



ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

# **Introdução aos Elementos de Máquinas**

**PMR 3320 - A03**

**Esforços e Diagramas de esforços**

**2020.2**



## Cronograma de aulas

Dia	S	Aula	Tópico	Prof.
17.08	2ª	A1	<b>Introdução a disciplina</b> Modelagem, carregamento e equilíbrio	RS
24.08	2ª	A2	Composição de tensões Estado plano de tensões - Círculo de <u>Mohr</u>	RS
31.08	2ª	A3	Composição de tensões Diagramas de esforços	RS
07.09	2ª	---	<b>Feriado - Independência do Brasil</b>	
14.09	2ª	A4	Teorias de Falha: 2) Falha por deformação permanente: von <u>Mises</u> , <u>Tresca</u> , <u>Coulomb-Mohr</u>	RS
21.09	2ª	A5	Teorias de Falha: 3) Falha por fadiga	RS
28.09	2ª	A6	Fixações cubo-eixo	NG
05.10	2ª	A7	Dimensionamento de Eixos	NG
12.10	2ª	---	<b>Feriado - Dia da Criança</b>	
19.10	2ª	A8	Especificação e dimensionamento de elementos de fixação: Rebites	NG
26.10	2ª	A9	Especificação e dimensionamento de elementos de fixação: Parafusos	NG
02.11	2ª	A10	Especificação e dimensionamento de elementos de transmissão: Fusos	NG
09.11	2ª	A11	Análise e dimensionamento de componentes mecânicos: Engrenagens: Parte - 1	RS
16.11	2ª	A12	Análise e dimensionamento de componentes mecânicos: Engrenagens: Parte - 2	RS
23.11	2ª	A13	Análise e dimensionamento de componentes mecânicos: Mancais	RS
30.11	2ª	A17	Análise e dimensionamento de componentes mecânicos: Molas	NG
07.12	2ª	A18	Análise e dimensionamento de componentes mecânicos: Acoplamentos e embreagens	NG
14.12	2ª		<b>Encerramento do semestre 2020-2</b>	

