# Derivadas Parciais de ordens superiores

Seja $f : \Omega \subset \mathbb{R}^2$ com derivadas parciais em relação a $x$e $y$ para todo $(x, y) \in A \subset D_f$. Deste modo, podemos considerar a função derivada parcial em relação a $x$ e $y$ das funções

$$\frac{\partial f}{\partial x} : A \to \mathbb{R}, \quad \frac{\partial f}{\partial y} : A \to \mathbb{R}$$

Denotamos por

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

## Exemplo

*Calcule todas as derivadas parciais de segunda ordem de* $f(x,y) = 3x^3y + 2xy^3 + x + y - 1$.

**solução:**

$$\frac{\partial f}{\partial x}(x,y) = 9x^2y + 2y^3 + 1, \quad \frac{\partial f}{\partial y}(x,y) = 3x^3 + 6xy^2 + 1$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = 18xy$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = 9x^2 + 6y^2$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = 12xy$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = 9x^2 + 6y^2$$

De maneira análoga, definimos as derivadas parciais de terceira ordem

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad \frac{\partial^3 f}{\partial y \partial^2 x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$\frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right), \quad \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right)$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right), \quad \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right), \quad \frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)$$

e assim por diante.

## Definição

*Seja* $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *e* $\Omega$ *aberto* $k \in \mathbb{N}$. *Dizemos que* $f$ *é de classe* $C^k$ *em* $\Omega$, *se existirem e forem contínuas todas as derivadas até a ordem* $k$ *de* $f$ *em todos os pontos de* $\Omega$. *Dizemos que* $f$ *é de classe* $C^\infty$ *se ela for* $C^k$ *para todo* $k$.

### Exemplo

*a) A função* $f(x, y) = x^4y^2 + x^2 + y^2 + 2xy$ *é de classe* $C^\infty$.

*b) A função* $f(x, y) = \frac{x^3+y^2}{x^2+y^2}$ *é de classe* $C^\infty$.

**Teorema**

*Seja* $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *e* $\Omega$ *aberto. Se* $f$ *for de classe* $C^2$ *em* $\Omega$ *então*

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$$

*para todo* $(x, y) \in \Omega$.

## Exemplo

*Seja* $f(x,y) = x + |y|$. *Temos que* $\frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$. *Por outro lado,* $\frac{\partial f}{\partial y}(0,0)$ *não existe e portanto* $\frac{\partial^2 f}{\partial y \partial x}(0,0)$ *também não existe.*

Seja

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

Mostre que

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0 \neq 1 = \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

**solução:** Para $(x, y) \neq (0, 0)$

$$\frac{\partial f}{\partial y}(x, y) = \frac{3xy^2(x^2 + y^2) - xy^3(2y)}{(x^2 + y^2)^2} = \frac{xy^4 + 3x^3y^2}{(x^2 + y^2)^2}$$

e

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y - 0} = 0$$

Deste modo,

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} \frac{xy^4 + 3x^3y^2}{(x^2+y^2)^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0) \end{cases}$$

Temos então

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x - 0} = \lim_{x \to 0} \frac{0}{x} = 0$$

Analogamente,

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y^5 - x^2 y^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

então

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y - 0} = \lim_{y \to 0} \frac{y}{y} = 1$$