O spin do elétron

A equação de Dirac é $\left[p_0 - \rho_1 \vec{\Sigma}\cdot\vec{p} - \rho_3 m c\right]\psi = 0$

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \qquad \rho_3 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$p_\mu = i\hbar \frac{\partial}{\partial x^\mu}$$

Podemos escrevê-la como

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = H\psi}$$

Com

$$\boxed{H = c p_0 = c\rho_1 \vec{\Sigma}\cdot\vec{p} + \rho_3 m c^2}$$

$$H = c\,\vec{\alpha}\cdot\vec{p} + \rho_3 m c^2 \qquad \alpha_i = \rho_1 \Sigma_i$$

Representações de Heisenberg e Schrödinger

$$\Omega^{(H)}(t) = e^{iHt/\hbar}\,\Omega^{(S)}\,e^{-iHt/\hbar} \qquad \Omega^{(H)}(0) = \Omega^{(S)}$$

e

$$i\hbar \frac{d\Omega^{(H)}}{dt} = [\Omega^{(H)}, H] + i\hbar \frac{\partial \Omega^{(H)}}{\partial t}$$

$$\boxed{[A^{(H)}, H] = e^{iHt/\hbar}[A^{(S)}, H]\,e^{-iHt/\hbar}}$$

Momento angular orbital:

$$\vec{L} = \vec{x} \wedge \vec{p}$$

$$[\vec{L}, H] = [\vec{x}\wedge\vec{p},\ c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2] = [\vec{x}\wedge\vec{p},\ c\,\vec{\alpha}\cdot\vec{p}]$$

$$\boxed{(\vec{A}\wedge\vec{B})_i = \varepsilon_{ijk} A_j B_k}$$

$$[\varepsilon_{ijk} x_j p_k,\ \alpha_l p_l] = \alpha_l\, \varepsilon_{ijk}[x_j, p_l] p_k$$

$$= i\hbar\, \alpha_j\, \varepsilon_{ijk} p_k$$

$$= i\hbar\, (\vec{\alpha}\wedge\vec{p})_i$$

$$\boxed{\frac{d\vec{L}^{(H)}}{dt} = c\, \vec{\alpha}^{(H)} \wedge \vec{p}}$$

Momento angular orbital não é conservado

Operador dinâmico p/ Σ_i:

$$\Sigma_i^{(H)}(t) = e^{iHt/\hbar}\,\Sigma_i\, e^{-iHt/\hbar}$$

$$i\hbar \frac{d\Sigma_i^{(H)}}{dt} = [\Sigma_i^{(H)}, H] = e^{iHt/\hbar}\,[\Sigma_i, H]\,e^{-iHt/\hbar}$$

$$[\vec{\Sigma}, H] = [\vec{\Sigma}, c\,\vec{\alpha}\cdot\vec{p} + \rho_3 mc^2] = c\rho_1 [\vec{\Sigma}, \vec{\Sigma}\cdot\vec{p}]$$

mas $[\Sigma_i, \Sigma_j] = 2i\varepsilon_{ijk}\Sigma_k$

e $[\Sigma_i, \vec{H}] = 2ic\rho_1\,\varepsilon_{ijk}\Sigma_k p_j = -2ic\rho_1(\vec{\Sigma}\wedge\vec{p})_i$

$$i\hbar \frac{d\vec{\Sigma}^{(H)}}{dt} = e^{iHt/\hbar}(-2ic\rho_1\,\vec{\Sigma}\wedge\vec{p})\,e^{-iHt/\hbar} =$$

$$= -2ic\,\rho_1^{(H)}\,\vec{\Sigma}^{(H)}\wedge\vec{p} = -2ic\,\vec{\alpha}^{(H)}\wedge\vec{p}$$

Temos $$i\hbar \frac{d\vec{\Sigma}^{(H)}}{dt} = -2ic\,\vec{\alpha}^{(H)} \wedge \vec{P}$$

$$\frac{d\vec{L}^{(H)}}{dt} = c\,\vec{\alpha}^{(H)} \wedge \vec{P}$$

Logo $$\boxed{\vec{J}^{(H)} = \vec{L}^{(H)} + \frac{\hbar}{2}\vec{\Sigma}^{(H)}}$$

