Fórmula de Prostaférese para senos

São fórmulas de transformação de soma em produto:

$\operatorname{sen} x + \operatorname{sen} y$

1) $\operatorname{sen}(a+b) = \operatorname{sen}a \cdot \cos b + \operatorname{sen}b \cdot \cos a$

$\operatorname{sen}(a-b) = \operatorname{sen}a \cdot \cos b - \operatorname{sen}b \cdot \cos a$

$\operatorname{sen}(a+b) + \operatorname{sen}(a-b) = \operatorname{sen}a \cdot \cos b + \cancel{\operatorname{sen}b \cdot \cos a} + \operatorname{sen}a \cos b - \cancel{\operatorname{sen}b \cos a}$

$\operatorname{sen}(\underbrace{a+b}_{p}) + \operatorname{sen}(\underbrace{a-b}_{q}) = 2 \operatorname{sen}a \cdot \cos b$

$\begin{cases} a + \cancel{b} = p \\ a - \cancel{b} = q \end{cases}$

$2a = p + q$

$\boxed{a = \frac{p+q}{2}}$ //

$a + b = p$

$b = p - a$

$b = p - \frac{p-q}{2}$

$b = \frac{2p - p - q}{2}$

$\boxed{b = \frac{p-q}{2}}$ //

$2 \operatorname{sen}a \cdot \cos b = \operatorname{sen}p + \operatorname{sen}q = 2 \operatorname{sen}\frac{p+q}{2} \cdot \cos\frac{p-q}{2}$

2) $\operatorname{sen}x - \operatorname{sen}y$

$\operatorname{sen}(a+b) - \operatorname{sen}(a-b) = \cancel{\operatorname{sen}a \cdot \cos b} + \operatorname{sen}b \cdot \cos a - \cancel{\operatorname{sen}a \cos b} + \operatorname{sen}b \cdot \cos a$

$\operatorname{sen}(\underbrace{a+b}_{p}) - \operatorname{sen}(\underbrace{a-b}_{q}) = 2 \operatorname{sen}b \cdot \cos a$

$2 \operatorname{sen}b \cdot \cos a = \operatorname{sen}p - \operatorname{sen}q = 2 \operatorname{sen}\frac{p-q}{2} \cdot \cos\frac{p+q}{2}$

3) $\cos x + \cos y$

$\cos(a+b) = \cos a \cdot \cos b - \operatorname{sen}a \cdot \operatorname{sen}b$

$\cos(a-b) = \cos a \cdot \cos b + \operatorname{sen}a \cdot \operatorname{sen}b$

$$\cos(a+b) + \cos(a-b) = \cos a \cdot \cos b - \cancel{\operatorname{sen} a \cdot \operatorname{sen} b} + \cos a \cdot \cos b + \cancel{\operatorname{sen} a \cdot \operatorname{sen} b}$$

$$\cos(\underbrace{a+b}_{p}) + \cos(\underbrace{a-b}_{q}) = 2 \cos a \cdot \cos b$$

$$2\cos a \cdot \cos b = \cos p + \cos q = 2 \cos \frac{p+q}{2} \cdot \cos \frac{p-q}{2}$$

4) $\cos x - \cos y$

$$\cos(a+b) - \cos(a-b) = \cos \cancel{a \cdot \cos} b - \operatorname{sen} a \cdot \operatorname{sen} b - \cos \cancel{a \cdot \cos} b - \operatorname{sen} a \cdot \operatorname{sen} b$$

$$\cos(\underbrace{a+b}_{p}) - \cos(\underbrace{a-b}_{q}) = -2 \operatorname{sen} a \cdot \operatorname{sen} b$$

$$-2 \operatorname{sen} a \cdot \operatorname{sen} b = \cos p - \cos q = -2 \operatorname{sen} \frac{p+q}{2} \cdot \operatorname{sen} \frac{p-q}{2}$$

Para a tangente:

5) $$\operatorname{tg} p + \operatorname{tg} q = \frac{\operatorname{sen} p}{\cos p} + \frac{\operatorname{sen} q}{\cos q}$$
$$= \frac{\operatorname{sen} p \cdot \cos q + \operatorname{sen} q \cdot \cos p}{\cos p \cdot \cos q}$$
$$= \frac{\operatorname{sen}(p+q)}{\cos p \cdot \cos q}$$

Considere os seguintes exemplos.

a) $$\lim_{x \to 0} \frac{\cos 5x - \cos 3x}{x} = \frac{0}{0}$$

$$\cos(a+b) - \cos(a-b) = -2 \operatorname{sen} a \cdot \operatorname{sen} b = -2 \operatorname{sen} \frac{p+q}{2} \cdot \operatorname{sen} \frac{p-q}{2}$$

$$= \lim_{x \to 0} \frac{-2 \operatorname{sen} \frac{8x}{2} \cdot \operatorname{sen} \frac{2x}{2}}{x}$$

$$= \lim_{x \to 0} \frac{-2 \operatorname{sen} 4x \cdot \operatorname{sen} x}{x}$$

lim fundamental

$$= \underbrace{\lim_{x \to 0} -2 \operatorname{sen} 4x}_{0} \cdot \underbrace{\lim_{x \to 0} \frac{\operatorname{sen} x}{x}}_{1} = 0 \cdot 1 = 0$$

b) $\lim_{x \to a} \dfrac{\text{sen}\, x - \text{sen}\, a}{x - a}$ $\qquad \text{sen}\, p - \text{sen}\, q = 2\, \text{sen}\, \dfrac{p-q}{2} \cdot \cos \dfrac{p+q}{2}$

$$\lim_{x \to a} \dfrac{2\, \text{sen} \dfrac{(x-a)}{2} \cdot \cos \dfrac{(x+a)}{2}}{x-a} \quad (\div 2)$$

$$= \lim_{x \to a} \dfrac{\text{sen} \dfrac{(x-a)}{2} \cdot \cos \dfrac{(x+a)}{2}}{\dfrac{x-a}{2}} = \cos a$$

