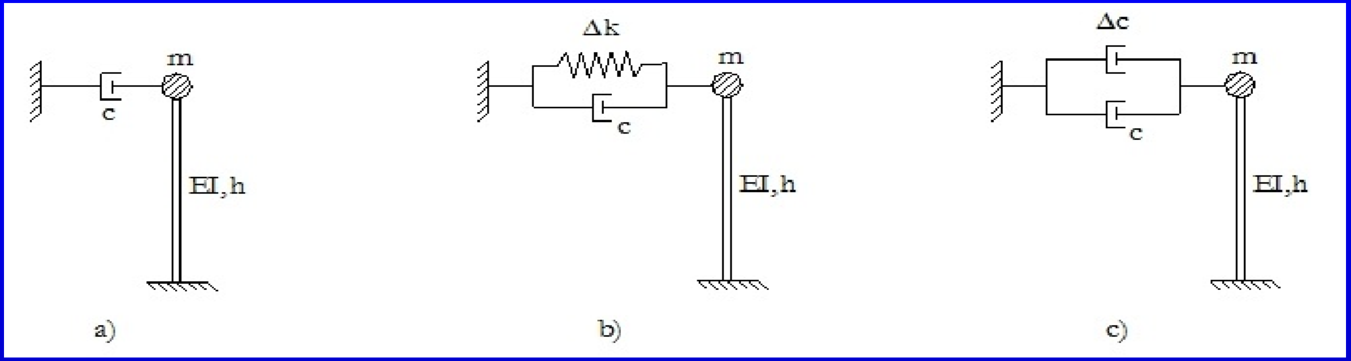
$$\Leftarrow \beta_b = 0,719 = \frac{7,746}{\omega_b}$$

$$\Downarrow$$

$$\omega_b = 10,773\,rad/s \Rightarrow k_b = k_a + \Delta k = m\,(\omega_b)^2 = 10^6 \times (10,773)^2 = 116,055 \times 10^6\,N/m \Rightarrow \Delta k = 56,055 \times 10^6\,N/m$$

# Vibração forçada com carregamento harmônico

Exemplo: carregamento harmônico ressonante $\beta = 1$




$m = 10^6\, kg,\ h = 15\, m,\ E = 3 \times 10^{10}\, Nm^{-2},\ I = 2,25\, m^4$

Caso c: quanto vale $\Delta c$ para $D_c = 2$ ?

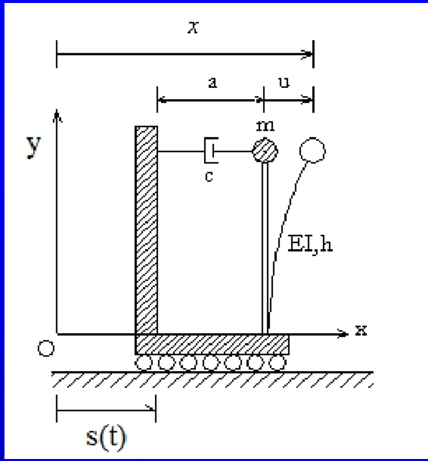
$D_c = 2 = \frac{1}{2\xi_c}$ ➡ $\xi_c = 0,25$ ➡ $c_c = c_a + \Delta c = 2m\omega_a\xi_c = 3,873 \times 10^6\, Ns/m$

⬇

$\Delta c = 1,937 \times 10^6\, Ns/m$

# Vibração forçada com carregamento harmônico

## Exemplo: excitação de suporte harmônica




Caso a $\omega = 7,746\,rad/s$ → $\beta = \frac{7}{7,746} = 0,9037$, $\xi = \frac{c}{2m\omega} = 0,125$ → $D = 3,437$

$c = 1,937 \times 10^6\,Ns/m,\ h = 15\,m,\ m = 10^6\,kg,\ s_0 = 0,010\,m,\ E = 3 \times 10^{10}\,Nm^{-2},\ \bar{\omega} = 7\,rad\,s^{-1},\ I = 2,25\,m^4$

Qual o máximo momento fletor na base da coluna?

$\ddot{u}_T = \ddot{s} + \ddot{u}$ → $m(\ddot{u} + \ddot{s}) + c\dot{u} + ku = 0$ →

$$m\ddot{u} + c\dot{u} + ku = p_{eq}(t),$$

$$p_{eq}(t) = -m\ddot{s} = m\bar{\omega}^2 s_0 \operatorname{sen} \bar{\omega}t$$

↓

$$u_e = \frac{m\bar{\omega}^2 s_0}{k} = 0,0082\,m$$

$$D = 3,437$$

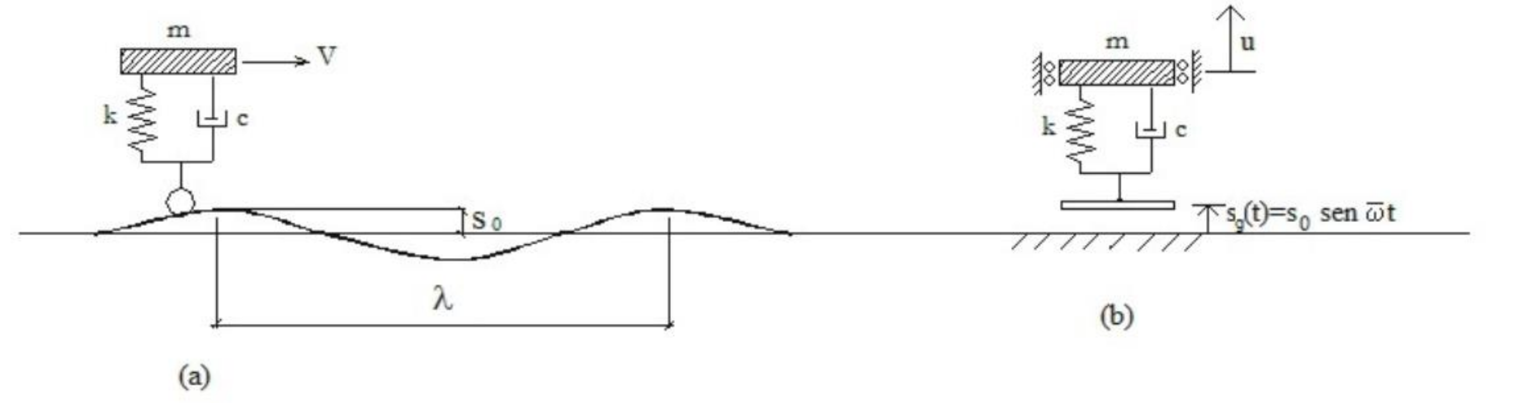
←

$$u_{max} = D\,u_e \cong 0,028\,m$$

$$M_{max} = k u_{max} h \cong 25200\,kNm$$

# Vibração forçada com carregamento harmônico

## Exemplo: transmissibilidade de deslocamentos




$$
\begin{aligned}
k &= 1,4865 \times 10^5\, N/m \\
m &= 1200\, kg \\
V &= 72\, km/h \\
s_0 &= 0,03\, m \\
\lambda &= 12\, m \\
u_{Tm\acute{a}x} &= 5\, cm
\end{aligned}
$$