é conservado $$\boxed{\frac{d\vec{J}^{(H)}}{dt} = 0}$$

$\frac{\hbar}{2}\vec{\Sigma} \equiv$ momento angular de spin

A teoria do pósitron - antimatéria

Hamiltoniana de Dirac: $\boxed{H = \rho_1 \vec{\Sigma} \cdot \vec{p} + \rho_3 m c}$

Note: ρ_2 comuta com Σ_i e anti-comuta com ρ_1 e ρ_3

Logo se

$$\boxed{H\psi = \lambda\psi}$$

energias positivas e negativas

segue que $\psi' = \rho_2 \psi$ satisfaz $\boxed{H\psi' = -\lambda\psi'}$

pois $H\rho_2\psi = -\rho_2 H\psi = -\lambda\rho_2\psi$

Além disso $(p_0 - \rho_1 \vec{\Sigma} \cdot \vec{p} - \rho_3 m c)\psi = 0$ implica

$(p_0^2 - \vec{p}^2 - \beta^2)\psi = 0$ que tem soluções de energia negativa

Energias negativas e conjugação de carga

Eq. Dirac acoplada ao campo EM

$$\left[\left(P_0+\frac{e}{c}A_0\right)-\alpha_i\left(P_i+\frac{e}{c}A_i\right)-\alpha_m mc\right]\psi=0 \qquad (*)$$

$\alpha_m=\rho_3 \qquad \alpha_i=\rho_1\Sigma_i$

Troca de "representação": α_i reais e α_m imaginário

$$\alpha_1=\rho_1\Sigma_1=\begin{pmatrix}0&\sigma_1\\ \sigma_1&0\end{pmatrix} \qquad \alpha_2=\begin{pmatrix}I&0\\0&-I\end{pmatrix} \qquad \alpha_3=\begin{pmatrix}0&\sigma_3\\ \sigma_3&0\end{pmatrix} \qquad \alpha_m=\begin{pmatrix}0&\sigma_2\\ \sigma_2&0\end{pmatrix}$$

Tome complexo conjugado de $(*)$ $\left(p_\mu=i\hbar\frac{\partial}{\partial x^\mu}\rightarrow -p_\mu\right)$

$$\left[\left(-P_0+\frac{e}{c}A_0\right)-\alpha_i\left(-P_i+\frac{e}{c}A_i\right)+\alpha_m mc\right]\psi^*=0 \qquad (**)$$

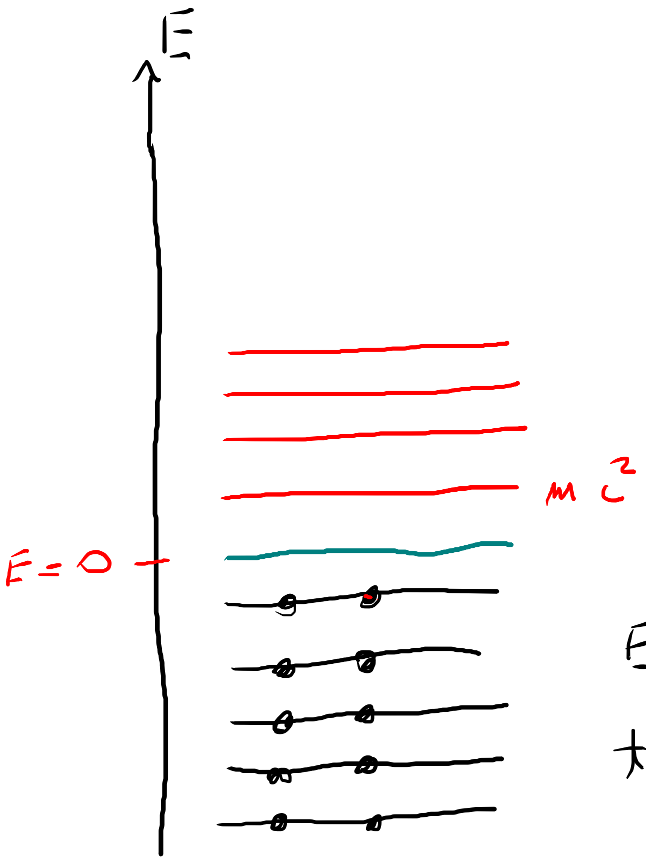
Daí se $\quad (cP_0+eA_0)\psi=E\psi \rightarrow (cP_0-eA_0)\psi^*=-E\psi^*$

