



# Fundamentos de Escoamentos Reativos Turbulentos

Aula 2-3

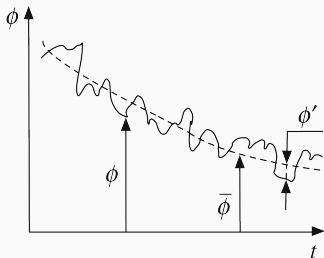
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Prof. Dr. Guenther Carlos Krieger Filho

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Escola Politécnica da USP - LETE/CRC - Combustion Research Centre

# Descrição de escoamentos turbulentos



## Média no tempo

$$\bar{\Phi} = \frac{1}{\Delta t} \int_0^{\Delta t} \Phi(t) dt$$

## Ergodicidade

Média conjunto = Média no tempo  
(Média estatística)

## Decomposição de Reynolds

$$\Phi(t) = \bar{\Phi} + \Phi'(t)$$

$$\frac{1}{\Delta t} \int_0^{\Delta t} \Phi(t) dt = \frac{1}{\Delta t} \int_0^{\Delta t} \bar{\Phi} dt + \frac{1}{\Delta t} \int_0^{\Delta t} \Phi' dt$$

$$\bar{\Phi} = \bar{\Phi} + \bar{\Phi}' \rightarrow \boxed{\bar{\Phi}' = 0}$$

# Descrição de escoamentos turbulentos

## Variância

$$\overline{(\Phi')^2} = \frac{1}{\Delta t} \int_0^{\Delta t} (\Phi')^2 dt$$

## RMS/Desvio Padrão

$$\Phi_{RMS} = \sqrt{\overline{(\Phi')^2}}$$

## Energia Cinética Turbulenta

$$K = \frac{1}{2} \left[ \overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2} \right]$$

## Intensidade Turbulenta

$$I = \frac{\left(\frac{2}{3}K\right)^{1/2}}{U_{ref}}$$

# Momentos (2ª ordem) de 2 variáveis

$$\phi = \bar{\phi} + \phi' ; \psi = \bar{\psi} + \psi' ; \bar{\phi}' = \bar{\psi}' = 0 ; \overline{\phi'\psi'} = \frac{1}{\Delta t} \int_0^{\Delta t} \phi'\psi' dt$$

Para velocidade:  $\begin{cases} u'v' \\ u'w' \\ v'w' \end{cases}$  Se  $= 0 \Rightarrow$  grandezas não correlacionadas

## Autocorrelação temporal

$$R_{\phi'\phi'}(\tau) = \overline{\phi'_{(t)}\phi'_{(t+\tau)}} = \frac{1}{\Delta t} \int_0^{\Delta t} \phi'_{(t)}\phi'_{(t+\tau)} dt$$

## Autocorrelação espacial

$$R_{\phi'\phi'}(\tau) = \overline{\phi'_{(t)}\phi'_{(t+\tau)}} = \frac{1}{\Delta t} \int_0^{\Delta t} \phi'_{(t)}\phi'_{(t+\tau)} dt$$

# Reynolds Averaged Navier Stokes

## Lembrando...

$\rho = cte$  ; decomposição de Reynolds:  $\Phi_{(t)} = \bar{\Phi} + \Phi'_{(t)}$

- $\frac{\partial \bar{\Phi}}{\partial s} \equiv \frac{1}{\Delta t} \int_0^{\Delta t} \frac{\partial \Phi}{\partial s} dt = \frac{\partial \bar{\Phi}}{\partial s}$
- $\overline{\int \Phi ds} = \int \bar{\Phi} ds$
- $\overline{\Phi + \Psi} = \bar{\Phi} + \bar{\Psi}$
- $\overline{\Phi \Psi} = \overline{(\bar{\Phi} + \Phi')(\bar{\Psi} + \Psi')} = \bar{\Phi} \bar{\Psi} + \overline{\Phi' \Psi'} + \underbrace{\overline{\Phi \Psi'}}_{=0} + \underbrace{\overline{\Phi' \Psi}}_{=0}$

# Reynolds Averaged Navier Stokes

Para um vetor  $a$  :

$$a = \bar{a} + a' \text{ ou } a_i = \bar{a}_i + a'_i$$

- $\overline{\text{div } a} = \text{div } \bar{a}$  ou  $\overline{\nabla \cdot a} = \nabla \cdot \bar{a}$  ou  $\frac{\partial \overline{a_i}}{\partial x_j} = \frac{\partial \bar{a}_i}{\partial x_j}$

