Substituição Trigonométrica

Considere para este caso o seguinte exemplo:

1) $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+1}}$; expoente $(x^2+1)^{-1/2}$

O problema é a $\sqrt{x^2+1}$, vamos utilizar as fórmulas trigonométricas que nos auxiliam neste caso:

a) $\operatorname{sen}^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \operatorname{sen}^2\theta$

b) $\operatorname{tg}^2\theta + 1 = \sec^2\theta \Rightarrow \operatorname{tg}^2\theta = \sec^2\theta - 1$

Exemplos para trabalhar com as relações dos itens a e b:

2) $\sqrt{x^2+1} \Rightarrow \sec^2\theta = \operatorname{tg}^2\theta + 1$

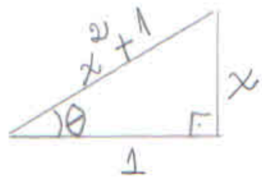
$$\underbrace{x^2+1}_{\sec^2\theta} = (\underbrace{\operatorname{tg}\theta}_{x})^2 + 1$$

$$\sqrt{x^2+1} = \sqrt{\sec^2\theta} = \sec\theta$$

$x = \operatorname{tg}\theta \Rightarrow \operatorname{tg}\theta = \dfrac{x}{1}$

$\dfrac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$



3) $\sqrt{x^2+6} \Rightarrow \sec^2\theta = \operatorname{tg}^2\theta + 1$

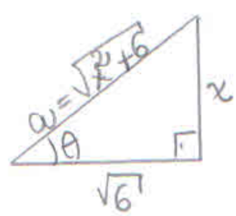
$$x^2 + 6 = (\operatorname{tg}\theta)^2 + 1$$

$$\underbrace{x^2 + 6}_{\sec^2\theta} = (\underbrace{\sqrt{6}\,\operatorname{tg}\theta}_{x})^2 + 6 = 6\operatorname{tg}^2\theta + 6 = 6(\operatorname{tg}^2\theta + 1)$$

$$= 6\sec^2\theta$$

$x = \sqrt{6}\operatorname{tg}\theta \longrightarrow \operatorname{tg}\theta = \dfrac{x}{\sqrt{6}}$

$dx = \sqrt{6}\sec^2\theta\, d\theta$



$a^2 = x^2 + (\sqrt{6})^2$

$a^2 = x^2 + 6$

$a = \sqrt{x^2+6}$

4) $\sqrt{3-5x^2}$

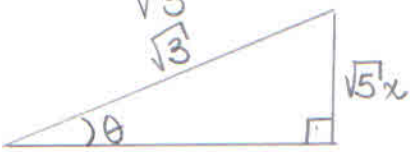
$\operatorname{sen}^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \operatorname{sen}^2\theta$

$3-5x^2 = 3-5\Big(\underbrace{\frac{\sqrt{3}\operatorname{sen}\theta}{\sqrt{5}}}_{x}\Big)^2 = 3-\cancel{5}\Big(\frac{3\operatorname{sen}^2\theta}{\cancel{5}}\Big) = 3\big(\overbrace{1-\operatorname{sen}^2\theta}^{=\cos\theta}\big) = 3\cos\theta$

$x = \frac{\sqrt{3}\operatorname{sen}\theta}{\sqrt{5}} \longrightarrow \operatorname{sen}\theta = \frac{\sqrt{5}x}{\sqrt{3}}$

$dx = \frac{\sqrt{3}\cos\theta}{\sqrt{5}}$



$(\sqrt{3})^2 = (\sqrt{5}x)^2 + a^2$

$3 = 5x^2 + a^2$

$a = \sqrt{3-5x^2}$

5) $\sqrt{7x^2-9}$

↳ sinal negativo

$\sec^2\theta = \operatorname{tg}^2\theta + 1$

$\sec^2\theta - 1 = \operatorname{tg}^2\theta$

$\sec^2\theta - 1 = \operatorname{tg}^2\theta$

$7\Big(\underbrace{\frac{\sqrt{9}\sec\theta}{\sqrt{7}}}_{x}\Big)^2 - 9 = \operatorname{tg}^2\theta$

