AULAS ANTERIORES 16/09/2020

Base

Um conjunto $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ será base de um EV V se, e somente se:

* B é LI. $\rightarrow \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \vec{0}, \quad \alpha_i = 0$
* B gera V. $\rightarrow \forall \vec{v} \in V \ \exists \ \vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$

Slide 37

$$\vec{v} = x_1 \vec{u}_1 + x_2 \vec{u}_2 + \dots + x_n \vec{u}_n \qquad (1) \qquad \vec{v} \text{ como CL de } A$$

$$\vec{v} = y_1 \vec{w}_1 + y_2 \vec{w}_2 + \dots + y_n \vec{w}_n \qquad (2) \qquad \vec{v} \text{ como CL de } B$$

Vetores de A como CL dos vetores de B:

$$\begin{aligned}
\vec{u}_1 &= a_{11} \vec{w}_1 + a_{21} \vec{w}_2 + \dots + a_{n1} \vec{w}_n \\
\vec{u}_2 &= a_{12} \vec{w}_1 + a_{22} \vec{w}_2 + \dots + a_{n2} \vec{w}_n \\
&\vdots \\
\vec{u}_n &= a_{1n} \vec{w}_1 + a_{2n} \vec{w}_2 + \dots + a_{nn} \vec{w}_n
\end{aligned} \qquad (3)$$

$(3) \longrightarrow (1)$:

$$\begin{aligned}
\vec{v} = \ & x_1 (a_{11} \vec{w}_1 + a_{21} \vec{w}_2 + \dots + a_{n1} \vec{w}_n) + \\
& x_2 (a_{12} \vec{w}_1 + a_{22} \vec{w}_2 + \dots + a_{n2} \vec{w}_n) + \dots + \\
& x_n (a_{1n} \vec{w}_1 + a_{2n} \vec{w}_2 + \dots + a_{nn} \vec{w}_n)
\end{aligned}$$

$$\vec{v} = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)\vec{w_1} +$$
$$(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)\vec{w_2} + \cdots + \qquad (4)$$
$$(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n)\vec{w_n}$$

Comparando (4) e (2) : $\vec{v} = y_1\vec{w_1} + y_2\vec{w_2} + \cdots + y_n\vec{w_n}$

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

Ou, na forma matricial:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

ou

$$[\vec{v}]_B = [I]_B^A [\vec{v}]_A$$

** matriz de mudança de base de A para B ou matriz de transição de A para B.

** relaciona as coord de um dado vetor $\vec{v}$ na base A com as coord desse mesmo $\vec{v}$ na base B.

Slide 40 - Observações

1) Cada vetor coluna em $[I]_B^A$ contém os componentes da CL dos vetores de A em relação a B.

$$[I]_B^A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \cdots \quad \downarrow$$

$$[\vec{u}_1]_B \quad [\vec{u}_2]_B \quad \cdots \quad [\vec{u}_n]_B$$

3) $[I]_B^A$ relaciona vetores da base A em vetores da base B, ou seja, vetores LI em vetores LI. Por isso, $[I]_B^A$ é inversível.

Da equação:

$$[\vec{v}]_B = [I]_B^A \, [\vec{v}]_A$$

$$\left([I]_B^A\right)^{-1} [\vec{v}]_B = \left([I]_B^A\right)^{-1} [I]_B^A \, [\vec{v}]_A$$

Obtém-se:

$$[\vec{v}]_A = \underbrace{\left([I]_B^A\right)^{-1}} [\vec{v}]_B$$

$$\left([I]_B^A\right)^{-1} = [I]_A^B$$

Ou seja: a inversa da matriz mudança base (MMB) de A para B é a MMB de B para A.

