Lista de exercícios 4 – Pontos singulares (do livro de Boyce e Di Prima)

Problems

In each of Problems 1 through 6:

a. Show that the given differential equation has a regular singular point at $x = 0$.
b. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation.
c. Find the series solution ($x > 0$) corresponding to the larger root.
d. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$

2. $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

3. $xy'' + y = 0$

4. $xy'' + y' - y = 0$

5. $x^2y'' + xy' + (x - 2)y = 0$

6. $xy'' + (1 - x)y' - y = 0$

7. The Legendre equation of order α is

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

The solution of this equation near the ordinary point $x = 0$ was discussed in Problems 17 and 18 of Section 5.3. In Example 4 of Section 5.4, it was shown that $x = \pm 1$ are regular singular points.

a. Determine the indicial equation and its roots for the point $x = 1$.
b. Find a series solution in powers of $x - 1$ for $x - 1 > 0$.
Hint: Write $1 + x = 2 + (x - 1)$ and $x = 1 + (x - 1)$. Alternatively, make the change of variable $x - 1 = t$ and determine a series solution in powers of t.

8. The Chebyshev equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant; see Problem 8 of Section 5.3.

a. Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.
b. Find two solutions about $x = 1$.

9. The Laguerre[13] differential equation is

$$xy'' + (1 - x)y' + \lambda y = 0.$$

a. Show that $x = 0$ is a regular singular point.
b. Determine the indicial equation, its roots, and the recurrence relation.
c. Find one solution (for $x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized, this polynomial is known as the **Laguerre polynomial**, $L_m(x)$.

10. The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0.$$

a. Show that $x = 0$ is a regular singular point.
b. Show that the roots of the indicial equation are $r_1 = r_2 = 0$.
c. Show that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

The function J_0 is known as the **Bessel function of the first kind of order zero**.
d. Show that the series for $J_0(x)$ converges for all x.

11. Referring to Problem 10, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.
Hint: If $y_2(x) = J_0(x)v(x)$, then

$$y_2(x) = J_0(x) \int \frac{dx}{x\left(J_0(x)\right)^2}.$$

Find the first term in the series expansion of $\dfrac{1}{x\left(J_0(x)\right)^2}$.

12. The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

a. Show that $x = 0$ is a regular singular point.
b. Show that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$.
c. Show that one solution for $x > 0$ is

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!\,n!\,2^{2n}}.$$

The function J_1 is known as the **Bessel function of the first kind of order one**.
d. Show that the series for $J_1(x)$ converges for all x.
e. Show that it is impossible to determine a second solution of the form

$$x^{-1} \sum_{n=0}^{\infty} b_n x^n, \quad x > 0.$$

[13] Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879. He is also known for an algorithm for calculating roots of polynomial equations.