03.1 A DERIVADA DE GÂTEAUX, VISTA EM TERMOS DE $\mathcal{J}:\mathbb{R}\to\mathbb{R}$, $\varepsilon \mapsto \mathcal{J}(\varepsilon) = J(f+\varepsilon h)$, É UMA DERIVADA DO CÁLCULO BÁSICO, CUJAS PROPRIEDADES SÃO CONHECIDAS: $\delta J(f,h) = \mathcal{J}'(0)$

$$\int d^n x \,.\, h(x) \frac{\delta(c_1 J_1 + c_2 J_2)}{\delta f(x)} = (c_1 \mathcal{J}_1 + c_2 \mathcal{J}_2)'(0) =$$

$$= c_1 \mathcal{J}_1'(0) + c_2 \mathcal{J}_2'(0)$$

$$= c_1 \int d^n x \,.\, h(x) \frac{\delta J_1}{\delta f(x)} + c_2 \int d^n x \,.\, h(x) \frac{\delta J_2}{\delta f(x)}$$

$$\therefore \frac{\delta(c_1 J_1 + c_2 J_2)}{\delta f(x)} = c_1 \frac{\delta J_1}{\delta f(x)} + c_2 \frac{\delta J_2}{\delta f(x)} .$$

Q.E.D.

$\boxed{03.2}$ $\displaystyle\int d^n x \,.\, h(x) \frac{\delta(J_1 . J_2)}{\delta f(x)} =$

$$= (f_1 . f_2)'(0) = f_1'(0) . f_2(0) + f_1(0) . f_2'(0) =$$

$$= f_1'(0) . J_2(f) + J_1(f) . f_2'(0) =$$

$$= \left[\int d^n x \,.\, h(x) \frac{\delta J_1}{\delta f(x)}\right] . J_2(f) \;+$$

$$+ \left[\int d^n x \,.\, h(x) \frac{\delta J_2}{\delta f(x)}\right] . J_1(f)$$

$$\therefore \quad \frac{\delta(J_1 . J_2)}{\delta f(x)} = \frac{\delta J_1}{\delta f(x)} . J_2 + J_1 \frac{\delta J_2}{\delta f(x)} .$$

Q.E.D.

$\boxed{03.3}$ $F: A \to \mathbb{R} \rightsquigarrow \mathcal{F}: \mathbb{R} \to \mathbb{R},\ \varepsilon \mapsto \mathcal{F}(\varepsilon) \equiv F(f + \varepsilon h)$

$$\tilde{\Phi}: A \to \mathbb{R} \rightsquigarrow \Omega: \mathbb{R} \to \mathbb{R},\ \varepsilon \mapsto \Omega(\varepsilon) \equiv \tilde{\Phi}(f + \varepsilon h) = \Phi[F(f + \varepsilon h)] = (\Phi \circ \mathcal{F})(\varepsilon) = \Phi[\mathcal{F}(\varepsilon)]$$

$$\int d^n x . h(x) \frac{\delta \tilde{\Phi}}{\delta f(x)} = \int d^n x . h(x) \frac{\delta (\Phi \circ F)}{\delta f(x)} =$$

$$= (\Phi \circ \mathcal{F})'(0) = \Phi'[\mathcal{F}(0)] . \mathcal{F}'(0)$$

$$= \Phi'[F(f)] . \int d^n x . h(x) \frac{\delta F}{\delta f(x)}$$

$$\therefore \frac{\delta \tilde{\Phi}}{\delta f(x)} = \frac{d\Phi}{dF}(f) \cdot \frac{\delta F}{\delta f(x)} .$$

Q.E.D.

$\boxed{03.4}$ $J(f) = \int dx\, [f(x)]^n \Rightarrow$

$$\delta J^{\varepsilon} = J(f+\varepsilon h) - J(f)$$
$$= \int dx \left\{ [f(x)+\varepsilon h(x)]^n - [f(x)]^n \right\}$$
$$= \int dx \,.\, h(x) [f(x)]^{n-1} . n . \varepsilon + O(\varepsilon^2)$$

$$\delta J(f,h) = \lim_{\varepsilon \to 0} \frac{\delta J^{\varepsilon}}{\varepsilon} = \int dx \,.\, h(x) \left\{ n [f(x)]^{n-1} \right\}$$

$$\therefore \boxed{\frac{\delta J}{\delta f(x)} = n [f(x)]^{n-1}}$$

03.5 ANTES, PROPRIEDADES DA DELTA DE DIRAC E DA SUA DERIVADA. SABEMOS QUE $\int dx\, \delta(x) f(x) = f(0)$ E $\int dx\, \delta(x-y) f(x) = f(y)$. TAMBÉM VALE $\delta(-x) = \delta(+x)$. DEFINE-SE $\delta'(x)$ POR $\int dx\, \delta'(x) f(x) = -f'(0)$. VALEM TAMBÉM (VER NO FIM) $\boxed{\int dx\, \delta'(x-y) f(x) = -f'(y)}$ E