- $\overline{\text{div } \Phi a} = \text{div } (\overline{\Phi a}) = \text{div } (\overline{\Phi} \bar{a}) + \text{div } (\overline{\Phi' a'})$

$$\frac{\partial (\overline{\Phi a_i})}{\partial x_j} = \frac{\partial}{\partial x_j} (\overline{\Phi a_i}) = \frac{\partial}{\partial x_j} (\overline{\Phi} \bar{a}_i) + \frac{\partial}{\partial x_j} (\overline{\Phi' a'_i})$$

- $\overline{\text{div grad } a} = \text{div grad } \bar{a}$  ou  $\overline{\nabla \cdot (\nabla a)} = \nabla \cdot (\nabla \bar{a}) = \nabla^2 \bar{a}$

$$\frac{\partial}{\partial x_j} \left( \frac{\partial \overline{a_i}}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\partial \bar{a}_i}{\partial x_j} \right) = \frac{\partial^2 \bar{a}_i}{\partial x_j \partial x_j}$$

# Reynolds Averaged Navier-Stokes

## Continuidade

$$\frac{\partial \bar{u}_i}{\partial x_i} \text{ ou } \nabla \cdot \bar{\mathbf{u}} = 0$$

## Quantidade de movimento

$$\text{Instantânea: } \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (uu) = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u$$

$$\text{Média: } \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j} (\overline{u'_i u'_j}) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$$

$$\nabla \cdot (\bar{\mathbf{u}} \bar{\mathbf{u}}) + \nabla \cdot (\overline{u' u'}) = -\frac{1}{\rho} \nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}}$$

# Reynolds Average Navier-Stokes

$$-\rho \overline{u'_i u'_j} = -\rho \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{pmatrix} \Rightarrow \text{Tensor de Reynolds}$$

$$\underbrace{\text{Tr} \left( \begin{array}{c} \text{Tensor de} \\ \text{Reynolds} \end{array} \right)}_{\text{traço}} = 2K = -2\rho \left( \overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right)$$

$K \rightarrow$  Energia cinética turbulenta



## Transporte de Escalar

$$\text{Instantânea: } \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x_j} (\Phi u_j) = \frac{D}{\rho} \frac{\partial^2 \Phi}{\partial x_j \partial x_j} + S_\Phi$$

$$\frac{\partial \Phi}{\partial t} + \nabla \cdot (\Phi \mathbf{u}) = \frac{D}{\rho} \nabla^2 \Phi + S_\Phi$$

$$\text{Média: } \frac{\partial}{\partial x_j} (\overline{\Phi \bar{u}_j}) + \frac{\partial}{\partial x_j} (\overline{\Phi' u'_j}) = \frac{D}{\rho} \frac{\partial^2 \bar{\Phi}}{\partial x_j \partial x_j} + \overline{S_\Phi}$$

$$\nabla \cdot (\overline{\Phi \bar{\mathbf{u}}}) + \nabla \cdot (\overline{\Phi' \mathbf{u}'}) = \frac{D}{\rho} \nabla^2 \bar{\Phi} + \overline{S_\Phi}$$

## Exemplo: Placa plana $c/\rho = \text{cte}$ (Turns cap. 11)

Quantidade de movimento em  $x$   $\left(\frac{\partial p}{\partial x} = 0\right)$

$$\underbrace{\frac{\partial v_x}{\partial t}}_{(1)} + \underbrace{\frac{\partial}{\partial x}(v_x v_x)}_{(2)} + \underbrace{\frac{\partial}{\partial y}(v_x v_y)}_{(3)} = \nu \underbrace{\frac{\partial^2 v_x}{\partial y \partial y}}_{(4)}$$

$$(1) : \overline{\frac{\partial}{\partial t} (\bar{v}_x + v'_x)} = \frac{\partial \bar{v}_x}{\partial t} + \frac{\partial \bar{v}'_x}{\partial t} = 0$$

$$(2) : \overline{\frac{\partial}{\partial x} (\bar{v}_x + v'_x) (\bar{v}_x + v'_x)} = \frac{\partial}{\partial x} (\bar{v}_x \bar{v}_x) + \frac{\partial}{\partial x} (\overline{v_x'^2})$$

$$(3) : \overline{\frac{\partial}{\partial y} (\bar{v}_x + v'_x) (\bar{v}_y + v'_y)} = \frac{\partial}{\partial y} (\bar{v}_x \bar{v}_y) + \frac{\partial}{\partial y} (\overline{v'_x v'_y})$$

$$(4) : \nu \overline{\frac{\partial^2 v_x}{\partial y \partial y}} = \nu \frac{\partial^2 \bar{v}_x}{\partial y \partial y}$$

