Exercícios Extras

Considere os exemplos:

a) $\int sen^{6} 2x \cdot cos^{6} 2x \, dx$

Multiplicidade de arcos, tem-se

$sen\,2x = 2\,sen\,x \cdot cos\,x$

$sen\,2x = \frac{1}{2}\,2\,sen\,x \cdot cos\,x = sen\,2x = \frac{1}{2}\,sen\,4x$

$= \int \left(\frac{1}{2}\,sen\,4x\right)^6 dx = \frac{1}{64}\int (sen^2 4x)^3 dx \qquad sen^2 x = \frac{1-cos\,2x}{2}$

$= \frac{1}{64}\int \left[\frac{1}{2}(1-cos\,8x)^3\right] dx = \frac{1}{64}\int \frac{1}{8}\left(1 - 3cos\,8x + 3cos^2 8x - cos^3 8x\right) dx$

$= \frac{1}{512}\left[\int dx - \frac{1}{8}\cdot 3\int cos\,8x\,8dx + \underbrace{3\int cos^2 8x}_{I_1} - \underbrace{\int cos^3 8x\,dx}_{I_2}\right] \quad (1)$

$u = 8x$
$du = 8dx$

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$I_1 = 3\int cos^2 8x\,dx \qquad cos^2 x = \frac{1+cos\,2x}{2}$

$I_1 = 3\int \left(\frac{1+cos\,16x}{2}\right) dx$

$= \frac{3}{2}\int dx + \frac{1}{16}\frac{3}{2}\int cos\,16x\,16dx \qquad \int cos\,u\,du = sen\,u + C$

$u = 16x$
$du = 16dx$

$$\boxed{I_1 = \frac{3}{2}x + \frac{3}{32}\,sen\,16x}$$

$I_2 = \int cos^2 8x \cdot cos\,8x\,dx = \int (1 - sen^2 8x) \cdot cos\,8x\,dx$

$= \frac{1}{8}\int cos\,8x\,8dx - \frac{1}{8}\int sen^2 8x\,8\,cos\,8x\,dx$

$u = 8x \qquad u = sen\,8x$
$du = 8dx \qquad du = 8\,cos\,8x\,dx$

$I_2 = \frac{1}{8}\,sen\,8x - \frac{1}{8}\,\frac{sen^3 8x}{3}$

$$\boxed{I_2 = \frac{1}{8}\,sen\,8x - \frac{1}{24}\,sen^3 8x}$$

Retornando na expressão (1):

$$= \frac{1}{512}x - \frac{3}{4096}\operatorname{sen}8x + \frac{3x}{1024} + \frac{3}{16384}\operatorname{sen}16x - \frac{1}{4096}\operatorname{sen}8x$$

$$+ \frac{1}{12288}\operatorname{sen}^3 8x + C$$

$$= \boxed{\frac{1}{512}x - \frac{\operatorname{sen}8x}{1024} + \frac{3\operatorname{sen}16x}{16384} + \frac{\operatorname{sen}^3 8x}{12288} + C}$$

b) $\int \operatorname{sen}^3 5x \cdot \cos^3 5x\, dx$

$\operatorname{sen}2x = 2\operatorname{sen}x \cdot \cos x$

$\operatorname{sen}2x = \frac{1}{2}2\operatorname{sen}x \cdot \cos x = \operatorname{sen}2x = \frac{1}{2}\operatorname{sen}10x$

$$= \int\left(\frac{1}{2}\operatorname{sen}10x\right)^3 dx = \int \frac{1}{8}\operatorname{sen}^3 10x\, dx = \frac{1}{8}\int \operatorname{sen}^2 10x \cdot \operatorname{sen}10x\, dx$$

$\operatorname{sen}^2 x + \cos^2 x = 1$

$$= \frac{1}{8}\int(1 - \cos^2 10x)\cdot \operatorname{sen}10x\, dx$$

$$= \frac{1}{10}\cdot\frac{1}{8}\int \operatorname{sen}10x\, 10\, dx \qquad - \left(-\frac{1}{10}\right)\frac{1}{8}\int \cos^2 10x\,(-10)\operatorname{sen}10x\, dx$$

$u = 10x$ $\quad$ $u = \cos 10x$

$du = 10dx$ $\quad$ $du = -10\operatorname{sen}10x\, dx$

$$= -\frac{1}{80}\cos 10x + \frac{1}{80}\frac{\cos^3 10x}{3} + C$$

$$= \boxed{-\frac{1}{80}\cos 10x + \frac{1}{240}\cos^3 10x + C}$$

c) $\int \operatorname{sen}^5 6x \cdot \cos^7 6x\, dx$

d) $\int \operatorname{sen} 7x \cdot \cos 9x \, dx$

Para este caso deve usar a fórmula de Prostaferese:

$$\begin{cases} \operatorname{sen} p + \operatorname{sen} q = 2 \operatorname{sen} \dfrac{p+q}{2} \cdot \cos \dfrac{p-q}{2} & (1) \\ \operatorname{sen} p - \operatorname{sen} q = 2 \operatorname{sen} \dfrac{p-q}{2} \cdot \cos \dfrac{p+q}{2} & (2) \end{cases}$$