(1)

$\boxed{\delta'(-x) = -\delta'(+x)}$. (2)

~~A' COMO "FUNCIONAL DE SI MESMA"~~

SEJA $J(f) = f'(x)$. ENTÃO

$$\delta J(f,h) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ [f(x) + \epsilon h(x)]' - [f(x)]' \right\} = h'(x) \overset{(1)}{=}$$

$$\overset{(1)}{=} -\int dy\, \delta'(y-x) h(y) = \int dy \,.\, h(y) [-\delta'(y-x)] .$$

ASSIM, $$\frac{\delta f'(x)}{\delta f(y)} = -\delta'(y-x) \overset{(2)}{=} -[-\delta'(x-y)] .$$

$$\therefore \frac{\delta f'(x)}{\delta f(y)} = \delta'(x-y) \quad \text{Q.E.D.}$$

(2): $$\int_{-\infty}^{+\infty} dx\, \delta'(-x) f(x) = \int_{+\infty}^{-\infty} d(-x) . \delta'(x) f(-x) = \int_{-\infty}^{+\infty} dx\, \delta'(x) f(-x) =$$

$$= -\left\{ [f(-x)]' \right\}_{x=0} = -[-f'(0)] = f'(0) \quad \therefore \delta'(-x) = -\delta'(+x) .$$

$\boxed{03.5}$ (CONT.)

"PARÂMETRO"

(1): SEJA $g(x+y) \equiv f(x)$. ENTÃO $f(0) = g(y)$ E $g'(y) = f'(0)$.

DEFINIÇÃO!

ENTÃO $-f'(0) = \int dx \, . \, \delta'(x) f(x) =$

$= \int dx \, \delta'(x) . g(x+y) = \int d\xi \, \delta'(\xi - y) \, g(\xi)$.

MAS $f'(0) = g'(y)$...

$\therefore \int d\xi \, \delta'(\xi - y) \, g(\xi) = -g'(y)$.

$\boxed{03.6}$ $J(f) = \int d^n y\, d^n z\, K(y,z)\, f(y) f(z) \Rightarrow$

$$J(f+\varepsilon h) = \int d^n y\, d^n z\, K(y,z)\, [f(y) + \varepsilon h(y)] \cdot [f(z) + \varepsilon h(z)]$$

$$\delta J^\varepsilon \equiv J(f+\varepsilon h) - Jf$$

$$\delta J(f,h) = \lim_{\varepsilon \to 0} \frac{\delta J^\varepsilon}{\varepsilon} = \int d^n y\, d^n z\, K(y,z) \cdot [f(y) h(z) + f(z) h(y)] =$$

$$= \int d^n y\, d^n z\, K(y,z) f(y) h(z) + \boxed{\int d^n y\, d^n z\, K(y,z) f(z) h(y)} = \int d^n y\, d^n z\, K(z,y) \cdot f(y) h(z)$$

$y \leftrightarrow z$

$$\parallel$$

$$\int d^n z\, h(z) \underbrace{\left\{ \int d^n y\, f(y) [K(y,z) + K(z,y)] \right\}}_{\equiv \frac{\delta J}{\delta f(z)}}$$

Q.E.D.

$\boxed{03.7}$ SEJA $F(f) = \int_{\mathbb{R}^n} d^n x \, [f(x)]^2$. FOI DADO QUE $\dfrac{\delta F}{\delta f(x)} = 2 f(x)$.

É FÁCIL VER QUE $J(f) = C \cdot e^{-\frac{1}{2} F(f)} =$

$= \Phi[F(f)]$, COM $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(z) = C \cdot e^{-\frac{1}{2} z}$ E

$$\Phi'(z) = -\frac{1}{2} C \cdot e^{-\frac{1}{2} z} = -\frac{1}{2} \Phi(z).$$

PELA REGRA DA CADEIA,

$$\frac{\delta J}{\delta f(x)} = \Phi'[F(f)] \cdot \frac{\delta F}{\delta f(x)} =$$

$$= \left[-\frac{1}{\cancel{2}} \Phi(F(f)) \right] \cdot \cancel{2} f(x) = -f(x) \cdot J(f).$$

$$\therefore \frac{\delta J}{\delta f(x)} + f(x) \cdot J = 0.$$

Q.E.D.