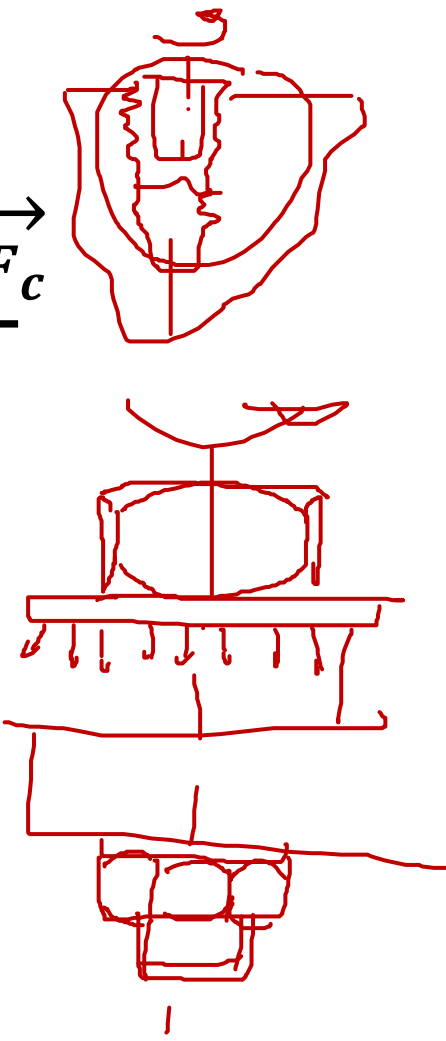
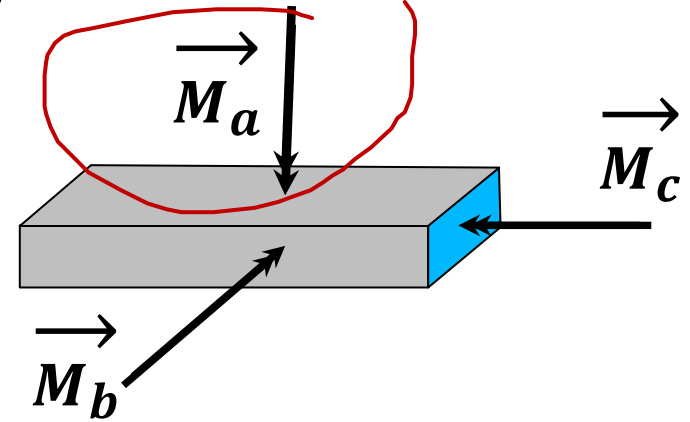
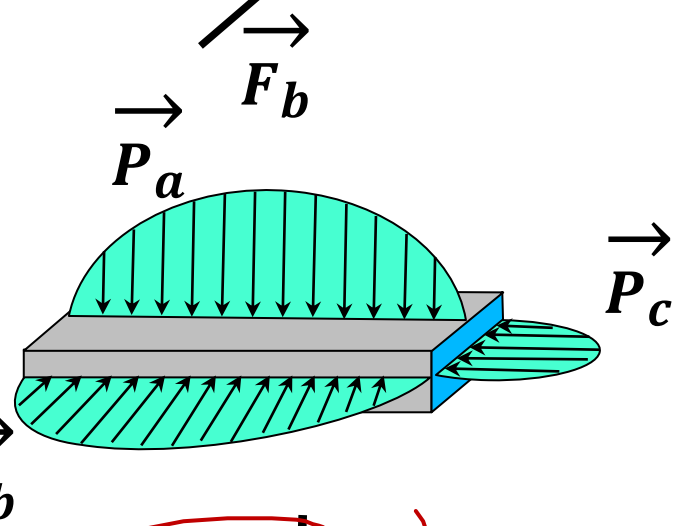
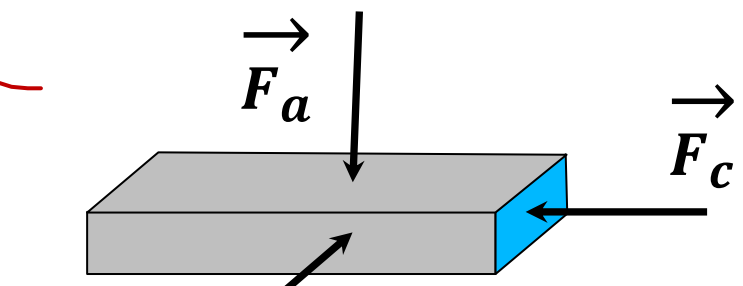
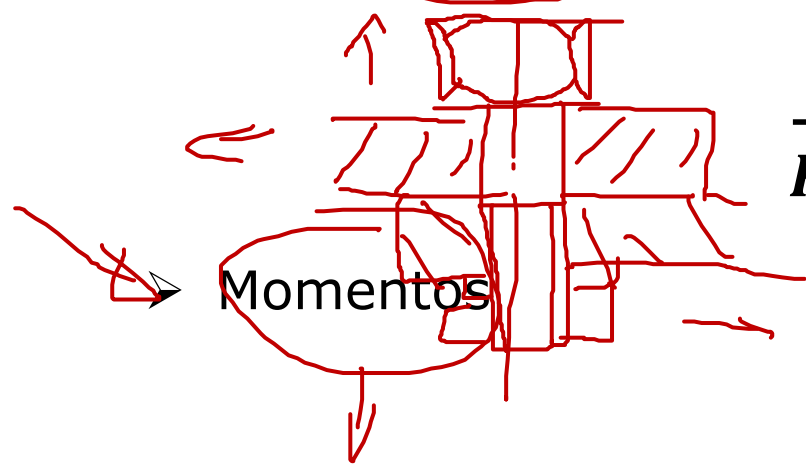
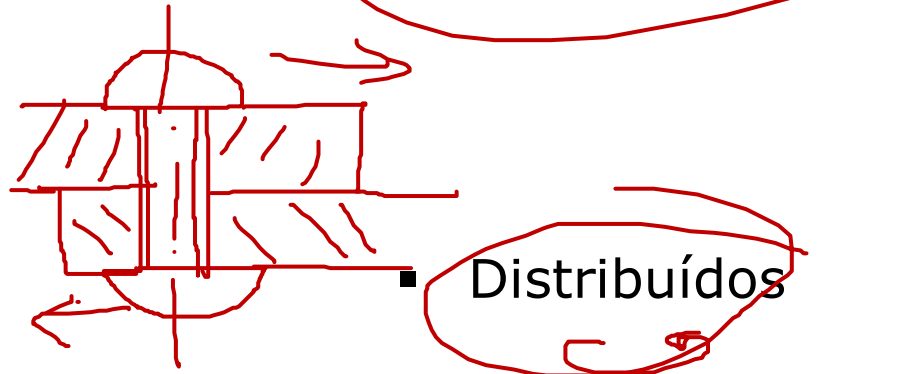
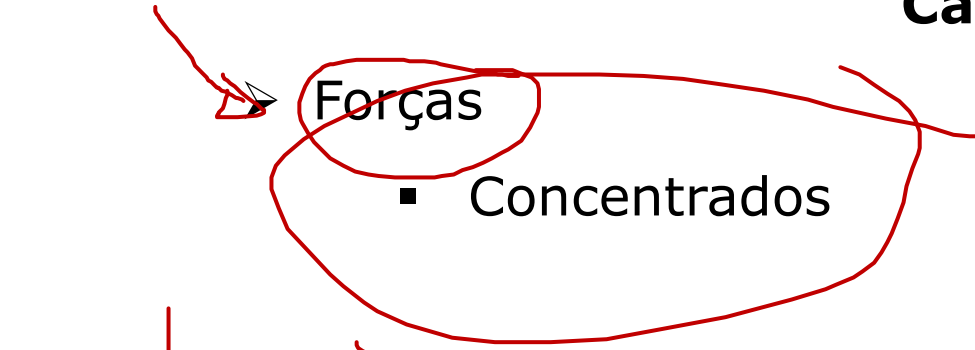