Slide 42 - Exercícios

1) $A = \{ \underset{\vec{u_1}}{(1,1)}, \underset{\vec{u_2}}{(0,-1)} \}$ e $B = \{ \underset{\vec{w_1}}{(2,-3)}, \underset{\vec{w_2}}{(-3,5)} \}$

a) $[I]_B^A = ?$

$$[I]_B^A = \begin{bmatrix} [\vec{u_1}]_B & [\vec{u_2}]_B \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$$

* $\vec{u_1} = \alpha_1 \vec{w_1} + \alpha_2 \vec{w_2}$

$(1,1) = \alpha_1(2,-3) + \alpha_2(-3,5)$

$$\begin{cases} 2\alpha_1 - 3\alpha_2 = 1 \\ -3\alpha_1 + 5\alpha_2 = 1 \end{cases}$$

$\alpha_1 = 8$; $\alpha_2 = 5$

* $\vec{u_2} = \beta_1 \vec{w_1} + \beta_2 \vec{w_2}$

$(0,-1) = \beta_1(2,-3) + \beta_2(-3,5)$

$$\begin{cases} 2\beta_1 - 3\beta_2 = 0 \\ -3\beta_1 + 5\beta_2 = -1 \end{cases}$$

$\beta_1 = -3$; $\beta_2 = -2$

$$\therefore [I]_B^A = \begin{bmatrix} 8 & -3 \\ 5 & -2 \end{bmatrix}$$

b) $\vec{v_A} = (2,3)$; $\vec{v_B} = ?$

$$[\vec{v}]_B = [I]_B^A \, [\vec{v}]_A$$

$$[\vec{v}]_B = \underbrace{\begin{bmatrix} 8 & -3 \\ 5 & -2 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{2\times 1} = \begin{bmatrix} 16-9 \\ 10-6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$\therefore \vec{v_B} = (7,4)$

c) $[I]_A^B = \left([I]_B^A\right)^{-1}$

Matriz de Cofatores : $M^{-1} = \dfrac{1}{\det M} (\text{cof } M)^T$

$M^{-1} M = I$

$M = [I]_B^A$

$\det M = -16 + 15 = -1 \neq 0 \therefore \exists M^{-1}$

$\text{cof } M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$; $A_{ij} = (-1)^{i+j} \det \begin{pmatrix} \text{matriz} \\ \text{sem linha } i \\ \text{sem coluna } j \end{pmatrix}$

$A_{11} = (-1)^{1+1} \det(-2) = -2$

$A_{12} = (-1)^{1+2} \det(5) = -5$

$A_{21} = (-1)^{1+2} \det(-3) = 3$

$A_{22} = (-1)^{2+2} \det(8) = 8$

$\text{cof } M = \begin{bmatrix} -2 & -5 \\ 3 & 8 \end{bmatrix} \longrightarrow (\text{cof } M)^T = \begin{bmatrix} -2 & 3 \\ -5 & 8 \end{bmatrix}$

$\therefore M^{-1} = \frac{1}{(-1)} \begin{bmatrix} -2 & 3 \\ -5 & 8 \end{bmatrix}$ $\therefore [I]_A^B = \begin{bmatrix} 2 & -3 \\ 5 & -8 \end{bmatrix}$ //

2) $[I]_B^A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. $[\vec{v}]_B = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$, $[\vec{v}]_A = ?$

Sabe-se que: $[\vec{v}]_B = [I]_B^A [\vec{v}]_A$

Portanto:

$$[\vec{v}]_A = [I]_A^B [\vec{v}]_B$$

Como $[I]_A^B = ([I]_B^A)^{-1}$, será necessário calcular a inversa de $M = [I]_B^A$.

Matriz de Cofatores : $M^{-1} = \frac{1}{\det M} (\text{cof } M)^T$

$\det M = -1 \neq 0 \therefore \exists M^{-1}$

$$\text{cof } M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \; ; \quad A_{ij} = (-1)^{i+j} \det \begin{pmatrix} \text{matriz} \\ \text{sem linha } i \\ \text{sem coluna } j \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \qquad A_{23} = (-1)^{2+3} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \qquad A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \qquad A_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \qquad A_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$\text{cof } M = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow (\text{cof } M)^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\therefore \quad M^{-1} = [I]_A^B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Assim:

$$[\vec{v}]_A = [I]_A^B \; [\vec{v}]_B$$

$$[\vec{v}]_A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3-2 \\ 3 \\ 2 \end{bmatrix} \quad \therefore \quad [\vec{v}]_A = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$