Projetar o amortecedor

Critério de projeto

$$u(t) = \bar{\rho}\,\text{sen}\,\left(\bar{\omega}t - \bar{\theta}\right)$$

$$\bar{\rho} = D\frac{m\bar{\omega}^2 s_0}{k} = D\beta^2 s_0$$

$$\bar{\theta} = \arctan\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

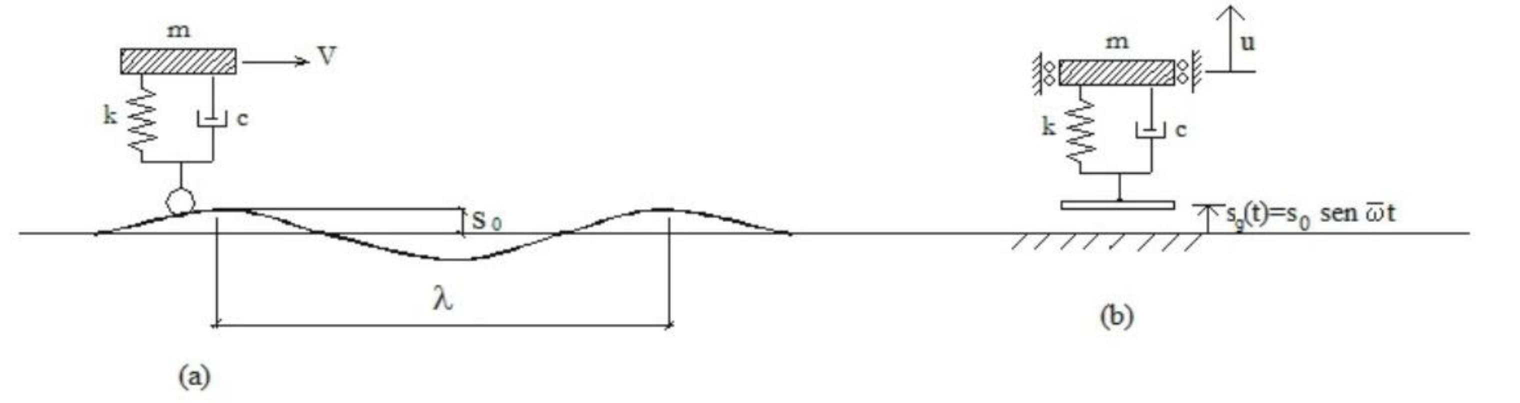
$$
\begin{aligned}
u_T(t) &= u(t) + s(t) = D\beta^2 s_0\,\text{sen}\,\left(\bar{\omega}t - \bar{\theta}\right) + s_0\,\text{sen}\,(\bar{\omega}t) = \\
& s_0\left(1 + D\beta^2 \cos\bar{\theta}\right)\text{sen}\,\bar{\omega}t - s_0 D\beta^2 \text{sen}\,\bar{\theta}\cos\bar{\omega}t,
\end{aligned}
$$

$$u_{Tm\acute{a}x} = TR\; s_0$$

$$TR = D\sqrt{1 + (2\xi\beta)^2}$$

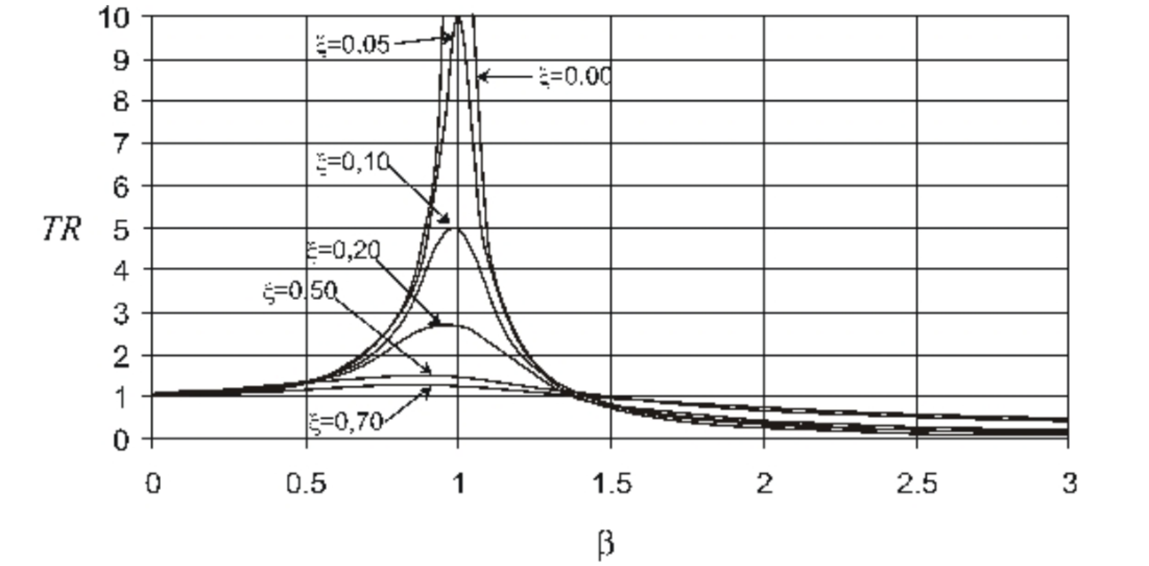
# Vibração forçada com carregamento harmônico

## Exemplo: transmissibilidade de deslocamentos



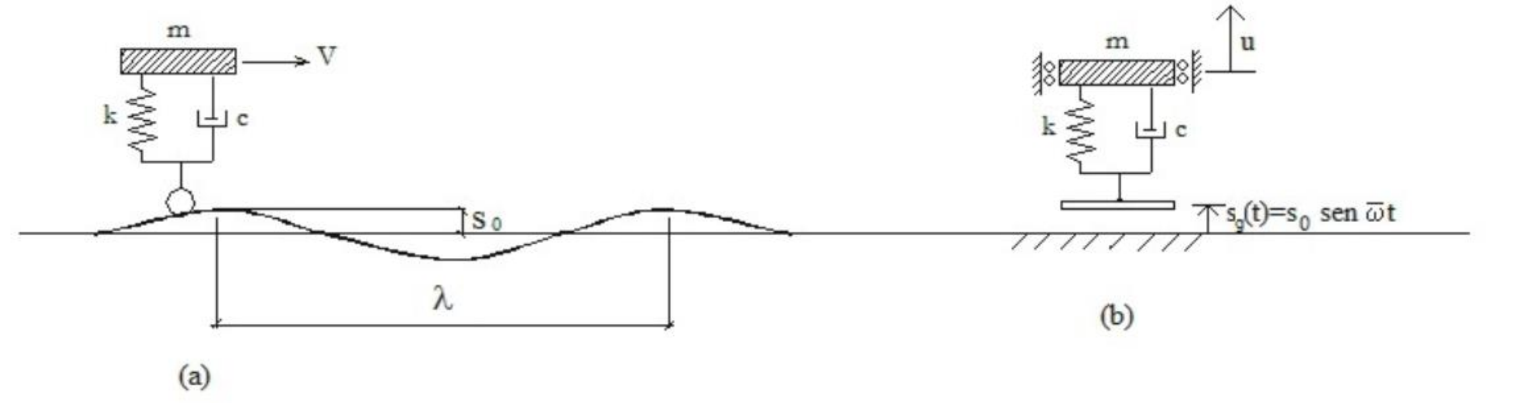

$$u_{Tmáx} = TR\ s_0$$

$$TR = D\sqrt{1+(2\xi\beta)^2}$$



# Vibração forçada com carregamento harmônico

## Exemplo: transmissibilidade de deslocamentos




$$u_{Tmáx} = TR\ s_0$$

$$TR = D\sqrt{1+(2\xi\beta)^2}$$

$$\frac{5}{3} = D\sqrt{1+(2\xi\beta)^2} = \sqrt{\frac{1+(2\xi\beta)^2}{(1-\beta^2)^2+(2\xi\beta)^2}} \Rightarrow \xi = \frac{\sqrt{9-25\,(1-\beta^2)^2}}{8\beta}$$

$$\omega = \sqrt{\frac{k}{m}} = 11,130\,rad/s \Rightarrow T = \frac{2\pi}{\omega} = 0,565\,s \Rightarrow \beta = \frac{\bar{\omega}}{\omega} = \frac{T}{\bar{T}} = 0,941$$

$$V = 72\,km/h = 20\,m/s \Rightarrow \bar{T} = \frac{\lambda}{V} = 0,6\,s \Rightarrow \bar{\omega} = \frac{2\pi}{\bar{T}} = 10,472\,rad/s$$