## Exemplo: Placa plana $c/\rho = \text{cte}$ (Turns cap. 11)

### Quantidade de movimento em x

$$\frac{\partial}{\partial x} (\overline{v_x v_x}) + \frac{\partial}{\partial y} (\overline{v_x v_y}) + \frac{\partial}{\partial y} (\overline{v'_x v'_y}) + \frac{\partial}{\partial x} (\overline{v'_x v'_x}) = \nu \frac{\partial^2 \overline{v_x}}{\partial y \partial y}$$

### Continuidade

$$\frac{\partial \overline{v_x}}{\partial x} + \frac{\partial \overline{v_y}}{\partial y} = 0$$

- p/ jato (análogo à placa plana)

$$\rho \left( \overline{v_x} \frac{\partial \overline{v_x}}{\partial x} + \overline{v_r} \frac{\partial \overline{v_x}}{\partial r} \right) = \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \overline{v_x}}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} (\rho r \overline{v'_x v'_r})$$

# Eddy viscosity ou hipótese de Boussinesq (1877)

$$\frac{1}{r} \frac{\partial}{\partial r} [r (\tau_{\text{lam}} + \tau_{\text{turb}})]$$

$$\tau_{\text{lam}} = \mu \frac{\partial \bar{v}_x}{\partial r}$$

$$\tau_{\text{turb}} = -\rho \nu_t \frac{\partial \bar{v}_x}{\partial r} ; \nu_t = \frac{\mu_t}{\rho}$$

$$\mu_{\text{eff}} = \mu_{\text{molecular}} + \mu_{\text{turb}}$$

1. Como determinar  $\mu_{\text{turb}}$ ?
2.  $\mu_{\text{molecular}}$  é propriedade termodinâmica de transporte  
 $\mu_{\text{turb}}$  é dependente do "padrão" do escoamento
3. Nem sempre é função apenas do grad da velocidade.

# Eddy viscosity ou hipótese de Boussinesq (1877)

## Comprimento de mistura de Prandtl $l_m$

$$\mu_t = \rho \nu_t = \rho l_m v_{turb} = \rho l_m^2 \left| \frac{\partial v_x}{\partial x_r} \right|$$

- Jatos livres:  $v_{turb} \propto \bar{v}_{x,max} - \bar{v}_{x,min}$

## Comprimento de mistura de Prandtl $l_m$

$$\mu_t = \underbrace{0,1365}_{experimental} \rho l_m^2 (\bar{v}_{x,max} - \bar{v}_{x,min})$$

# Eddy viscosity ou hipótese de Boussinesq (1877)

- Jatos livres:  $v_{turb} \propto \bar{v}_{x,max} - \bar{v}_{x,min}$

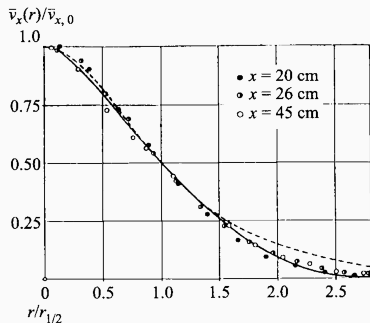
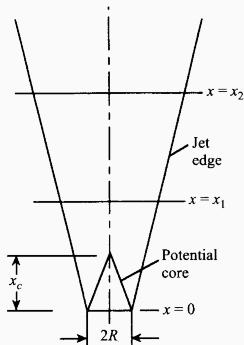
## Quantidade de movimento

$$\rho \left( \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_r \frac{\partial \bar{v}_x}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r(\mu + \mu_t) \frac{\partial \bar{v}_x}{\partial r} \right]$$

## Continuidade

$$\frac{\partial}{\partial x} (\bar{v}_x r) + \frac{\partial}{\partial r} (\bar{v}_r r) = 0$$