## Tópicos

- ▶ Composição de tensões
- ▶ Método das seções
- ▶ Diagramas de esforços e flexão



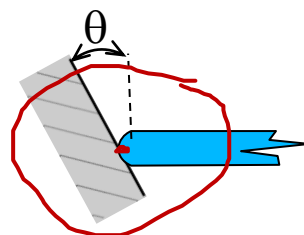
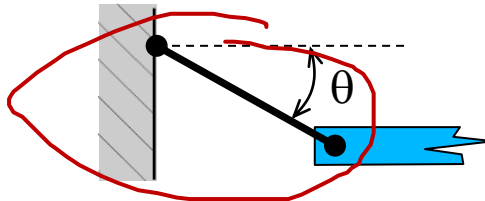
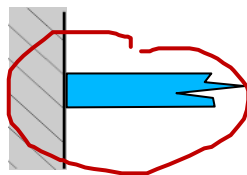
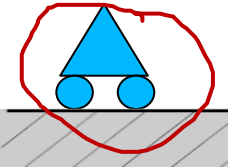
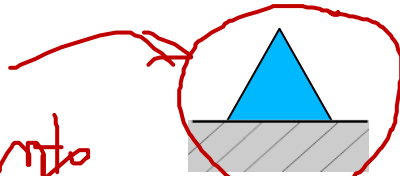
## Carregamentos