$$\Rightarrow \xi = 0,391$$

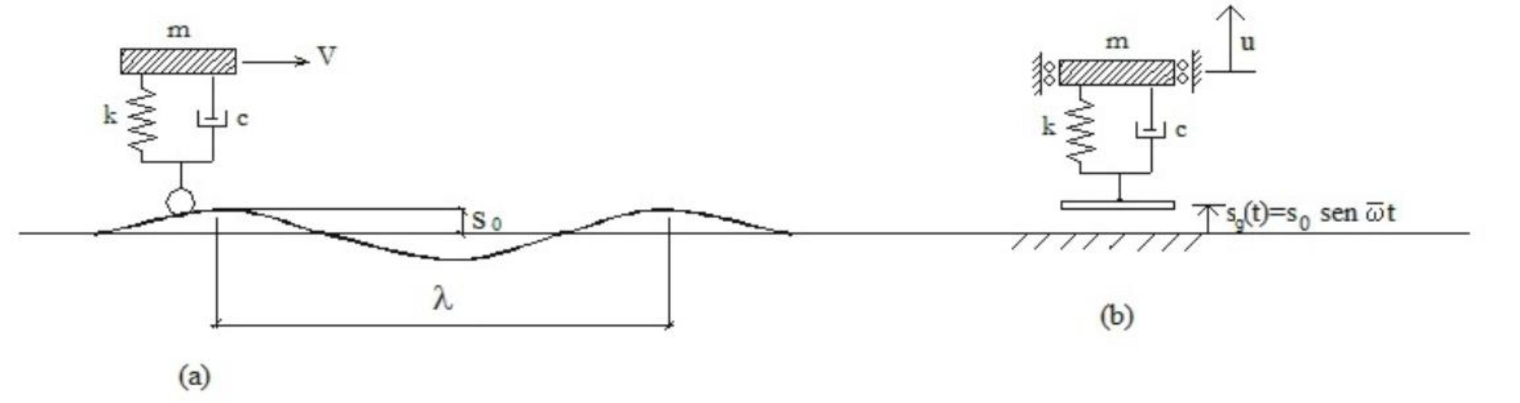
$$c = 2\xi m\omega = 10450\,Ns/m$$

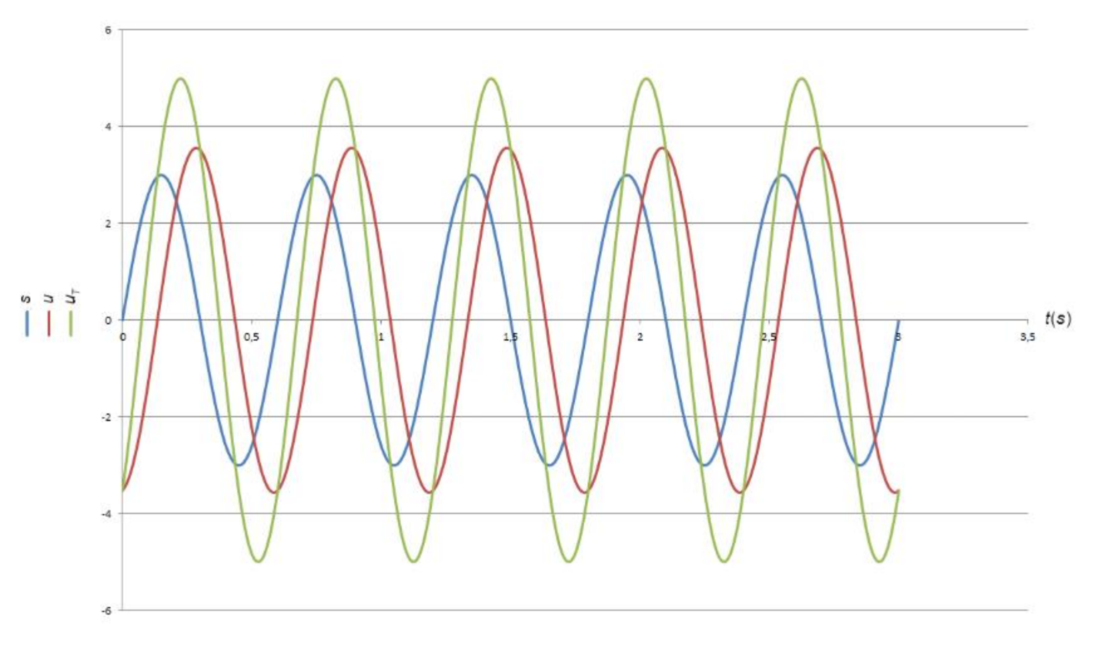
$$D = 1,342$$

$$\bar{\rho} = D\beta^2 s_0 = 3,56cm$$

# Vibração forçada com carregamento harmônico

## Exemplo: transmissibilidade de deslocamentos

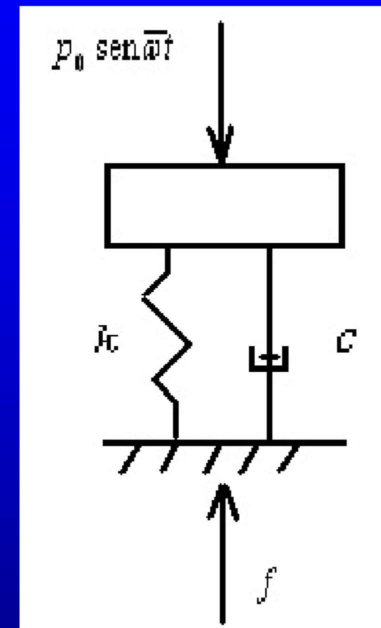


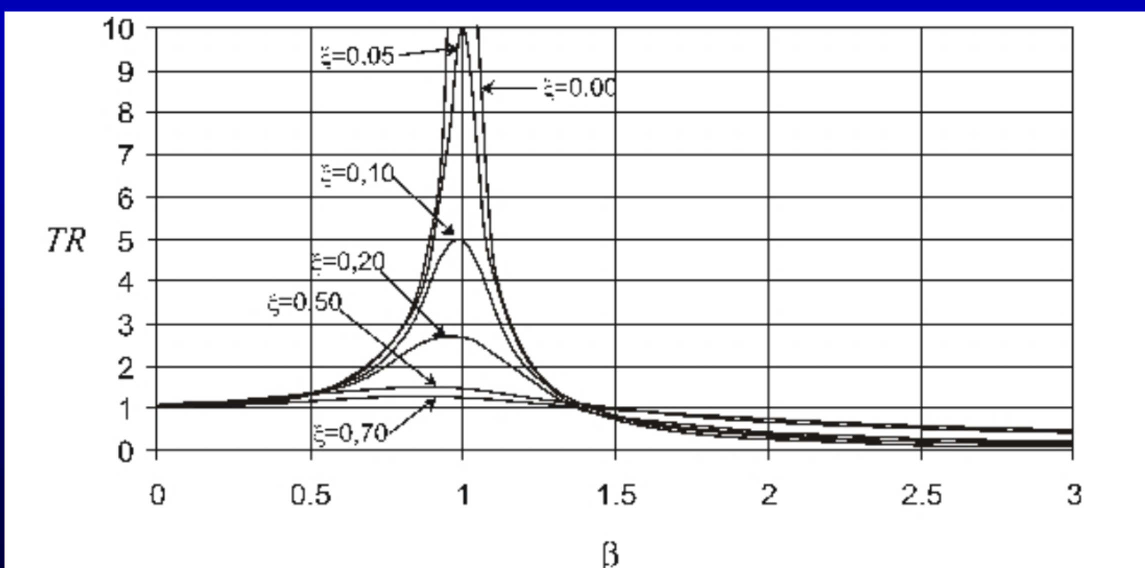
# Vibração forçada com carregamento harmônico

## Exemplo: transmissibilidade de forças

$$\begin{aligned} f_{mola} &= ku = Dp_0 \text{sen}\left(\bar{\omega}t - \bar{\theta}\right) \\ f_{amort} &= c\dot{u} = 2\xi\beta D p_0 \cos\left(\bar{\omega}t - \bar{\theta}\right) \end{aligned}$$




$$f_{\max} = \sqrt{(Dp_0)^2 + (2\xi\beta Dp_0)^2} = p_0 D\sqrt{1 + (2\xi\beta)^2} = TR\ p_0$$