# Eddy viscosity ou hipótese de Boussinesq (1877)



$$J_e = \rho_e v_e^2 \pi R^2 ; l_m = 0,075 \delta_{99\%}$$

$$\xi = \left[ \frac{3J_e}{16\rho_e\pi} \right] \frac{1}{\nu_t} \frac{r}{x}$$

$$\delta_{99\%} \rightarrow \text{raio no qual } \frac{\bar{v}_x(r)}{\bar{v}_{x,0}} = 1\%$$

$$a) \nu_t = 0,0102 \delta_{99\%} \bar{v}_{x,0}(x) \approx cte$$

$$b) r_{1/2} \propto x$$

$$\bar{v}_{x,0} \propto x^{-1}$$

## Eddy viscosity ou hipótese de Boussinesq (1877)

Aplicando a solução analítica p/ jato laminar:

$$\mu_{lam} \Rightarrow \rho\nu_t$$

$$\bar{v}_x = \frac{3}{8\pi} \frac{J_e}{\rho\nu_t x} \left[ 1 + \frac{\xi^2}{4} \right]^{-2} \quad \bar{v}_r = \left[ \frac{3J_e}{16\pi\rho_e} \right]^{1/2} \frac{1}{x} \frac{\xi - \xi^3/4}{[1 + \xi^2/4]^2}$$

Fazendo  $\frac{\bar{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1 + \xi^2/4]^{-2}$ , no centro  
 $r = 0 \Rightarrow \xi = 0$

$$\frac{\bar{v}_x}{v_e} = 0,375(v_e R/\nu_t)(x/R)^{-1}[1 + \xi^2/4]^{-2} \quad (1)$$

$$\frac{\bar{v}_x}{\bar{v}_{x,0}} = \frac{1}{2} = [1 + \xi^2/4]^{-2} \rightarrow \xi = 1,287 = \frac{3}{16} \left[ \frac{\rho_e v_e^2 \pi R^2}{\rho_e \pi} \right]^{1/2} \frac{1}{\nu_t} \frac{r_{1/2}}{x}$$
$$\rightarrow 1,287 = 0,43 v_e R \frac{1}{\nu_t} \frac{r_{1/2}}{x} \rightarrow r_{1/2} = 2,97 \left( \frac{v_e R}{\nu_t x} \right)^{-1} \quad (2)$$



## Eddy viscosity ou hipótese de Boussinesq (1877)

Resolvendo 1, 2 e a)

$$\frac{\bar{v}_{x,0}}{v_e} = 13,15(x/R)^{-1} \quad \nu_t = 0,0285v_eR$$
$$\frac{r_{1/2}}{x} = 0,08468$$

Lembrando no caso laminar  $\frac{\bar{v}_{x,0}}{v_e} \propto Re_{jet}$  e  $\frac{r_{1/2}}{x} \propto Re_{jet}^{-1}$ . Portanto no caso turbulento existe independência do número de Reynolds.

## Equação de transporte $p / -\rho \overline{u'_i u'_j}$

### Quantidade de movimento instantânea

$$\frac{\partial}{\partial x_j} (\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad (3)$$

Aplicando decomposição de Reynolds e a média temporal

$$\frac{1}{\Delta t} \int_0^t \left[ \frac{\partial}{\partial x_j} \rho (\bar{u}_i \bar{u}_j + \bar{u}_i u'_j + u'_i \bar{u}_j + u'_i u'_j) = -\frac{\partial}{\partial x_i} (\bar{p} + p') + \frac{\partial}{\partial x_j} (\bar{\tau}_{ij} + \tau'_{ij}) \right] dt$$

$$\frac{\partial}{\partial x_j} \rho (\bar{u}_i \bar{u}_j + \overline{u'_i u'_j}) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ij}}{\partial x_j}$$

### Quantidade de movimento média

$$\frac{\partial}{\partial x_j} (\rho \bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} (\bar{\tau}_{ij} - \rho \overline{u'_i u'_j}) \quad (4)$$

## Equação de transporte $p/ -\overline{\rho u'_i u'_j}$

Multiplicando a equação 3 por  $u_j$  e lembrando a continuidade

$\frac{\partial}{\partial x_i} (\rho u_i) = 0$ , temos para o termo advectivo

$$\begin{aligned} u_j \frac{\partial}{\partial x_k} (\rho u_i u_k) + u_i \frac{\partial}{\partial x_k} (\rho u_j u_k) &\rightarrow u_j \left[ \underbrace{u_i \frac{\partial}{\partial x_k} (\rho u_k)}_{=0} + \rho u_k \frac{\partial u_i}{\partial x_k} \right] \\ &+ u_i \left[ \underbrace{u_j \frac{\partial}{\partial x_k} (\rho u_k)}_{=0} + \rho u_k \frac{\partial u_j}{\partial x_k} \right] \\ &\rightarrow \rho u_j u_k \frac{\partial u_i}{\partial x_k} + \rho u_i u_k \frac{\partial u_j}{\partial x_k} \rightarrow \rho u_k \left[ u_j \frac{\partial u_i}{\partial x_k} + u_i \frac{\partial u_j}{\partial x_k} \right] \\ &\rightarrow \rho u_k \frac{\partial}{\partial x_k} (u_i u_j) \rightarrow \boxed{\frac{\partial}{\partial x_k} (\rho u_i u_j u_k)} = \rho u_k \frac{\partial}{\partial x_k} (u_i u_j) + \rho u_i u_j \underbrace{\frac{\partial u_k}{\partial x_k}}_{=0} \end{aligned}$$