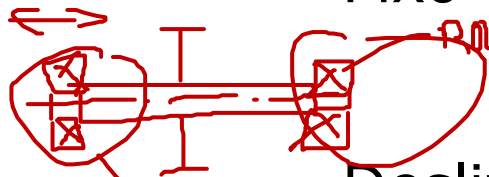
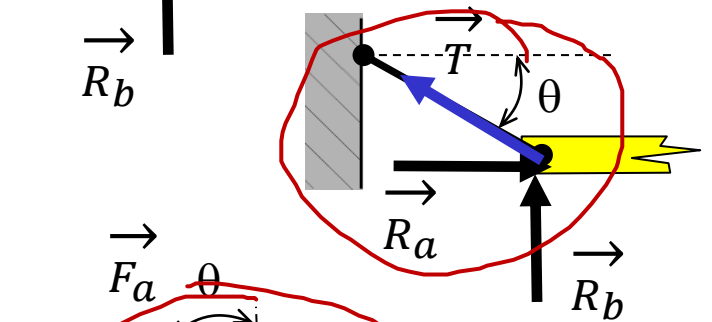
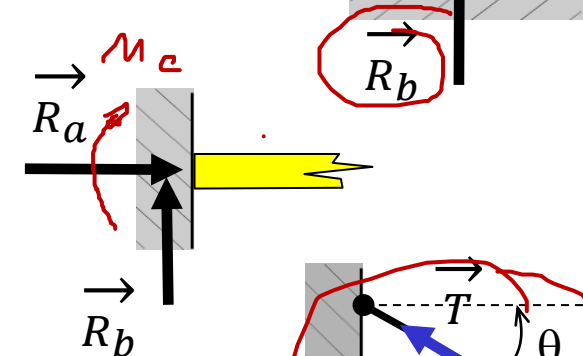
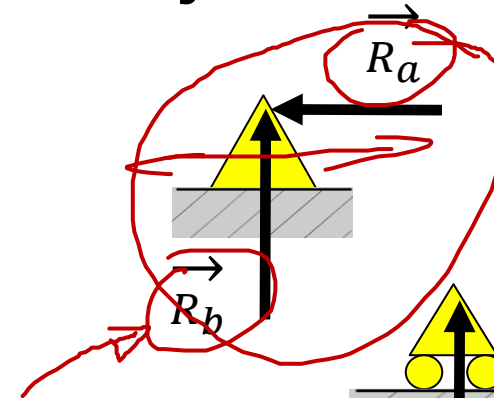