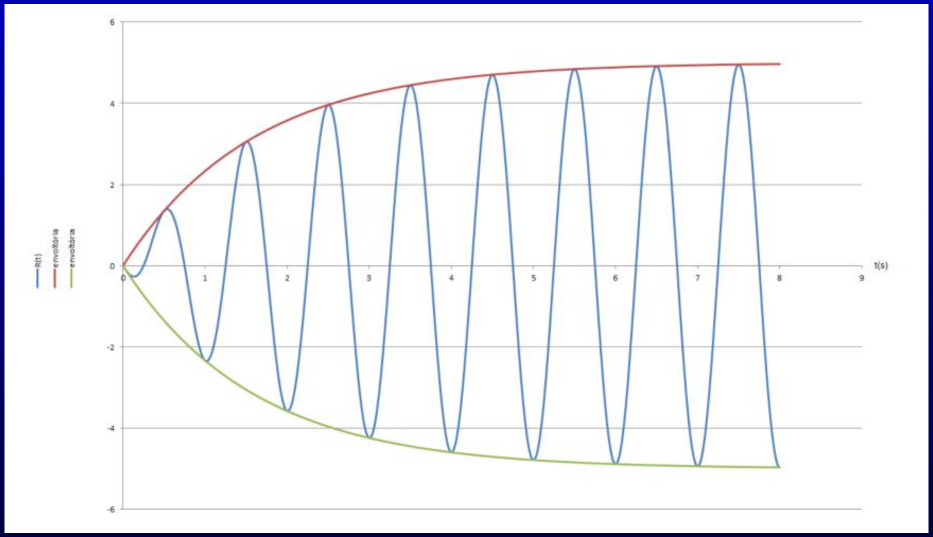
# Vibração forçada com carregamento harmônico

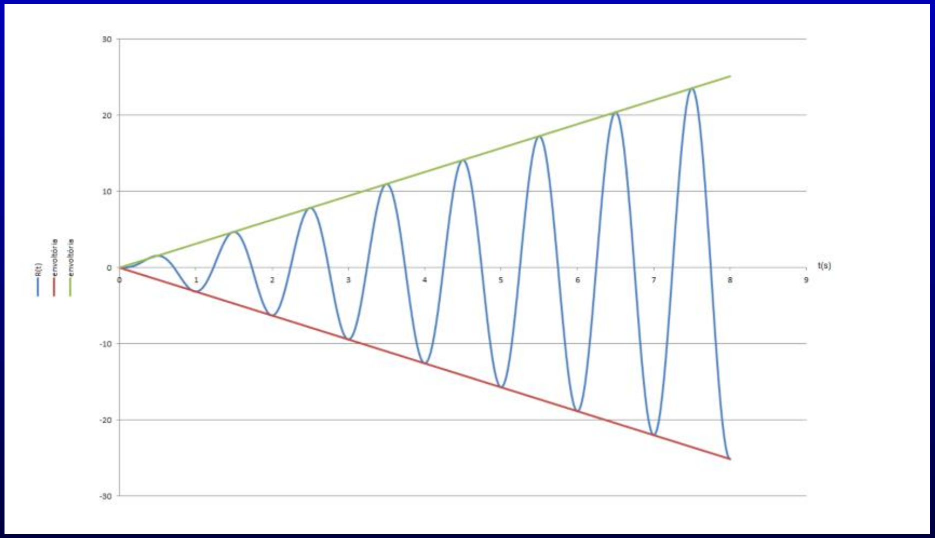
## Resposta transitória ressonante

$$R(t) = \frac{u\,(t)}{u_e} = \frac{1}{2\xi}\left[\frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega t}\cos\left(\omega_D t - \theta\right) + \text{sen}\,\left(\omega t - \bar{\theta}\right)\right]$$
$$= \frac{1}{2\xi}\left[\frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega t}\cos\left(\omega_D t - \theta\right) - \cos\,\omega t\right]$$

$$R(t) = \frac{u\,(t)}{u_e} \cong \frac{1}{2\xi}\left[e^{-\xi\omega t} - 1\right]\cos\,\omega t$$

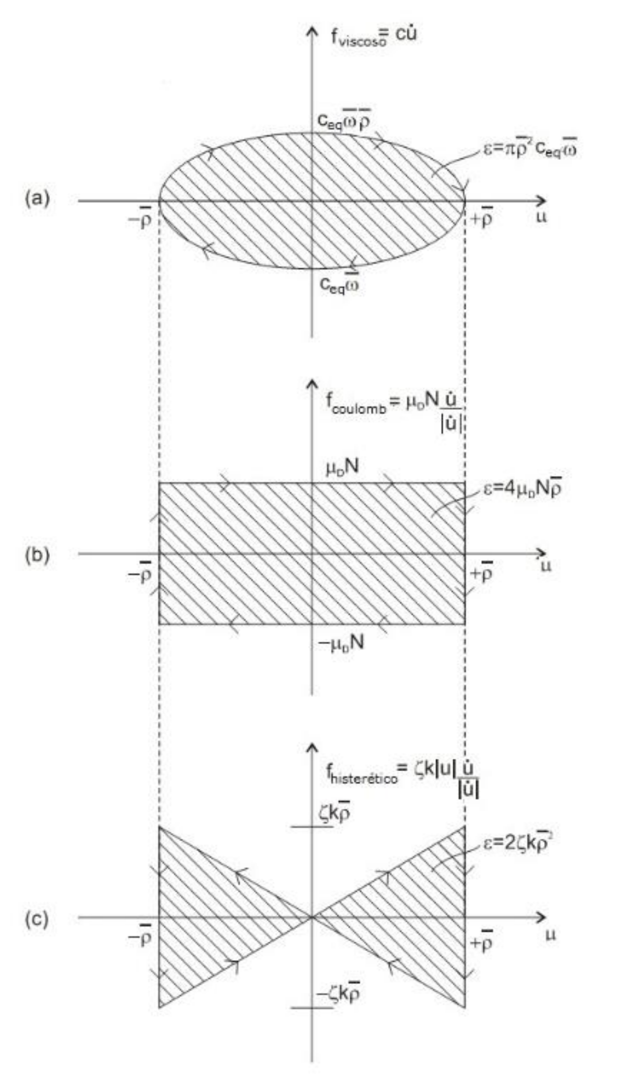
$$\lim_{\xi\to 0} R(t) = \lim_{\xi\to 0}\frac{u\,(t)}{u_e} = \frac{1}{2}\left[\text{sen}\omega t - \omega t\cos\,\omega t\right]$$



# Vibração forçada com carregamento harmônico

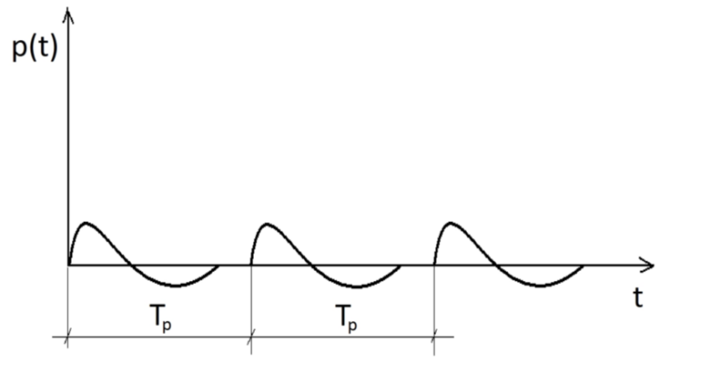
## Amortecedor viscoso linear equivalente




$$\xi_{eq} = \frac{2\mu_D N}{\pi m \omega \bar{\omega} \bar{\rho}} = \frac{2\mu_D N}{\pi k \beta \bar{\rho}}$$

$$\xi_{eq} = \frac{\zeta}{\pi \beta}$$

# Resposta a carregamento periódico




Série de Fourier

$$p(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \overline{\omega}_n t + \sum_{n=1}^{\infty} b_n \text{sen}\, \overline{\omega}_n t$$