## Equação de transporte $p / -\rho \overline{u'_i u'_j}$

Assim a equação 3 multiplicada por  $u_j$  resulta

$$\frac{\partial}{\partial x_k} (\rho u_i u_j u_k) = -u_j \frac{\partial p}{\partial x_i} - u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial \tau_{ik}}{\partial x_k} + u_i \frac{\partial \tau_{jk}}{\partial x_k}$$

Decomposição de Reynolds

$$\begin{aligned} \frac{\partial}{\partial x_k} [\rho (\bar{u}_i + u'_i)(\bar{u}_j + u'_j)(\bar{u}_k + u'_k)] &= -(\bar{u}_j + u'_j) \frac{\partial (\bar{p} + p')}{\partial x_i} \\ &\quad - (\bar{u}_i + u'_i) \frac{\partial (\bar{p} + p')}{\partial x_j} \\ &\quad + (\bar{u}_j + u'_j) \frac{\partial}{\partial x_k} (\bar{\tau}_{ik} + \tau'_{ik}) \\ &\quad + (\bar{u}_i + u'_i) \frac{\partial}{\partial x_k} (\bar{\tau}_{jk} + \tau'_{jk}) \end{aligned}$$

# Equação de transporte $p / -\rho \overline{u'_i u'_j}$

Organizando e aplicando a média

$$\begin{aligned}
 & \frac{\partial}{\partial x_k} \left[ \overline{\rho \bar{u}_i \bar{u}_j \bar{u}_k} + \overline{\bar{u}_i \rho u'_j u'_k} + \overline{\bar{u}_j \rho u'_i u'_k} + \overline{\bar{u}_k \rho u'_i u'_j} + \overline{\rho u'_i u'_j u'_k} \right] \\
 = & -\bar{u}_j \frac{\partial \bar{p}}{\partial x_i} - \bar{u}_j \frac{\partial p'}{\partial x_i} - u'_j \frac{\partial \bar{p}}{\partial x_i} - u'_j \frac{\partial p'}{\partial x_i} - \bar{u}_i \frac{\partial \bar{p}}{\partial x_j} - \bar{u}_i \frac{\partial p'}{\partial x_j} - u'_i \frac{\partial \bar{p}}{\partial x_j} - u'_i \frac{\partial p'}{\partial x_j} \\
 & -\bar{u}_j \frac{\partial \bar{\tau}_{ik}}{\partial x_k} - \bar{u}_j \frac{\partial \tau'_{ik}}{\partial x_k} - u'_j \frac{\partial \bar{\tau}_{ik}}{\partial x_k} - u'_j \frac{\partial \tau'_{ik}}{\partial x_k} - \bar{u}_i \frac{\partial \bar{\tau}_{jk}}{\partial x_k} - \bar{u}_i \frac{\partial \tau'_{jk}}{\partial x_k} - u'_i \frac{\partial \bar{\tau}_{jk}}{\partial x_k} - u'_i \frac{\partial \tau'_{jk}}{\partial x_k} \\
 & \quad \quad \quad \text{(A)} \\
 & \frac{\partial}{\partial x_k} \left[ \overline{\rho \bar{u}_i \bar{u}_j \bar{u}_k} + \overbrace{\overline{\rho \bar{u}_i u'_j u'_k} + \overline{\rho \bar{u}_j u'_i u'_k} + \overline{\rho \bar{u}_k u'_i u'_j}} + \overline{\rho u'_i u'_j u'_k} \right] \\
 = & -\bar{u}_j \frac{\partial \bar{p}}{\partial x_i} - u'_j \frac{\partial p'}{\partial x_i} - \bar{u}_i \frac{\partial \bar{p}}{\partial x_j} - u'_i \frac{\partial p'}{\partial x_j} - \bar{u}_j \frac{\partial \bar{\tau}_{ik}}{\partial x_k} - u'_j \frac{\partial \tau'_{ik}}{\partial x_k} - \bar{u}_i \frac{\partial \bar{\tau}_{jk}}{\partial x_k} - u'_i \frac{\partial \tau'_{jk}}{\partial x_k} \\
 & \quad \quad \quad (5)
 \end{aligned}$$