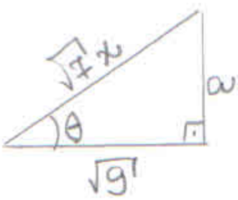
$\cancel{7}\Big(\frac{9\sec^2\theta}{\cancel{7}}\Big) - 9 = \operatorname{tg}^2\theta$

$9\sec^2\theta - 9 = \operatorname{tg}^2\theta$

$9(\sec^2\theta - 1) = \operatorname{tg}^2\theta$

$x = \frac{\sqrt{9}\sec\theta}{\sqrt{7}} \longrightarrow \sec\theta = \frac{\sqrt{7}x}{\sqrt{9}}$

$dx = \sqrt{9}\sec\theta.\operatorname{tg}\theta\, d\theta$



$(\sqrt{7}x)^2 = (\sqrt{9})^2 + a^2$

$7x^2 = 9 + a^2$

$a^2 = 7x^2 - 9$

$a = \sqrt{7x^2-9}$

6) $\sqrt{x^2-12x-4}$

$x^2 - 12x - 4 = [(x)^2 - 2.6x + 36] - 36 - 4$

$(x-6)^2 - 40 \Rightarrow \operatorname{tg}^2\theta + 1 = \sec^2\theta$

$\sec^2\theta - 1 = \operatorname{tg}^2\theta$

$(\sec^2\theta - 1) = \operatorname{tg}^2\theta$

$(x-6)^2 - 40 = \operatorname{tg}^2\theta$

$(\sqrt{40}\sec\theta)^2 - 40 = 40\sec^2\theta - 40 = \underbrace{40(\sec^2\theta - 1)}_{40\operatorname{tg}^2\theta} =$

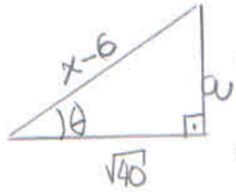
$x - 6 = \sqrt{40}\sec\theta$

$\sec\theta = \frac{x-6}{\sqrt{40}}$

$x - 6 = \sqrt{40}\sec\theta$

$x = \sqrt{40}\sec\theta + 6$

$dx = \sqrt{40}\sec\theta \cdot tg\,\theta\, d\theta$



$(x-6)^2 = a^2 + (\sqrt{40})^2$

$x^2 - 12x + 36 = a^2 + 40$

$x^2 - 12x - 4 = a^2$

$a = \sqrt{x^2 - 12x - 4}$

7) $\sqrt{-x^2 + 10x - 15}$

considero -25

$-x^2 + 10x - 15 = -[(x)^2 - 2.5x + 25] + 25 - 15$

$-(x-5)^2 + 10 \Rightarrow 10 - (x-5)^2$

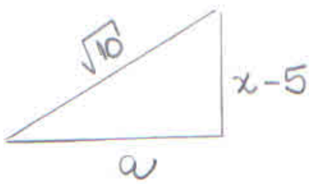
$10 - \underbrace{(x-5)}^2 = 10 - (\underbrace{\sqrt{10}\,sen\theta})^2 = 10 - 10\,sen^2\theta = 10(1 - sen^2\theta)$

$= 10\cos^2\theta$

$x - 5 = \sqrt{10}\,sen\theta \longrightarrow sen\theta = \dfrac{x-5}{\sqrt{10}}$

$x = \sqrt{10}\,sen\theta + 5$

$dx = \sqrt{10}\cos\theta\, d\theta$



$(\sqrt{10})^2 = (x-5)^2 + a^2$

$10 = x^2 - 10x + 25 + a^2$

$-x^2 + 10x - 15 = a^2$

$a = \sqrt{-x^2 + 10x - 15}$

8) $\sqrt{x^2 - 8x + 8}$ Obs.: Será desenvolvido durante a aula.

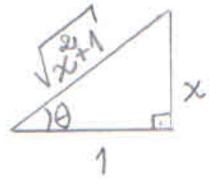
Exemplos de Integrais

a) $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

$\sqrt{x^2+1} = tg^2\theta + 1 = sec^2\theta = \sqrt{sec^2\theta} = sec\,\theta$

$x^2+1 = (tg\theta)^2 + 1$ $\qquad x = tg\theta \longrightarrow tg\theta = \dfrac{x}{1}$

$sec^2\theta = (\underbrace{tg\theta}_{x})^2 + 1$ $\qquad dx = sec^2\theta\, d\theta$