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t)\, dt$$

$$\overline{\omega}_n = n \left( \frac{2\pi}{T_p} \right)$$

$$\begin{aligned} a_n &= \frac{2}{T_p} \int_0^{T_p} p(t) \cos \overline{\omega}_n t \; dt \\ b_n &= \frac{2}{T_p} \int_0^{T_p} p(t) \,\text{sen}\, \overline{\omega}_n t \; dt \end{aligned}$$

# Resposta a carregamento periódico

Equação do movimento

$$m\ddot{u} + c\dot{u} + ku = a_0 + \sum_{n=1}^{\infty} a_n \cos \overline{\omega}_n t + \sum_{n=1}^{\infty} b_n \text{sen}\, \overline{\omega}_n t$$

Resposta em regime permanente

$$u(t) = \frac{a_0}{k} + \frac{1}{k}\sum_{n=1}^{\infty} D_n^2 \left[\left(1-\beta_n^2\right) b_n + \left(2\xi\beta_n\right) a_n\right] \text{sen}\, \overline{\omega}_n t + \frac{1}{k}\sum_{n=1}^{\infty} D_n^2 \left[\left(1-\beta_n^2\right) a_n - \left(2\xi\beta_n\right) b_n\right] \cos \overline{\omega}_n t]$$

$$\beta_n = \frac{\overline{\omega}_n}{\omega} = n\left(\frac{T}{T_p}\right)$$

$$D_n = \frac{1}{\sqrt{\left(1-\beta_n^2\right)^2 + \left(2\xi\beta_n\right)^2}}$$

$$u(t) = c_0 + \sum_{n=1}^{\infty} c_n \text{sen}\left(\overline{\omega}_n t - \bar{\theta}_n\right)$$

$$c_0 = \frac{a_0}{k}$$

$$c_n = \frac{D_n}{k}\sqrt{a_n^2 + b_n^2}\,, n = 1, 2, 3, \ldots$$

$$\bar{\theta}_n = \arctan\left[-\frac{\left(1-\beta_n^2\right) a_n - \left(2\xi\beta_n\right) b_n}{\left(1-\beta_n^2\right) b_n + \left(2\xi\beta_n\right) a_n}\right]$$

$$0 < \bar{\theta}_n < 2\pi$$

$$\cos(\bar{\theta}_n) = \frac{{D_n}^2}{kc_n}\left[\left(1-\beta_n^2\right) b_n + \left(2\xi\beta_n\right) a_n\right]$$

$$sen(\bar{\theta}_n) = -\frac{{D_n}^2}{kc_n}\left[\left(1-\beta_n^2\right) a_n - \left(2\xi\beta_n\right) b_n\right]$$

# Resposta a carregamento periódico

Notação exponencial complexa

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{i\varpi_n t}$$

$$P_0 = \frac{1}{T_p}\int_0^{T_p} p(t)\, dt = a_0$$

$$P_n = \frac{1}{T_p}\int_0^{T_p} p(t)\, e^{-i\varpi_n t} dt = \frac{1}{2}(a_n - ib_n), n \neq 0$$

Função complexa de resposta em frequência

$$m\ddot{u} + c\dot{u} + ku = \exp(i\varpi t)$$

$$u(t) = H(\varpi) e^{i\varpi t}$$

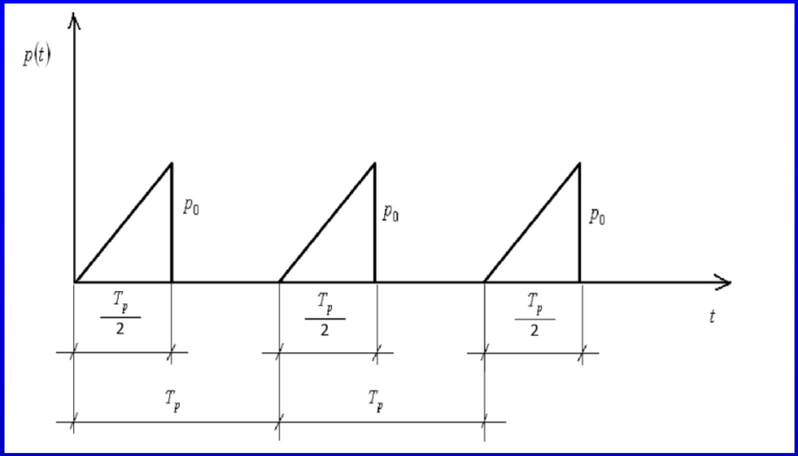
$$H(\varpi) = \frac{1}{-\varpi^2 m + i\varpi c + k} = \frac{(1-\beta^2)D^2}{k} - \frac{2\beta\xi D^2}{k} i$$

$$m\ddot{u} + c\dot{u} + ku = \sum_{n=-\infty}^{\infty} P_n e^{i\varpi_n t}$$

$$\begin{aligned}
u(t) &= \sum_{n=-\infty}^{\infty} P_n H(\varpi_n) e^{i\varpi_n t} = C_0 + \sum_{n=1}^{\infty} C_n \mathrm{sen}\left(\varpi_n t - \hat{\theta}_n\right) \\
C_0 &= \frac{a_0}{k} = c_0 \\
C_n &= \frac{D_n}{k}\sqrt{a_n^2 + b_n^2} = c_n \\
\hat{\theta}_n &= \arctan\left[-\frac{(1-\beta_n^2)\, a_n - (2\xi\beta_n)\, b_n}{(1-\beta_n^2)\, b_n + (2\xi\beta_n)\, a_n}\right] = \bar{\theta}_n
\end{aligned}$$

# Resposta a carregamento periódico

## Exemplo




$$a_0 = \frac{p_0}{4}$$

$$a_n = 0 \ \text{(para } n \neq 0 \text{ e par) ou } -\frac{2p_0}{(\pi n)^2} \ \text{(para } n \text{ ímpar)}$$

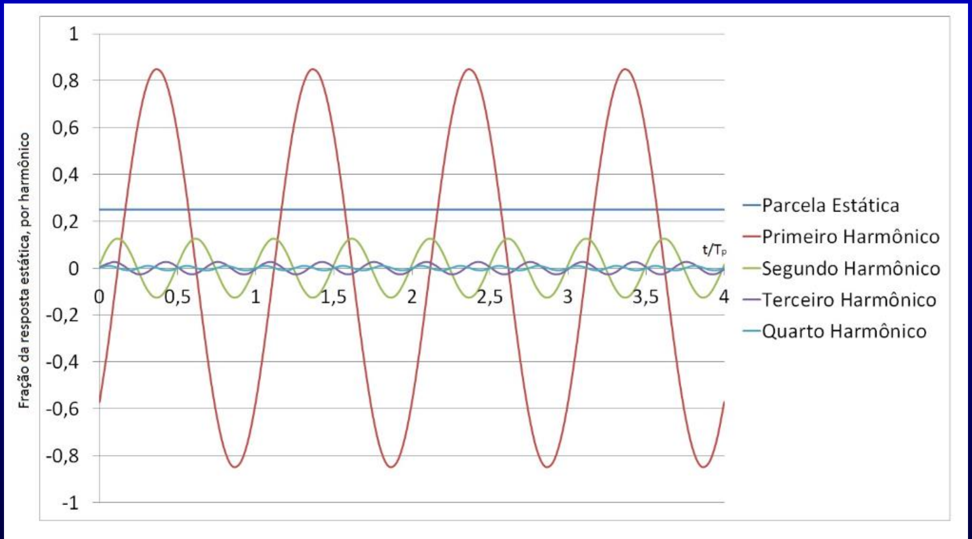
$$b_n = -\frac{p_0}{\pi n} \ \text{(para } n \neq 0 \text{ e par) ou } \frac{p_0}{\pi n} \ \text{(para } n \text{ ímpar)}$$