## Apoios

- Fixo
- Deslizante
- Engastado
- Com cabo
- Com contato



## Reações

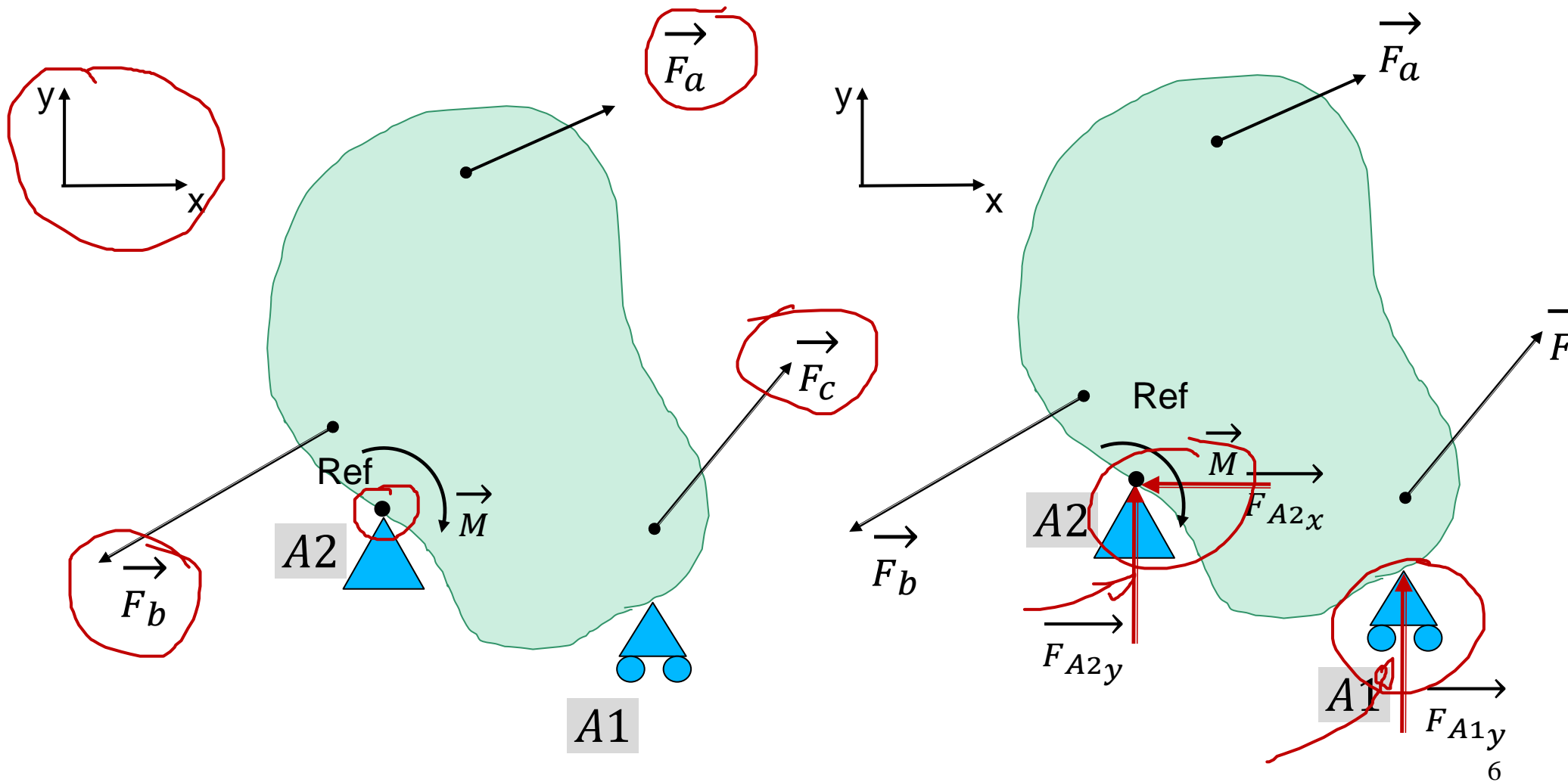


rolamento



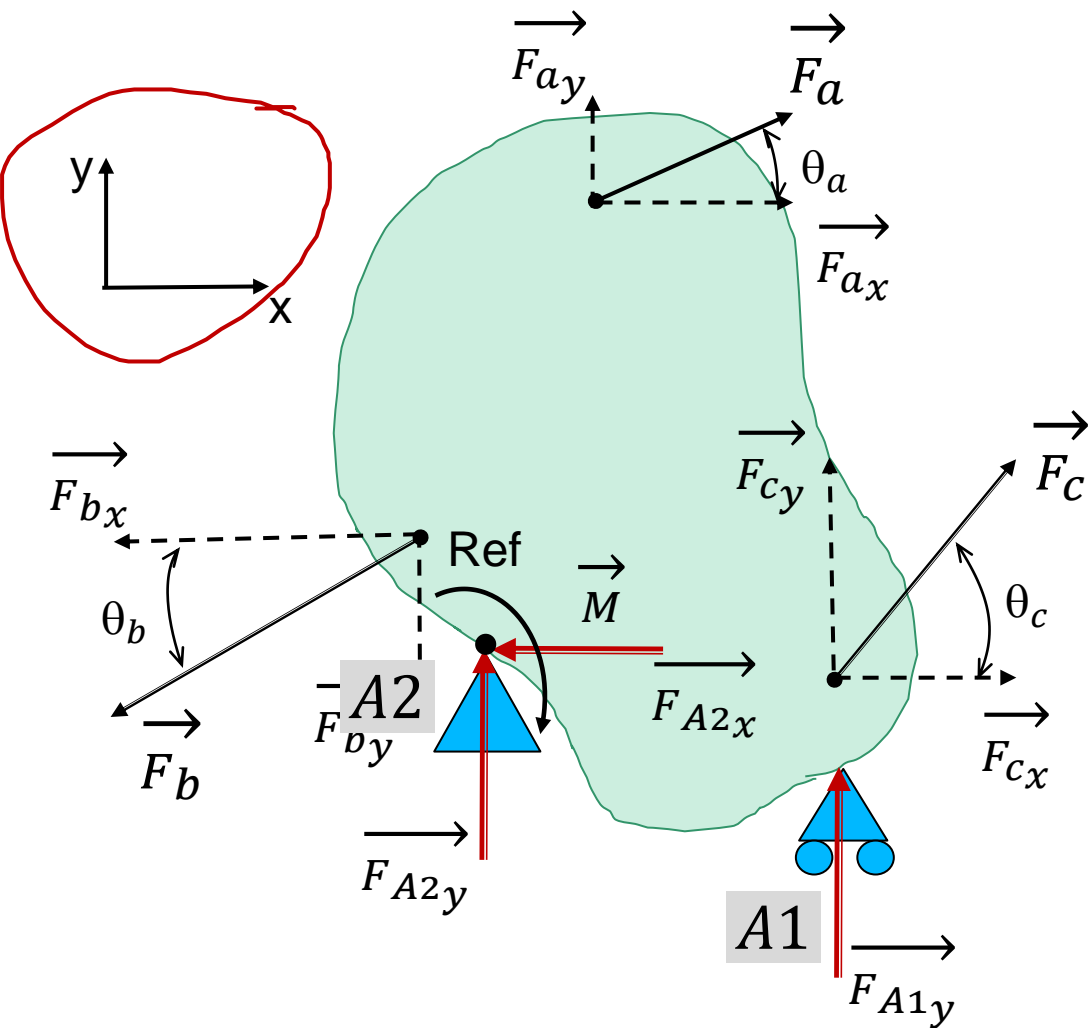


## Corpo apoiado sujeito a carregamentos





## Apoios e