$= \displaystyle\int \frac{tg^2\theta \cdot \cancel{sec^2}\theta\, d\theta}{\cancel{sec\,\theta}} = \int tg^2\theta \cdot sec\,\theta\, d\theta = \int (sec^2\theta - 1)\cdot sec\,\theta\, d\theta$

(1)

$= \underbrace{\displaystyle\int sec^3\theta\, d\theta}_{I_1} - \int sec\,\theta\, d\theta$

Integral por Partes

$\to \int sec^2 u\,du = tg\,u + C$

$I_1 = \displaystyle\int \underbrace{sec\,\theta}_{u} \cdot \underbrace{sec^2\theta\, d\theta}_{dv}$ $\qquad u = sec\,\theta$ $\qquad \int dv = \int sec^2\theta\, d\theta$

$du = sec\,\theta \cdot tg\theta\, d\theta$ $\qquad v = tg\theta$

$\int u\,dv = u \cdot v - \int v\,du$

$= \displaystyle\int sec\,\theta \cdot sec^2\theta\, d\theta = sec\,\theta \cdot tg\theta - \int tg^2\theta\, sec\,\theta\, d\theta$

$= sec\,\theta \cdot tg\theta - \displaystyle\int (sec^2\theta - 1)\, sec\,\theta\, d\theta$

$\to \int sec\,u\,du = \ln|sec\,u + tg\,u| + C$

$= \displaystyle\int sec^3\theta\, d\theta = sec\,\theta \cdot tg\theta - \int sec^3\theta\, d\theta + \int sec\,\theta\, d\theta$

$= 2\displaystyle\int sec^3\theta\, d\theta = sec\,\theta \cdot tg\theta + \ln|sec\,\theta + tg\,\theta|$ $\quad (\div 2)$

$= \displaystyle\int sec^3\theta\, d\theta = \frac{1}{2} sec\,\theta \cdot tg\theta + \frac{1}{2}\ln|sec\,\theta + tg\theta|$

$= \dfrac{1}{2} sec\,\theta \cdot tg\theta + \dfrac{1}{2}\ln|sec\,\theta + tg\theta| - \ln|sec\,\theta + tg\theta| + C$

$= \boxed{\dfrac{1}{2}\sqrt{x^2+1}\cdot x + \dfrac{1}{2}\ln\left|\sqrt{x^2+1} + x\right| - \ln\left|\sqrt{x^2+1} + x\right| + C}$

b) $\displaystyle\int \frac{dx}{(-x^2+6x+7)^{5/2}}$

$-x^2+6x+7 = -\left[x^2-6x-7\right]$

$-\left[(x)^2-2.3x+9\right] = -\left[(x)^2-6x+\overset{-9}{9}\right]+9+7$

$-(x-3)^2+16$

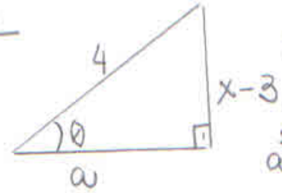
$-(x-3)^2+16 = 16-(x-3)^2$

$= 16-(4\,\text{sen}\,\theta)^2 = 16-16\,\text{sen}^2\theta = 16(1-\text{sen}^2\theta) = 16\cos^2\theta$

$x-3 = 4\,\text{sen}\,\theta \longrightarrow \text{sen}\,\theta = \dfrac{x-3}{4}$

$x = 4\,\text{sen}\,\theta+3$

$dx = 4\cos\theta\,d\theta$



$(4)^2 = (x-3)^2+a^2$

$16 = x^2-6x+9+a^2$

$a^2 = -x^2+6x+7$

$\sqrt{(-x^2+6x+7)^5} = \sqrt{(16\cos^2\theta)^5} = \sqrt{(2^4\cos^2\theta)^5} = 2^{10}\cos^5\theta$

$\underline{(2^4\cos^2\theta)^{5/2} = 2^{10}\cdot\cos^5\theta}$

$= \displaystyle\int \frac{\overset{2^2}{\cancel{4}}\,\cancel{\cos\theta}\,d\theta}{2^{\cancel{10}\,8}\cos^{\cancel{5}\,4}\theta} = \frac{1}{2^8}\int \sec^4\theta\,d\theta = \frac{1}{2^8}\int \sec^2\theta\cdot\sec^2\theta\,d\theta$