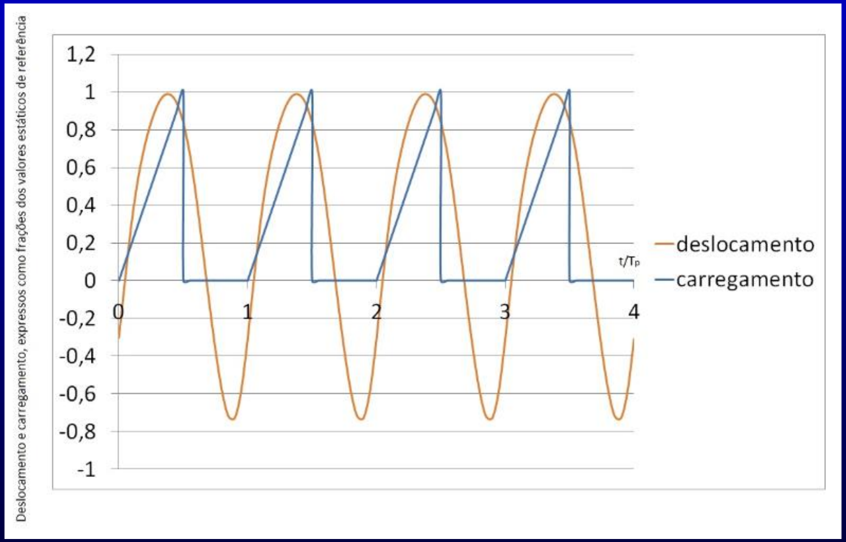
| $n$ | $\xi$ | $\beta_n$ | $D_n$ | $\left(\frac{1}{p_0}\right) a_n$ | $\left(\frac{1}{p_0}\right) b_n$ | $\left(\frac{k}{p_0}\right) c_n$ | $\bar{\theta}_n$ |
|---|---|---|---|---|---|---|---|
| 1 | 0,05 | 0,7500 | 2,2529 | -0,2026 | 0,3183 | 0,8501 | 0,7367 |
| 2 | 0,05 | 1,5000 | 0,7943 | 0,0000 | -0,1592 | 0,1264 | 6,1638 |
| 3 | 0,05 | 2,2500 | 0,2458 | 3,2954 | 0,1061 | 0,0267 | 3,2954 |
| 4 | 0,05 | 3,0000 | 0,1249 | 0,0000 | -0,0796 | 0,0099 | 6,2457 |

# Resposta a carregamento periódico

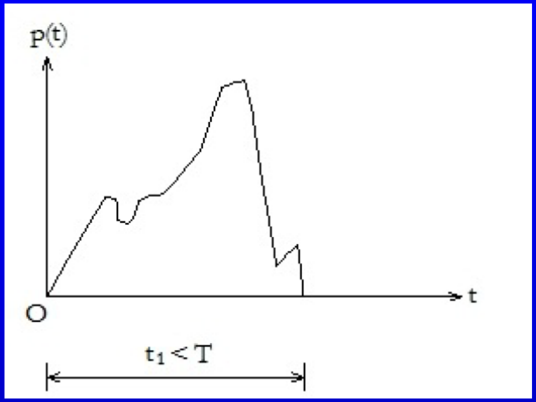
## Exemplo

$$\frac{u(t)}{\frac{p_0}{k}} = 0,2500 + 0,8501\text{sen}\left(\frac{2\pi}{T_p}t - 0,7367\right) + 0,1264\text{sen}\left(\frac{4\pi}{T_p}t - 6,1638\right) + 0,0267\text{sen}\left(\frac{6\pi}{T_p}t - 3,2954\right) + 0,0099\text{sen}\left(\frac{8\pi}{T_p}t - 6,2457\right)$$



# Resposta a carregamento impulsivo




É usual desprezar o amortecimento!

- fase 1 $(0 \leqslant t \leqslant t_1)$: vibrações forçadas;
- fase 2 $(t > t_1)$: vibrações livres, tendo como condições "iniciais" o deslocamento e a velocidade atingidos em $t_1$, quando cessa o carregamento.

# Resposta a carregamento impulsivo

Pulso retangular

$$p(t) = p_0, \text{ para } 0 \leqslant t \leqslant t_1$$
$$p(t) = 0, \text{ para } t > t_1$$

- **Fase 1** $(0 \leqslant t \leqslant t_1)$

$$m\ddot{u} + ku = p_0$$
$$u_0 = 0 \text{ e } \dot{u}_0 = 0$$

$\Rightarrow$

$$u(t) = \frac{p_0}{k}(1 - \cos\omega t) = u_e(1 - \cos\omega t)$$

$\frac{T}{2} \leqslant t_1$ $\Rightarrow$ $u_{max} = 2u_e$ $\Rightarrow$ $D = 2$

- **Fase 2** $(t > t_1)$

$$m\ddot{\bar{u}} + k\bar{u} = 0$$

Definem-se $\bar{t} = t - t_1 > 0$ e $\bar{u}(\bar{t}) = u(t)$

$$\bar{u}(0) = u(t_1) \text{ e } \dot{\bar{u}}(0) = \dot{u}(t_1)$$

$\Rightarrow$

$$\bar{u}(\bar{t}) = \bar{\rho}\cos\left(\omega\bar{t} - \bar{\theta}\right)$$

$$\bar{\theta} = \arctan\left[\frac{\dot{u}(t_1)}{\omega u(t_1)}\right] = \arctan\left(\frac{sen\omega t_1}{1 - cos\omega t_1}\right)$$

$$\bar{\rho} = \sqrt{[u(t_1)]^2 + \left[\frac{\dot{u}(t_1)}{\omega}\right]^2} = 2\,\text{sen}\left(\frac{\pi t_1}{T}\right)u_e$$

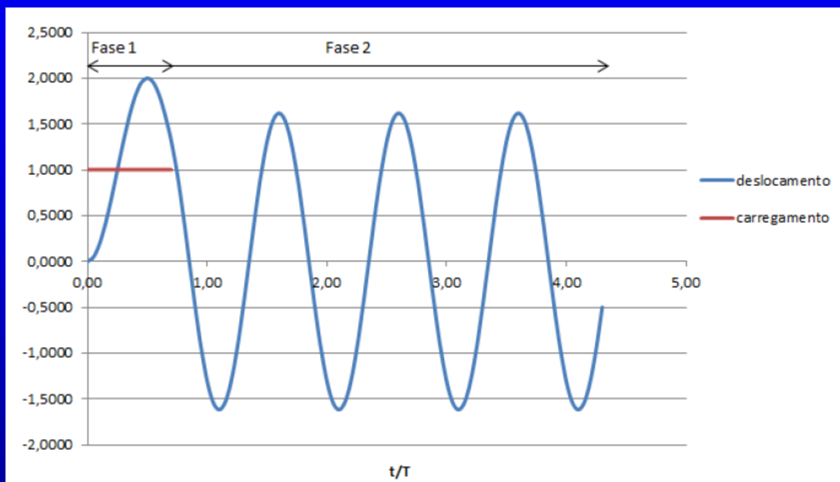
$\frac{t_1}{T} < \frac{1}{2}$ $\Rightarrow$ $D = \frac{\bar{\rho}}{u_e} = 2\,\text{sen}\left(\frac{\pi t_1}{T}\right)$