$1 \cdot \cos a$

c) $\lim_{x \to 0} \dfrac{\cos 6x - \cos 2x}{\text{sen}\, 4x}$ $\qquad \cos p - \cos q = -2\, \text{sen}\, \dfrac{p+q}{2} \cdot \text{sen}\, \dfrac{p-q}{2}$

$$= \lim_{x \to 0} \dfrac{-2\, \text{sen}\, \dfrac{8x}{2} \cdot \text{sen}\, \dfrac{4x}{2}}{\text{sen}\, 4x}$$

$$= \lim_{x \to 0} \dfrac{-2\, (\text{sen}\, 4x \cdot \text{sen}\, 2x)}{\text{sen}\, 4x} = -2 \cdot 1 \cdot 0 = 0$$

$= 1$

d) $\lim_{x \to +\infty} \sqrt{x^2 - 3x + 7} - \sqrt{x^2 + 1}$

$$\lim_{x \to +\infty} \dfrac{\left(\sqrt{x^2-3x+7} - \sqrt{x^2+1}\right) \cdot \left(\sqrt{x^2-3x+7} + \sqrt{x^2+1}\right)}{\left(\sqrt{x^2-3x+7} + \sqrt{x^2+1}\right)}$$

$$\lim_{x \to +\infty} \dfrac{\cancel{x^2} - 3x + 7 - \cancel{x^2} - 1}{\left(\sqrt{x^2-3x+7} + \sqrt{x^2+1}\right)} = \lim_{x \to +\infty} \dfrac{-3x + 6}{\left(\sqrt{x^2-3x+7} + \sqrt{x^2+1}\right)}$$

$$\lim_{x \to +\infty} \dfrac{x\left(-3 + \dfrac{6}{x}\right)}{\sqrt{x^2\left(1 - \dfrac{3}{x} + \dfrac{7}{x^2}\right)} + \sqrt{x^2\left(1 + \dfrac{1}{x^2}\right)}} = \lim_{x \to +\infty} \dfrac{\cancel{x}\left(-3 + \dfrac{6}{x}\right)}{\cancel{x}\left(\sqrt{1 - \dfrac{3}{x} + \dfrac{7}{x^2}} + \sqrt{1 + \dfrac{1}{x^2}}\right)} = \dfrac{-3}{2}$$

e) $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = \dfrac{0}{0}$

$$\lim_{x \to 0} \dfrac{(1 - \cos x) \cdot (1 + \cos x)}{x \cdot (1 + \cos x)} = \lim_{x \to 0} \dfrac{1 + \cancel{\cos x} - \cancel{\cos x} - \cos^2 x}{x \cdot (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\operatorname{sen}^2 x}{x\,(1+\cos x)} \qquad \operatorname{sen}^2 x + \cos^2 x = 1$$

$$= \lim_{x \to 0} \frac{x \cdot \operatorname{sen} x}{x} \cdot \frac{\operatorname{sen} x}{x\,(1+\cos x)} = 0 \cdot 1 \cdot 1 \cdot \frac{1}{2} = 0$$

f) $\displaystyle\lim_{x \to 0} \frac{\operatorname{sen} 5x - \operatorname{sen} 3x}{\operatorname{sen} x}$

$\operatorname{sen} p - \operatorname{sen} q = 2 \operatorname{sen} \frac{p-q}{2} \cos \frac{p+q}{2}$

$$= \lim_{x \to 0} \frac{2 \operatorname{sen} \frac{2x}{2} \cdot \cos \frac{8x}{2}}{\operatorname{sen} x}$$

$$= \lim_{x \to 0} \frac{2 \overbrace{\operatorname{sen} x}^{=1} \cdot \cos 4x}{\operatorname{sen} x} = 2 \cdot 1 \cdot 1 = 2$$

g) $\displaystyle\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{(x+2)}$ → esta adição veio de uma multiplicação de limites.

$$\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

$$= \underbrace{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x}}_{e} \cdot \underbrace{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{2}}_{1} = e \cdot 1 = e$$

h) $\displaystyle\lim_{x \to +\infty} \left(\frac{x+1}{x-1}\right)^{x}$

$$\lim_{x \to +\infty} \left[\frac{\cancel{x}\left(1 + \frac{1}{x}\right)}{\cancel{x}\left(1 - \frac{1}{x}\right)}\right]^{x} = \lim_{x \to +\infty} \frac{\left(1 + \frac{1}{x}\right)^{x}}{\left(1 - \frac{1}{x}\right)^{x}}$$

$$\frac{-1}{x} = \frac{1}{y} \qquad x = -y$$

$$= \frac{\displaystyle\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x}}{\displaystyle\lim_{y \to -\infty} \left(1 + \frac{1}{y}\right)^{-y}} = \frac{\displaystyle\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x}}{\displaystyle\lim_{y \to -\infty} \left[\left(1 + \frac{1}{y}\right)^{y}\right]^{-1}} \quad \text{aplicando o limite tem-se}$$

$$= \frac{e}{e^{-1}} = e \cdot e^{1} = e^{2}$$