reações no equilíbrio

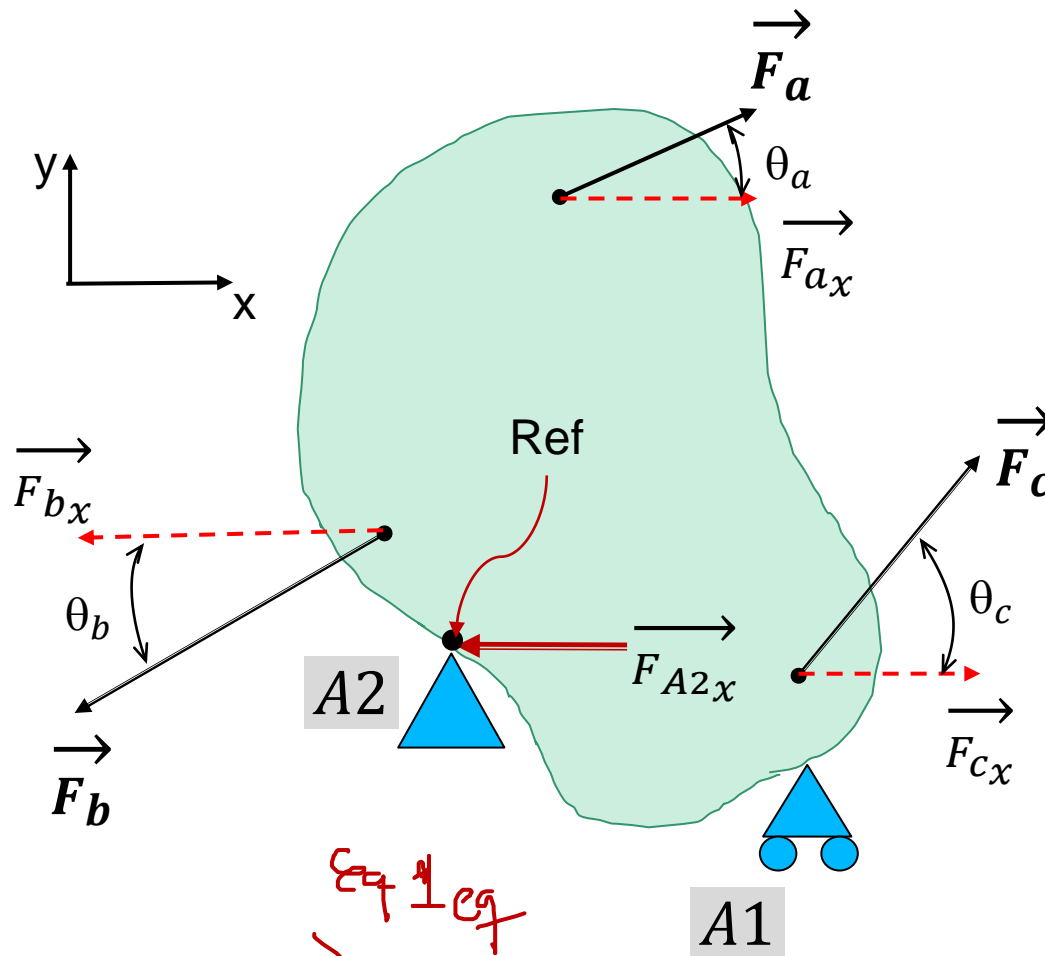


$$\sum_{Ref} \vec{F}_n = \vec{F} = 0$$

$$\sum_{Ref} \vec{M}_n = \vec{M} = 0$$



## Apoios e reações no equilíbrio



$$\sum \vec{F}_{ref(x)} = 0$$