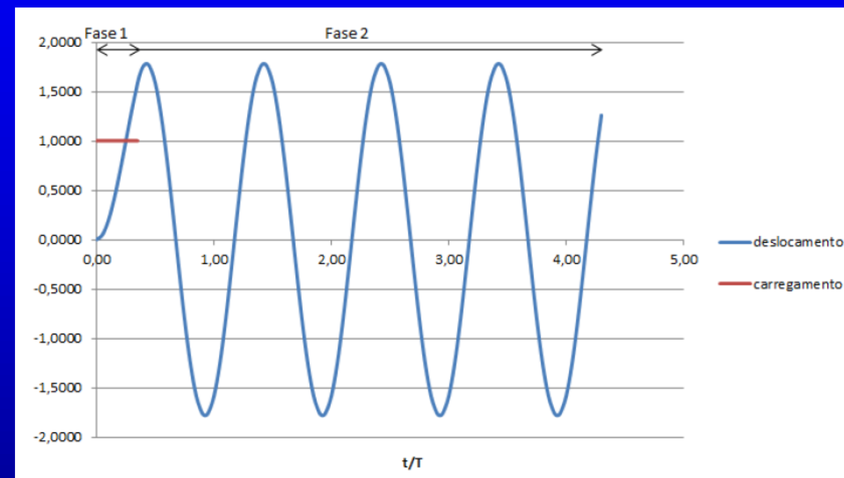
# Resposta a carregamento impulsivo

Pulso retangular

$$p(t) = p_0, \text{ para } 0 \leqslant t \leqslant t_1$$

$$p(t) = 0, \text{ para } t > t_1$$




$\frac{t_1}{T} = 0,70$ ➡ $D = 2$

$\frac{t_1}{T} = 0,35$ ➡ $D = 1,78$

# Resposta a carregamento impulsivo

Pulso senoidal

$$p(t) = p_0 \sin \overline{\omega} t \;\text{ para } 0 \leqslant t \leqslant t_1 = \frac{\pi}{\overline{\omega}}$$
$$p(t) = 0, \text{ para } t > t_1$$

- **Fase 1** $(0 \leqslant t \leqslant t_1)$

$$m\ddot{u} + ku = p_0 \sin \overline{\omega} t$$

$$u_0 = 0 \text{ e } \dot{u}_0 = 0$$

→ $$u(t) = \frac{1}{1-\beta^2}\left(\text{sen}\,\overline{\omega} t - \beta \text{sen}\,\omega t\right) u_e$$

$\beta = \frac{\overline{\omega}}{\omega} \leq 1$ → $$D = \frac{u_{\max}}{u_e} = \left|\frac{1}{1-\beta^2}\left[\text{sen}\left(\frac{2\pi\beta}{1+\beta}\right) - \beta\text{sen}\left(\frac{2\pi}{1+\beta}\right)\right]\right|$$

- **Fase 2** $(t > t_1)$

$$m\ddot{\bar{u}} + k\bar{u} = 0$$

Definem-se $\bar{t} = t - t_1 > 0$ e $\bar{u}(\bar{t}) = u(t)$

$$\bar{u}(0) = u(t_1) \text{ e } \dot{\bar{u}}(0) = \dot{u}(t_1)$$

→ $$\bar{u}(\bar{t}) = \bar{\rho} \cos\left(\omega \bar{t} - \bar{\theta}\right)$$

$$\bar{\theta} = \arctan\left[\frac{\dot{u}(t_1)}{\omega u(t_1)}\right]$$

$$\bar{\rho} = \sqrt{\left[u(t_1)\right]^2 + \left[\frac{\dot{u}(t_1)}{\omega}\right]^2}$$

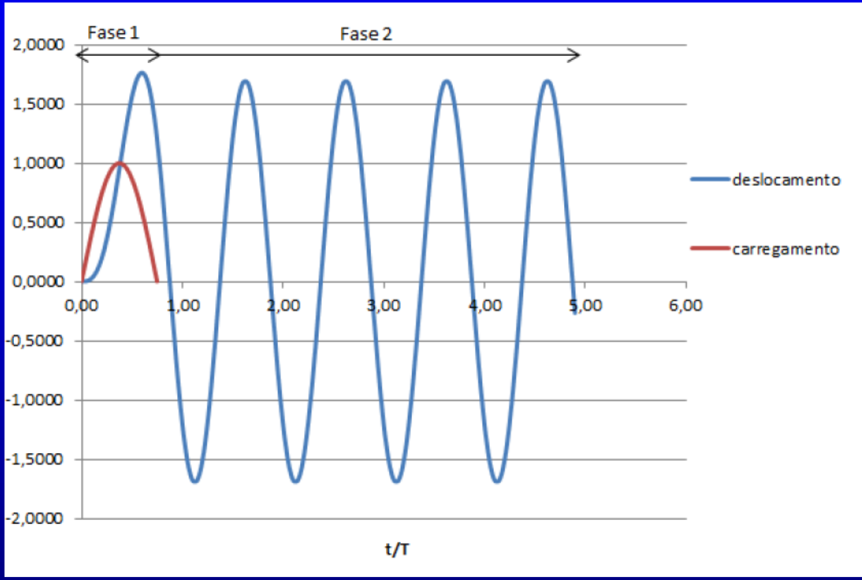
$\beta > 1$ → $$D = \frac{u_{max}}{u_e} = \left|\frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}\right|$$

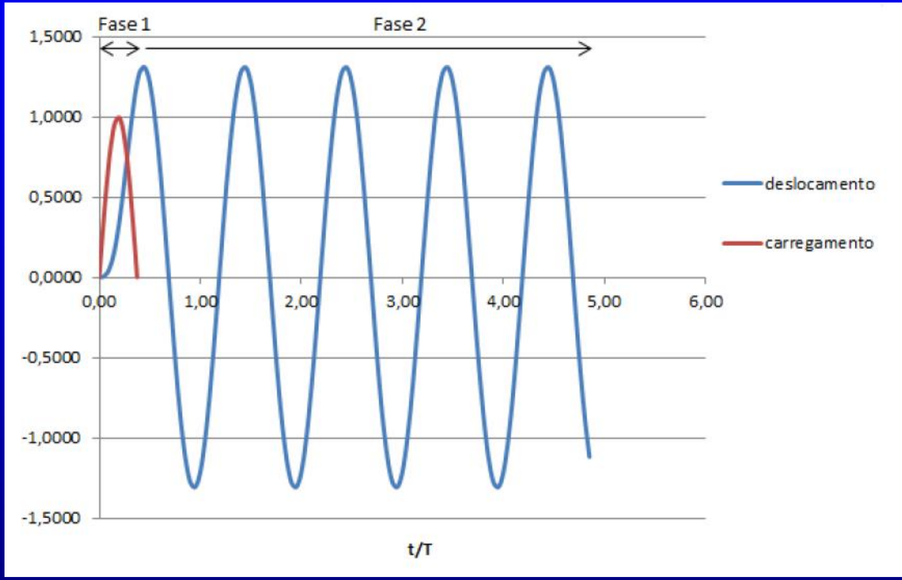
# Resposta a carregamento impulsivo

Pulso senoidal

$$p(t) = p_0 \sin \overline{\omega} t \;\; \text{para} \;\; 0 \leqslant t \leqslant t_1 = \frac{\pi}{\omega}$$

$$p\,(t) = 0, \;\text{para}\; t > t_1$$




$\beta = \frac{2}{3}$ ➡ $D = 1,76$

$\beta = \frac{4}{3}$ ➡ $D = 1,31$

# Resposta a carregamento impulsivo

## Pulso triangular

$$p(t) = p_0\left(1-\frac{t}{t_1}\right),\ \text{para } 0 \leqslant t \leqslant t_1$$
$$p(t) = 0,\ \text{para } t > t_1$$

- **Fase 1** $(0 \leqslant t \leqslant t_1)$

$$m\ddot{u} + ku = p_0\left(1-\frac{t}{t_1}\right)$$

$$u_0 = 0 \text{ e } \dot{u}_0 = 0$$

$$\begin{aligned} u(t) &= u_e\left(1-\frac{t}{t_1}\right) + \rho\cos(\omega t - \theta), \\ \rho &= u_e\sqrt{1+\left(\frac{1}{\omega t_1}\right)^2}, \\ \theta &= \arctan\left(\frac{-1}{\omega t_1}\right), 0 \leq \theta \leq \pi. \end{aligned}$$