ψ = energia E e carga e | ψ^* = energia $-E$ e carga $-e$

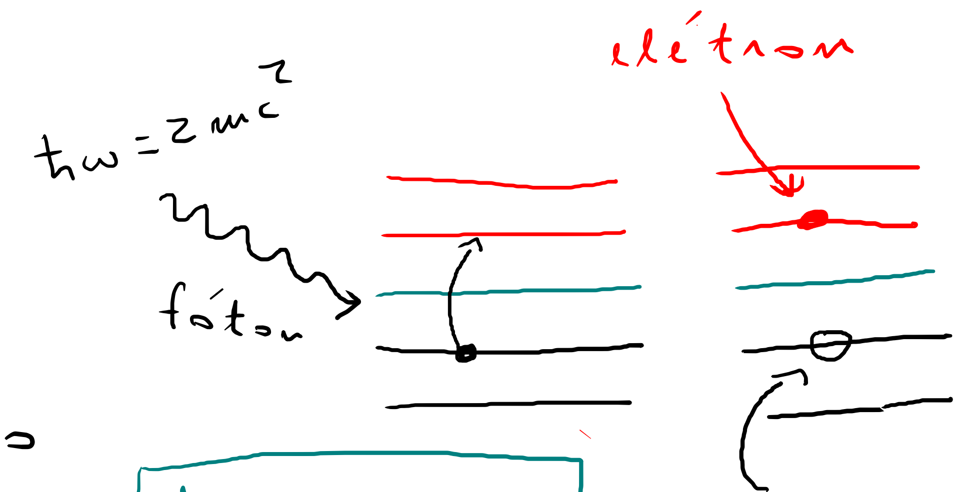
Teoria do Buraco de Dirac

Soluções c/ energia negativa têm carga oposta à do elétron (pósitrons)

Elétrons têm spin $1/2$, são férmions e satisfazem o princípio de exclusão de Pauli.



$$E_{obs} = E_{mar} - E_{buraco} > 0$$



Estados c/ $E<0$ todos ocupados (mar de Dirac)

$\boxed{\text{Anderson (1932) descobre o "buraco"}}$

buraco no mar parece ter carga oposta

Soluções de ondas planas

$$(\gamma^\mu p_\mu - mc)\psi = 0 \longrightarrow \left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \qquad p_\mu = i\hbar \frac{\partial}{\partial x^\mu}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Multiplique por $\gamma^\nu \partial_\nu$

$$i\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi - \frac{mc}{\hbar}\gamma^\nu \partial_\nu \psi = 0 \rightarrow \underbrace{\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu}_{\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu = \partial_\mu \partial^\mu} \psi + \left(\frac{mc}{\hbar}\right)^2 \psi = 0$$

$$\rightarrow \boxed{\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0}$$

4 eqs. desacopladas

$$\psi = u(\vec{p})\, e^{-ip_\mu x^\mu/\hbar} = u(\vec{p})\, e^{-i\frac{Et}{\hbar} + i\vec{p}\cdot\vec{x}/\hbar}$$

$u(\vec{p}) \equiv$ espinor de 4 componentes

$$\partial_\mu \partial^\mu \, e^{-i p_\mu x^\mu/\hbar} = -\frac{p_\mu p^\mu}{\hbar^2} \, e^{-i p_\mu x^\mu/\hbar}$$

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0 \longrightarrow -\frac{p^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2} = 0$$

$$-\frac{E^2}{c^2} + \vec{p}^{\,2} + m^2c^2 = 0 \longrightarrow E = \pm\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$$

energias positivas e negativas

Elas são auto-estados do momento e energia

$$i\hbar\frac{\partial\psi}{\partial t} = -i\frac{E}{\hbar}(i\hbar)\psi = E\psi$$

$$-i\hbar\nabla\psi = -i\hbar\, i\frac{\vec{p}}{\hbar}\psi = \vec{p}\psi$$

Referencial de repouso ($\vec{p}=0$)

$$i\gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar}\psi = 0 \longrightarrow i\frac{\gamma^0}{c}\frac{\partial \psi}{\partial t} = \frac{mc}{\hbar}\psi \qquad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Denota $\quad u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$\frac{i}{c}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} u_A(0) \\ u_A(0) \end{pmatrix}\left(-i\frac{E}{\hbar}\right) = \frac{mc}{\hbar}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} u_A \\ -u_B \end{pmatrix} = \frac{mc^2}{E}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \longrightarrow \begin{matrix} E = mc^2, & u_B = 0 \\ E = -mc^2, & u_A = 0 \end{matrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-i\frac{mc^2 t}{\hbar^2}} \quad (1) \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-i\frac{mc^2 t}{\hbar}} \quad (2) \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i\frac{mc^2 t}{\hbar}} \quad (3) \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imc^2 t/\hbar} \quad (4)$$

