Lista de exercícios 5 – Teoria de Frobenius (do livro de Boyce e Di Prima)

Problems

In each of Problems 1 through 8:

a. Find all the regular singular points of the given differential equation.

b. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$

2. $x^2 y'' - x(2+x)y' + (2+x^2)y = 0$

3. $y'' + 4xy' + 6y = 0$

4. $2x(x+2)y'' + y' - xy = 0$

5. $x^2 y'' + \frac{1}{2}(x + \sin x)y' + y = 0$

6. $x^2(1-x)y'' - (1+x)y' + 2xy = 0$

7. $(x-2)^2(x+2)y'' + 2xy' + 3(x-2)y = 0$

8. $(4-x^2)y'' + 2xy' + 3y = 0$

In each of Problems 9 through 12:

a. Show that $x = 0$ is a regular singular point of the given differential equation.

b. Find the exponents at the singular point $x = 0$.

c. Find the first three nonzero terms in each of two solutions (not multiples of each other) about $x = 0$.

9. $xy'' + y' - y = 0$

10. $xy'' + 2xy' + 6e^x y = 0$ (see Problem 1)

11. $xy'' + y = 0$

12. $x^2 y'' + (\sin x)y' - (\cos x)y = 0$

13. **a.** Show that

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$.

b. Determine the roots of the indicial equation at $x = 1$.

c. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n (x-1)^{r+n}$ corresponding to the larger root. You can assume $x - 1 > 0$.

d. What would you expect the radius of convergence of the series to be?

14. In several problems in mathematical physics, it is necessary to study the differential equation

$$x(1-x)y'' + (\gamma - (1+\alpha+\beta)x)y' - \alpha\beta y = 0, \qquad (25)$$

where α, β, and γ are constants. This equation is known as the **hypergeometric equation**.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are 0 and $1 - \gamma$.

b. Show that $x = 1$ is a regular singular point and that the roots of the indicial equation are 0 and $\gamma - \alpha - \beta$.

c. Assuming that $1 - \gamma$ is not a positive integer, show that, in the neighborhood of $x = 0$, one solution of equation (25) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots.$$

What would you expect the radius of convergence of this series to be?

d. Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left(1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right).$$

e. Show that the point at infinity is a regular singular point and that the roots of the indicial equation are α and β. See Problem 32 of Section 5.4.

15. Consider the differential equation

$$x^3 y'' + \alpha x y' + \beta y = 0,$$

where α and β are real constants and $\alpha \neq 0$.

a. Show that $x = 0$ is an irregular singular point.

b. By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear and that, consequently, there is only one formal solution of the assumed form.

c. Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of β/α, show that the formal series solution has a zero radius of convergence and so does not represent an actual solution in any interval.

16. Consider the differential equation

$$y'' + \frac{\alpha}{x^s}y' + \frac{\beta}{x^t}y = 0, \qquad (26)$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and s and t are positive integers that for the moment are arbitrary.

a. Show that if $s > 1$ or $t > 2$, then the point $x = 0$ is an irregular singular point.

b. Try to find a solution of equation (26) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad x > 0. \qquad (27)$$

Show that if $s = 2$ and $t = 2$, then there is only one possible value of r for which there is a formal solution of equation (26) of the form (27).

c. Show that if $s = 1$ and $t = 3$, then there are no solutions of equation (26) of the form (27).

d. Show that the maximum values of s and t for which the indicial equation is quadratic in r [and hence we can hope to find two solutions of the form (27)] are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a "weak singularity," or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution, we point out that although it is sometimes possible to obtain a formal series solution of the form (27) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 15 for an example.