$$\vec{F}_{ax} + \vec{F}_{cx} - \vec{F}_{bx} + \vec{F}_{A2x} = 0$$

$$\vec{F}_{ax} = \vec{F}_a \cdot \cos \theta_a,$$

$$\vec{F}_{cx} = \vec{F}_c \cdot \cos \theta_c,$$

$$-\vec{F}_{bx} = \vec{F}_b \cdot \cos \theta_b,$$

$$\vec{F}_{A2x}$$

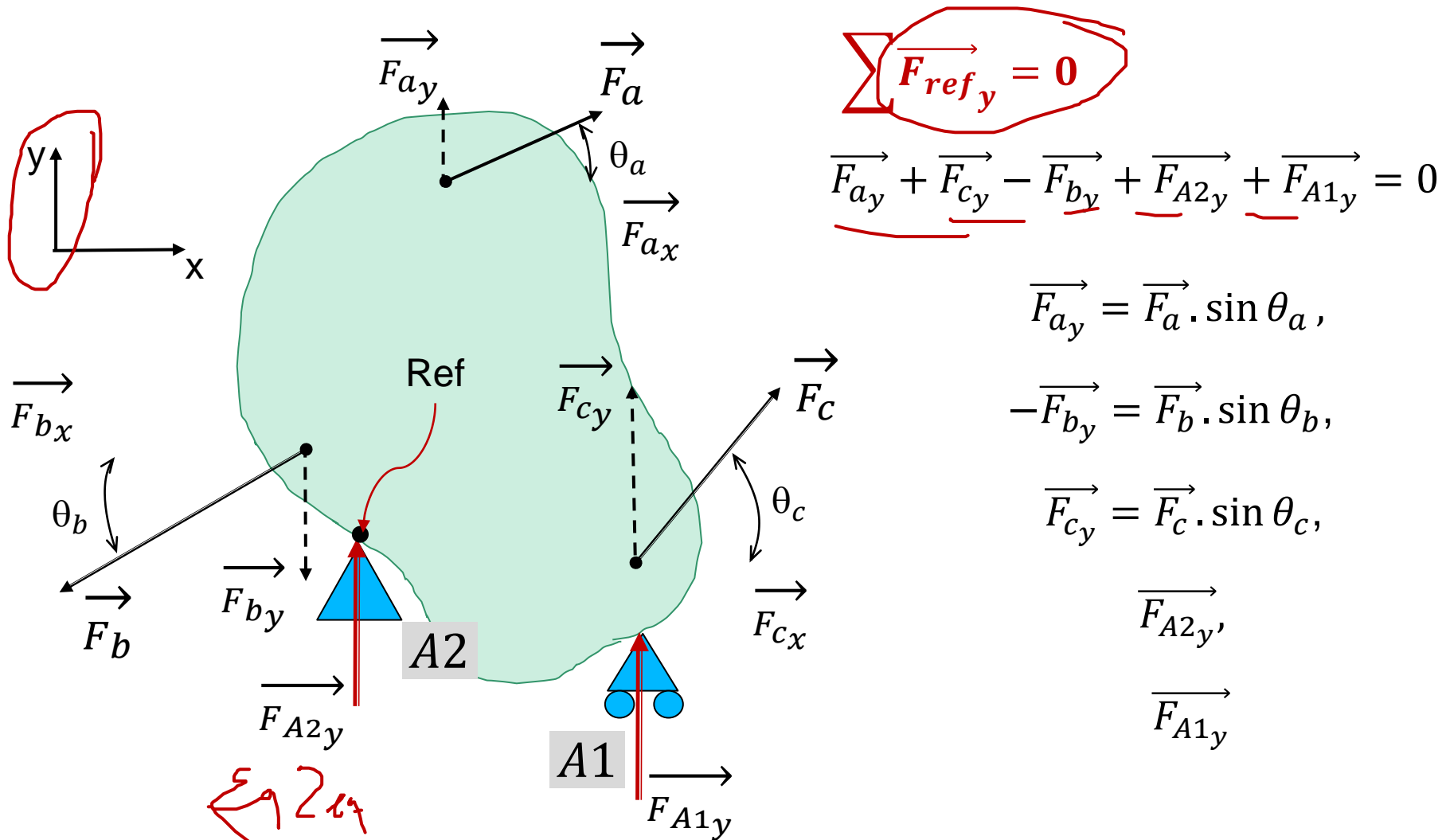