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

(1) $mc^2 \uparrow$, (2) $mc^2 \downarrow$, (3) $-mc^2 \uparrow$

(4) $-mc^2 \downarrow$

Caso $\vec{p} \neq 0$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$(\gamma^\mu p_\mu - mc)\psi = 0 \qquad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \qquad i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

$$\frac{E}{c}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = mc\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\Rightarrow \quad \vec{\sigma}\cdot\vec{p}\,\psi_A = \frac{1}{c}\left[E + mc^2\right]\psi_B \qquad \Bigg| \qquad \psi = \begin{pmatrix} u_A \\ u_B \end{pmatrix} e^{-i\frac{Et}{\hbar} + i\vec{p}\cdot\frac{\vec{x}}{\hbar}}$$

$$\vec{\sigma}\cdot\vec{p}\,\psi_B = \frac{1}{c}\left[E - mc^2\right]\psi_A$$

$$\rightarrow \quad u_A(\vec{p}) = \frac{c}{E - mc^2}\,\vec{\sigma}\cdot\vec{p}\,u_B(\vec{p})$$

$$u_B(\vec{p}) = \frac{c}{E + mc^2}\,\vec{\sigma}\cdot\vec{p}\,u_A(\vec{p})$$

$$\vec{\sigma}\cdot\vec{P} = \begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_3 \end{pmatrix}$$

1) Para $E > 0$ tomamos u_A como $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_B = \frac{c}{E+mc^2}\begin{pmatrix} P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E+mc^2}\begin{pmatrix} P_3 \\ P_1 + iP_2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E+mc^2}\begin{pmatrix} P_1 - iP_2 \\ -P_3 \end{pmatrix}$$

2) Para $E < 0$ tomamos u_B como $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

4 Soluções independentes

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ p_3 c/(E+mc^2) \\ (p_1 + i p_2) c/(E+mc^2) \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ (p_1 - i p_2) c/(E+mc^2) \\ -p_3 c/(E+mc^2) \end{pmatrix}$$

$$u^{(3)} = N \begin{pmatrix} -p_3 c/(E+mc^2) \\ -(p_1 + i p_2) c/(E+mc^2) \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)} = N \begin{pmatrix} -(p_1 - i p_2) c/(E+mc^2) \\ p_3 c/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

$$\psi^{(r)} = u^{(r)} e^{-iE\frac{t}{\hbar} + i\vec{p}\cdot\vec{x}/\hbar} \longrightarrow \boxed{(\gamma^\mu p_\mu - mc)\, u^{(r)} = 0} \leftarrow \text{Equação matricial}$$

Nota: p/ $\vec{p} = 0$ auto-estados de Σ_3

p/ $\vec{p} \neq 0$ auto-estados de $\left(\vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|}\right)$

↳ helicidade

$$H = c\,\rho_1\,\vec{\Sigma}\cdot\vec{p} + \rho_3 m c^2$$

$$[H, \vec{\Sigma}\cdot\hat{n}] = c\,\rho_1 [\vec{\Sigma}\cdot\vec{p}, \vec{\Sigma}\cdot\hat{n}]$$

$$= c\,\rho_1\, p_i \hat{n}_j [\Sigma_i, \Sigma_j] =$$

$$= c\,\rho_1\, p_i \hat{n}_j 2i \varepsilon_{ijk} \Sigma_k$$

$$= 2ic\,\rho_1\, \vec{\Sigma}\cdot(\vec{p}\wedge\hat{n})$$

$$= 0 \quad \text{se } \vec{p} \,/\!/\, \hat{n} \rightarrow \hat{n} = \pm \frac{\vec{p}}{|\vec{p}|}$$

$$\vec{\Sigma}\cdot\vec{p} = \begin{cases} 1 & \text{right handed} \\ -1 & \text{left handed} \end{cases}$$