## Equação de transporte $p / -\rho \overline{u'_i u'_j}$

Multiplicando a equação 4 por  $\bar{u}_j$  resulta

$$\frac{\partial}{\partial x_k} (\rho \bar{u}_i \bar{u}_j \bar{u}_k) = -\bar{u}_j \frac{\partial \bar{p}}{\partial x_i} - \bar{u}_i \frac{\partial \bar{p}}{\partial x_j} + \bar{u}_j \frac{\partial}{\partial x_k} (\bar{\tau}_{ik} - \rho \overline{u'_i u'_k}) + \bar{u}_i \frac{\partial}{\partial x_k} (\bar{\tau}_{jk} - \underbrace{\rho \overline{u'_j u'_k}}_{\textcircled{A}'}) \quad (6)$$

Subtraindo 6 de 5, sendo  $\textcircled{A} = \textcircled{A}' - \rho \overline{u'_j u'_k} \frac{\partial \bar{u}_i}{\partial x_k}$

### Equação de transporte de $\overline{u'_i u'_j}$

$$\begin{aligned} \frac{\partial}{\partial x_k} (\rho \bar{u}_k \overline{u'_i u'_j}) = & -\frac{\partial}{\partial x_k} (\rho \overline{u'_i u'_j u'_k}) - \overline{u'_j \frac{\partial p'}{\partial x_i}} - \overline{u'_i \frac{\partial p'}{\partial x_j}} \\ & + \overline{u'_j \frac{\partial \tau'_{ik}}{\partial x_k}} + \overline{u'_i \frac{\partial \tau'_{jk}}{\partial x_k}} - \rho \overline{u'_i u'_k} \frac{\partial \bar{u}_i}{\partial x_k} - \rho \overline{u'_j u'_k} \frac{\partial \bar{u}_i}{\partial x_k} \end{aligned} \quad (7)$$

# Equação de transporte $p/ -\rho \overline{u'_i u'_j}$

- Termos de tensões:

$$\overline{u'_j \frac{\partial \tau'_{ik}}{\partial x_k}} + \overline{u'_i \frac{\partial \tau'_{jk}}{\partial x_k}} ; \tau'_{ik} = \mu \frac{\partial u'_i}{\partial x_k}$$

$$\rightarrow \overline{u'_j \frac{\partial}{\partial x_k} \left( \mu \frac{\partial u'_i}{\partial x_k} \right)} + \overline{u'_i \frac{\partial}{\partial x_k} \left( \mu \frac{\partial u'_j}{\partial x_k} \right)}$$

$$\rightarrow \mu \left[ \frac{\partial}{\partial x_k} \left( \overline{u'_j \frac{\partial u'_i}{\partial x_k}} \right) - \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} + \frac{\partial}{\partial x_k} \left( \overline{u'_i \frac{\partial u'_j}{\partial x_k}} \right) - \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} \right]$$

$$\rightarrow \mu \left[ \frac{\partial}{\partial x_k} \left( \overline{\frac{\partial (u'_i u'_j)}{\partial x_k}} \right) - 2 \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} \right]$$

- Termo de dissipação:

$$\mu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_k}} = \varepsilon_{ik} = \frac{2}{3} \delta_{ik} \varepsilon$$

## Equação de transporte $p' / -\rho \overline{u'_i u'_j}$

- Termos de pressão:

$$\begin{aligned} -\overline{u'_j \frac{\partial p'}{\partial x_i}} - \overline{u'_i \frac{\partial p'}{\partial x_j}} &\rightarrow -\frac{\partial}{\partial x_k} \left( \overline{p' \delta_{ki} u'_j} \right) + \overline{p' \frac{\partial u'_j}{\partial x_i}} - \frac{\partial}{\partial x_k} \left( \overline{p' \delta_{kj} u'_i} \right) + \overline{p' \frac{\partial u'_i}{\partial x_j}} \\ &\rightarrow -\frac{\partial}{\partial x_k} \left( \overline{p' \delta_{ki} u'_j} + \overline{p' \delta_{kj} u'_i} \right) + \overline{p' \left( \frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j} \right)} \end{aligned}$$

A partir da equação 7, chegar na equação da energia cinética turbulenta

$$K = \left( \frac{1}{2} \overline{u'_i u'_i} \right).$$