- **Fase 2** $(t > t_1)$

$$m\ddot{\bar{u}} + k\bar{u} = 0$$

Definem-se $\bar{t} = t - t_1 > 0$ e $\bar{u}(\bar{t}) = u(t)$

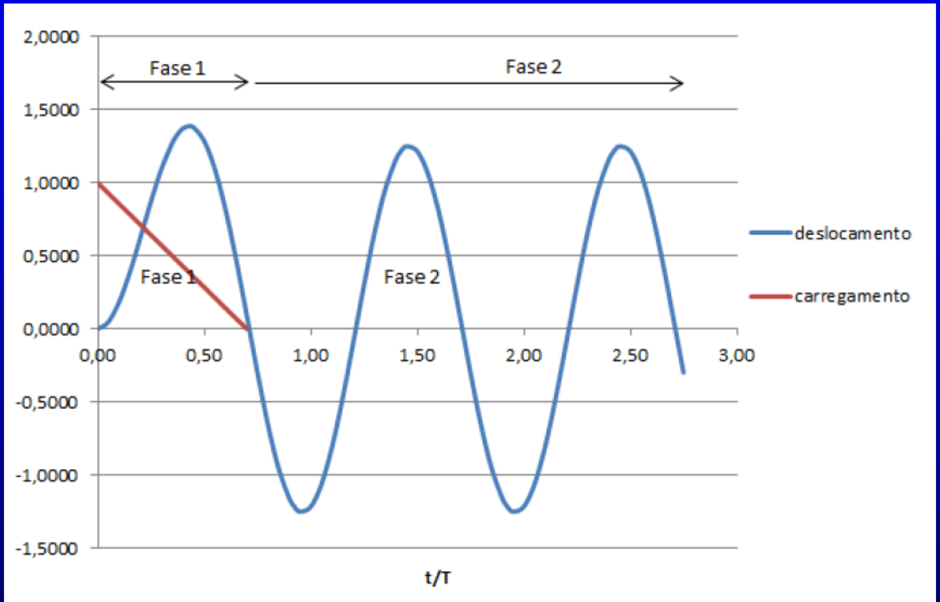
$$\bar{u}(0) = u(t_1) \text{ e } \dot{\bar{u}}(0) = \dot{u}(t_1)$$

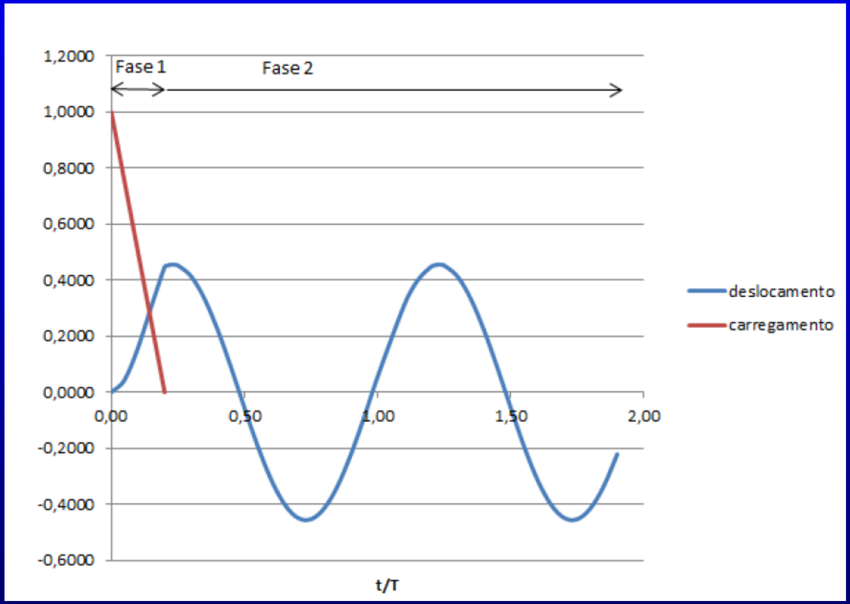
$$\begin{aligned} \bar{u}(\bar{t}) &= \bar{\rho}\cos\left(\omega\bar{t} - \bar{\theta}\right), \\ \bar{\theta} &= \arctan\left[\frac{\dot{u}(t_1)}{\omega u(t_1)}\right], \\ \bar{\rho} &= \sqrt{[u(t_1)]^2 + \left[\frac{\dot{u}(t_1)}{\omega}\right]^2}. \end{aligned}$$

# Resposta a carregamento impulsivo

Pulso triangular

$$p(t) = p_0\left(1 - \frac{t}{t_1}\right),\ \text{para}\ 0 \leqslant t \leqslant t_1$$
$$p(t) = 0,\ \text{para}\ t > t_1$$




$\frac{t_1}{T} = 0,70$ ➡ $D = 1,39$

$\frac{t_1}{T} = 0,20$ ➡ $D = 0,5981$

# Resposta a carregamento impulsivo de curtíssima duração

$\frac{t_1}{T} < 0,25$ ⇨ Além das forças dissipativas, as elásticas também são desprezadas

⇩

$$m\ddot{u} = p(t), \qquad 0 \leqslant t \leqslant t_1$$

⇩

$$\Delta Q = m\left[\dot{u}(t_1) - \dot{u}_0\right] = \int_0^{t_1} p(t)\,dt = I$$

⇨

$$\dot{u}(t_1) = \frac{I}{m}$$

⇩

$$u(t) \cong \frac{\dot{u}(t_1)}{\omega} \operatorname{sen} \omega(t - t_1) = \frac{I}{m\omega} \operatorname{sen} \omega(t - t_1)$$

Influência do amortecimento

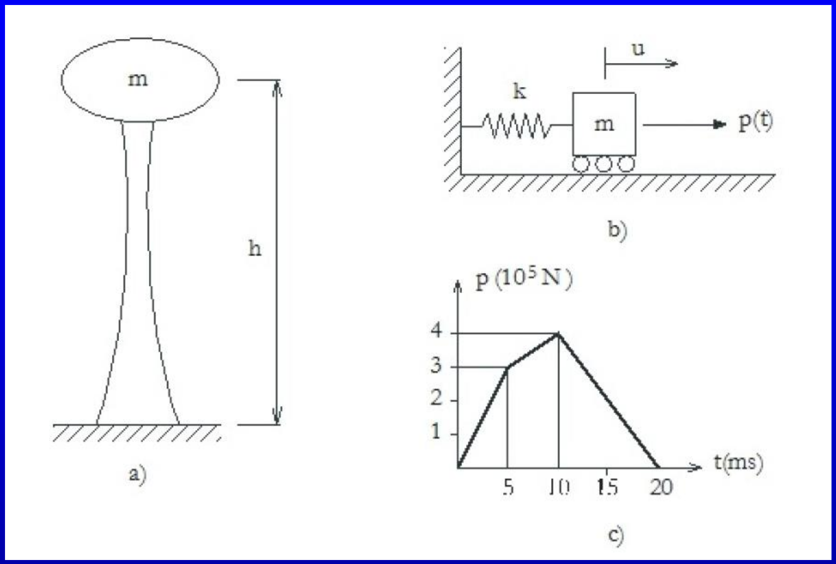
$$u(t) \cong \frac{I}{m\omega_D} e^{-\xi\omega(t-t_1)} \operatorname{sen} \omega_D (t - t_1)$$

⇨

$$u_{max} \cong u\left(t_1 + \frac{\pi}{2\omega_D}\right) = \frac{I}{m\omega_D} e^{-\frac{\xi\pi}{2}}, \text{ para } \xi \ll 1$$

# Resposta a carregamento impulsivo de curtíssima duração

## Exemplo




$$m = 7\times10^5\,kg,\ k = 7\times10^7\,N/m,\ h = 45\,m$$

$$\omega = \sqrt{\frac{k}{m}} = 10\,rad/s \quad \text{e} \quad T = \frac{2\pi}{\omega} = 0,6283\,s$$

$$\frac{t_1}{T} = \frac{0{,}02}{0{,}6283} = 0,0318 < 0,25$$

$$u_{max} = \frac{I}{m\omega} = \left(\frac{5\times3}{2} + \frac{5\times(3+4)}{2} + \frac{10\times4}{2}\right)\frac{10^{-3}\times10^5}{7\times10^5\times10} = 0,000643\,m$$

A máxima cortante no topo valerá: $F_{max} = k\,u_{max} = 4,5\times10^4 N = 45\,kN$
e o momento máximo na base será: $M_{max} = F_{max}h = 2025\,kNm$