Quantum Theory of Many-Body systems in Condensed Matter (4302112) 2020

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Today’s class: *Equations of motion and time-ordered correlation functions*

- Equations of motion for creation and destruction operators.
- Retarded and advanced correlation functions.
- Fourier transform

Equations of motion for operators

Heisenberg Picture: equation of motion:

$$\begin{cases} \dfrac{d\hat{a}_k(t)}{dt} = \dfrac{i}{\hbar}\left[\hat{H}, \hat{a}_k(t)\right]_- \\[2ex] \hat{a}_k(t) \equiv e^{+i\hat{H}t/\hbar}\hat{a}_k e^{-i\hat{H}t/\hbar} \end{cases}$$

Commutator/anti-commutator
at *different times*?

$$\left[\hat{a}_k(t_2), \hat{a}_m^\dagger(t_1)\right]_\pm = ?$$

Special case (diagonal Hamiltonian): [redacted]

$$\hat{H} = \sum_k \epsilon_k \hat{a}_k^\dagger \hat{a}_k \Rightarrow \begin{cases} \hat{a}_k(t) = e^{-i\epsilon_k t/\hbar}\hat{a}_k(0) \\[2ex] \left[\hat{a}_k(t_1), \hat{a}_m^\dagger(t_2)\right]_\pm = e^{-i(\epsilon_k t_1 - \epsilon_m t_2)/\hbar}\delta_{km} \end{cases}$$

Equations of motion for operators

"Tight-binding-like"?

$$\hat{H} = \sum_k \epsilon_k \hat{a}_k^\dagger \hat{a}_k - t \sum_{\substack{k_1,k_2 \\ (k_1 \neq k_2)}} \hat{a}_{k_1}^\dagger \hat{a}_{k_2} + \hat{a}_{k_2}^\dagger \hat{a}_{k_1}$$

In this case you can still solve ("close") the equations of motion (Assignment):

$$\begin{cases} \dfrac{d\hat{a}_k(t)}{dt} = \dfrac{i}{\hbar}\left[\hat{H}, \hat{a}_k(t)\right]_- \\ \dfrac{d\hat{a}_k^\dagger(t)}{dt} = \dfrac{i}{\hbar}\left[\hat{H}, \hat{a}_k^\dagger(t)\right]_- \end{cases}$$

In the *interacting* case, however, you cannot do it (Assignment):

$$\hat{H} = \sum_k \epsilon_k \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \sum_{k_1 k_2 q} \hat{a}_{k_1+q}^\dagger \hat{a}_{k_2-q}^\dagger \hat{a}_{k_2} \hat{a}_{k_1}$$

Time-dependent correlation functions

$$\langle \hat{A}(t)\hat{B}(t')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\hat{\rho}\hat{A}(t)\hat{B}(t')\right)$$

depends only on (t-t') [redacted]

$$= \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}e^{i\hat{H}(t-t')/\hbar}\hat{A}e^{-i\hat{H}(t-t')/\hbar}\hat{B}\right)$$

“Retarded”:

“Advanced”:

$$\begin{cases} C^{r}_{\hat{A},\hat{B}}(t-t') = -i\theta(t-t')\langle \hat{A}(t)\hat{B}(t')\rangle & t > t' \\ C^{a}_{\hat{A},\hat{B}}(t-t') = +i\theta(t'-t)\langle \hat{A}(t)\hat{B}(t')\rangle & t < t' \end{cases}$$

Fourier transform: retarded case

t > t'

FT of the retarded correlation function:

$$C^r_{\hat{A},\hat{B}}(\omega) = \int_{-\infty}^{\infty} d(t-t')\; e^{i\omega(t-t')} C^r_{\hat{A},\hat{B}}(t-t')$$

Ok, as long as:

$$\lim_{(t-t')\to\infty} C^r_{\hat{A},\hat{B}}(t-t') = 0$$

Physically: some relaxation mechanism

Or: $\omega \to \omega^+ = \omega + i\eta$ [redacted]

$$C^r_{\hat{A},\hat{B}}(\omega^+) = \int_{-\infty}^{\infty} dt\; e^{i\omega^+ t} C^r_{\hat{A},\hat{B}}(t)$$

$$C^r_{\hat{A},\hat{B}}(t) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; e^{-i\omega t} C^r_{\hat{A},\hat{B}}(\omega^+)$$

Fourier transform: advanced case

t < t'

FT of the advanced correlation function:

$$C^{a}_{\hat{A},\hat{B}}(\omega) = \int_{-\infty}^{\infty} d(t-t')\; e^{i\omega(t-t')} C^{a}_{\hat{A},\hat{B}}(t-t')$$

Ok, as long as:

$$\lim_{(t-t')\to-\infty} C^{a}_{\hat{A},\hat{B}}(t-t') = 0$$

Physically: some relaxation mechanism

Or: $\omega \to \omega^{-} = \omega - i\eta$ [redacted]

$$\begin{cases} C^{a}_{\hat{A},\hat{B}}(\omega^{-}) = \int_{-\infty}^{\infty} dt\; e^{i\omega^{-}t} C^{a}_{\hat{A},\hat{B}}(t) \\ C^{a}_{\hat{A},\hat{B}}(t) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; e^{-i\omega t} C^{a}_{\hat{A},\hat{B}}(\omega^{-}) \end{cases}$$