$$= \frac{1}{2}\, 2 \operatorname{sen} 7x \cdot \cos 9x$$

$$= \underbrace{\frac{1}{2}\, 2 \operatorname{sen} \frac{p-q}{2} \cdot \cos \frac{p+q}{2}}_{\operatorname{sen} p \,-\, \operatorname{sen} q \ (2)}$$

$$\begin{cases} \dfrac{p-q}{2} = 7x \\ \dfrac{p+q}{2} = 9x \end{cases} \Rightarrow \begin{cases} p - \cancel{q} = 14x \\ p + \cancel{q} = 18x \end{cases}$$

$$2p = 32x$$

$$p = 16x$$

$$q = 2x$$

$$= \frac{1}{2}(\operatorname{sen} 16x - \operatorname{sen} 2x)$$

$$= \frac{1}{2}\int (\operatorname{sen} 16x - \operatorname{sen} 2x)\, dx$$

$$= \frac{1}{16}\,\frac{1}{2}\int \operatorname{sen} 16x \; 16 \, dx \; - \frac{1}{2}\,\frac{1}{2}\int \operatorname{sen} 2x \; 2 \, dx$$

$u = 16x$

$du = 16dx$

$$\boxed{-\frac{1}{32}\cos 16 + \frac{1}{4}\cos 2x + C}$$

Outra forma de resolução

$$\frac{1}{2}\, 2 \operatorname{sen} 7x \cos 9x$$

$$\underbrace{\frac{1}{2}\, 2 \operatorname{sen} \frac{p+q}{2} \cos \frac{p-q}{2}}_{\operatorname{sen} p \,+\, \operatorname{sen} q \ (1)}$$

$$\begin{cases} \dfrac{p+q}{2} = 7x \\ \dfrac{p-q}{2} = 9x \end{cases} \Rightarrow \begin{cases} p + \cancel{q} = 14x \\ p - \cancel{q} = 18x \end{cases}$$

$$2p = 32x$$

$$p = 16x$$

$$q = -2x$$

$$= \frac{1}{2}\left[\operatorname{sen} 16x + \operatorname{sen}(-2x)\right]$$

$$= \frac{1}{16}\frac{1}{2}\int \operatorname{sen} 16x \, 16 dx \quad + -\frac{1}{2}\frac{1}{2}\int \operatorname{sen}(-2x)(-2)dx$$

$u = 16x$; $du = 16dx$ $\qquad$ $u = -2x$; $du = -2dx$

$$= \boxed{-\frac{1}{32}\cos 16x + \frac{1}{4}\cos(-2x) + C}$$

e) $\int tg^5 7x \, dx$

Obs.: A integral poderá estar relacionada com a tangente ou cotangente, consideraremos $tg^2 x$ ou $cotg^2 x$ através das seguintes relações:

$$\sec^2 x - tg^2 x = 1$$
$$cossec^2 x - cotg^2 x = 1$$

$$= \int tg^3 7x \cdot tg^2 7x \, dx$$

$$= \int \left[tg^3 7x \cdot (\sec^2 7x - 1)\right] dx$$

$$= \frac{1}{7}\int tg^3 7x \cdot 7\sec^2 7x \, dx - \underbrace{\int tg^3 7x \, dx}_{I_1} \qquad (1)$$

$u = tg\, 7x$; $du = \sec^2 7x \cdot 7\, dx$

$$I_1 = \int tg^3 7x \, dx = \int tg^2 7x \cdot tg 7x \, dx = \int tg 7x \cdot tg^2 7x \, dx$$

$$= \int tg 7x \, (\sec^2 7x - 1) \, dx$$

$$\boxed{\int tg\, u \, du = \ln \sec u + C}$$

$$= \frac{1}{7}\int tg 7x \cdot 7\sec^2 7x \, dx - \int tg 7x \, dx$$

$u = tg\, 7x$; $du = \sec^2 7x \cdot 7dx$

Retornando a expressão (1):

$$= \boxed{\frac{1}{28}\,tg^4 7x - \frac{1}{14}\,tg^2 7x + \ln|\sec 7x| + C}$$

f) $\int \cotg^6 x\, dx$

g) $\int tg^3 4x \cdot \sec^5 4x\, dx$

Obs.: $\int tg^n u \cdot \sec^n u\, du$ e $\int \cotg^n u \cdot \operatorname{cossec}^n u\, du$, será considerado um fator de cada para formar a diferencial. Para potência ímpar, ímpar

$$= \int tg^2 4x \cdot \sec^4 4x \cdot \sec 4x \cdot tg 4x\, dx$$

$$= \int (\sec^2 4x - 1) \sec^4 4x \cdot \sec 4x \cdot tg 4x\, dx$$

$$= \int (\sec^6 4x - \sec^4 4x) \cdot \sec 4x\, tg 4x\, dx$$

$$= \frac{1}{4}\int \sec^6 4x\; 4 \sec 4x\, tg 4x\, dx - \frac{1}{4}\int \sec^4 4x\; 4 \sec 4x\, tg 4x\, dx$$

$u = \sec 4x$

$du = 4 \sec 4x \cdot tg\, 4x\, dx$

$$= \frac{1}{4}\,\frac{\sec^7 4x}{7} - \frac{1}{4}\,\frac{\sec^5 4x}{5}$$

$$= \frac{\sec^7 4x}{28} - \frac{\sec^5 4x}{20} + C$$

h) $\int \cotg^7 2x \cdot \operatorname{cossec}^9 2x\, dx$