$\int \sec^2 u\,du = \text{tg}\,u + C$

$= \displaystyle\frac{1}{2^8}\int(\text{tg}^2\theta+1)\sec^2\theta\,d\theta = \frac{1}{2^8}\int \text{tg}^2\theta\,\sec^2\theta\,d\theta + \frac{1}{2^8}\int \sec^2\theta\,d\theta$

$u = \text{tg}\,\theta$

$du = \sec^2\theta\,d\theta$

$= \displaystyle\frac{1}{2^8}\,\frac{\text{tg}^3\theta}{3} + \frac{1}{2^8}\,\text{tg}\,\theta + C$

$$= \boxed{\frac{1}{2^8}\,\frac{\left(\dfrac{x-3}{\sqrt{-x^2+6x+7}}\right)^3}{3} + \frac{1}{2^8}\,\frac{x-3}{\sqrt{-x^2+6x+7}} + C}$$ //

c) $\int x^2 \cdot \sqrt{4-x^2}$

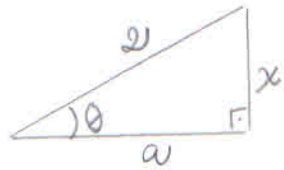
$\sqrt{4-x^2} = 1-\operatorname{sen}^2\theta = \cos^2\theta$

$\cos^2\theta = 1-\operatorname{sen}^2\theta$

$4-x^2 = 4-(2\operatorname{sen}\theta)^2 = 4-4\operatorname{sen}^2\theta = 4(1-\operatorname{sen}^2\theta) = 4\cos^2\theta$

$x = 2\operatorname{sen}\theta \longrightarrow \operatorname{sen}\theta = \frac{x}{2}$

$dx = 2\cos\theta\, d\theta$



$(2)^2 = x^2+a^2$

$4 = x^2+a^2$

$a=\sqrt{4-x^2}$

$$= \int 4\operatorname{sen}^2\theta \cdot 4\cos^2\theta\, d\theta = 16\int \operatorname{sen}^2\theta\cdot\cos^2\theta\, d\theta$$

$$= 16\int\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos2\theta}{2}\right)d\theta = 16\int\left(\frac{1}{2} - \frac{\cos2\theta}{2}\right)\left(\frac{1}{2}+\frac{\cos2\theta}{2}\right)d\theta$$

$$= 16\int\left(\frac{1}{4} + \cancel{\frac{\cos2\theta}{4}} - \cancel{\frac{\cos2\theta}{4}} - \frac{\cos^2 2\theta}{4}\right)d\theta$$

$$= \cancel{16}\cdot\frac{1}{\cancel{4}}\int d\theta - \cancel{16}\,\frac{1}{\cancel{4}}\int\cos^2 2\theta\, d\theta$$

$$= 4\int d\theta - 4\int\left(\frac{1+\cos4\theta}{2}\right)d\theta$$

$$= 4\theta - \cancel{4}.\frac{1}{\cancel{2}}\int d\theta - 4.\frac{1}{2}\,\frac{1}{4}\int\cos4\theta\, 4\, d\theta$$

$u = 4\theta$

$du = 4d\theta$

$$= \overbrace{4\theta - 2\theta}^{=2\theta} - \frac{1}{2}\operatorname{sen}4\theta + C$$

$$\operatorname{sen}4\theta = \operatorname{sen}2.2\theta = 2\operatorname{sen}2\theta\cos2\theta$$

$$= 2.2\operatorname{sen}\theta\cos\theta\,(\cos^2\theta - \operatorname{sen}^2\theta)$$

$$= 4\operatorname{sen}\theta\cos^3\theta - 4\operatorname{sen}^3\theta\cos\theta$$

$$= 4\left(\frac{x}{2}\right)\cdot\left(\frac{\sqrt{4-x^2}}{2}\right)^3 - 4\left(\frac{x}{2}\right)^3\cdot\left(\frac{\sqrt{4-x^2}}{2}\right)$$

$$= 2\operatorname{arc\,sen}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + 2\left(\frac{x^3}{8}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)$$

$$= \boxed{2\operatorname{arc\,sen}\left(\frac{x}{2}\right) - \frac{x\left(\sqrt{4-x^2}\right)^3}{2} + \frac{x^3\sqrt{4-x^2}}{8} + C}\ //$$