eq 1 eq

$$\vec{F}_a \cdot \cos \theta_a + \vec{F}_c \cdot \cos \theta_c - \vec{F}_b \cdot \cos \theta_b + \vec{F}_{A2x} = 0$$





## Apoios e reações no equilíbrio



Eq 2.17

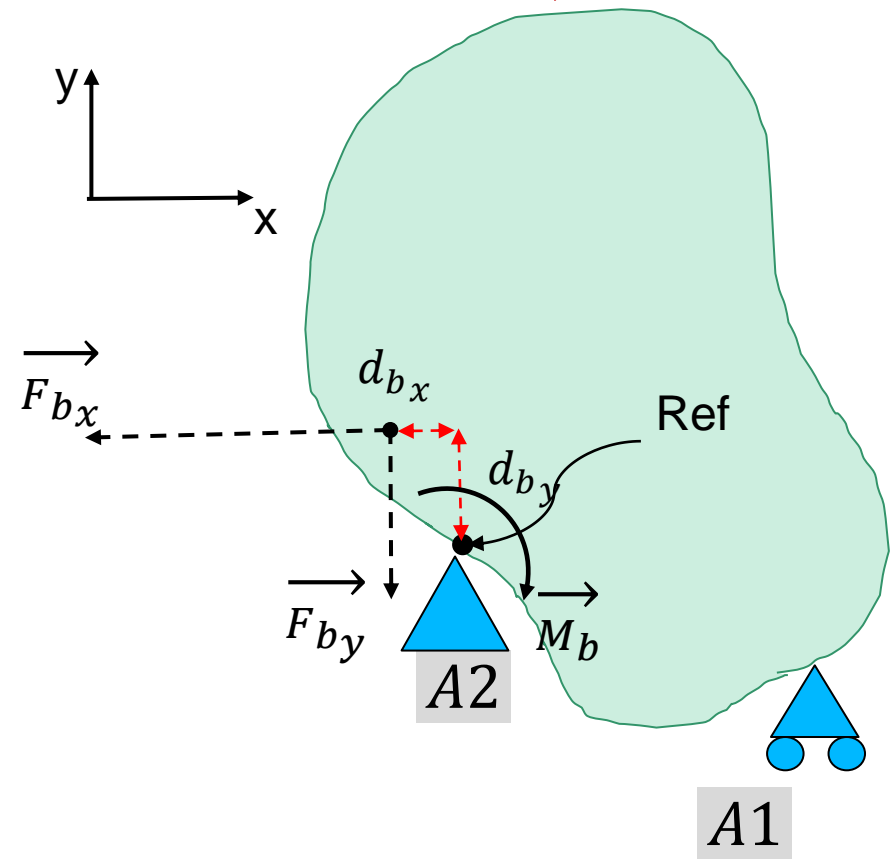
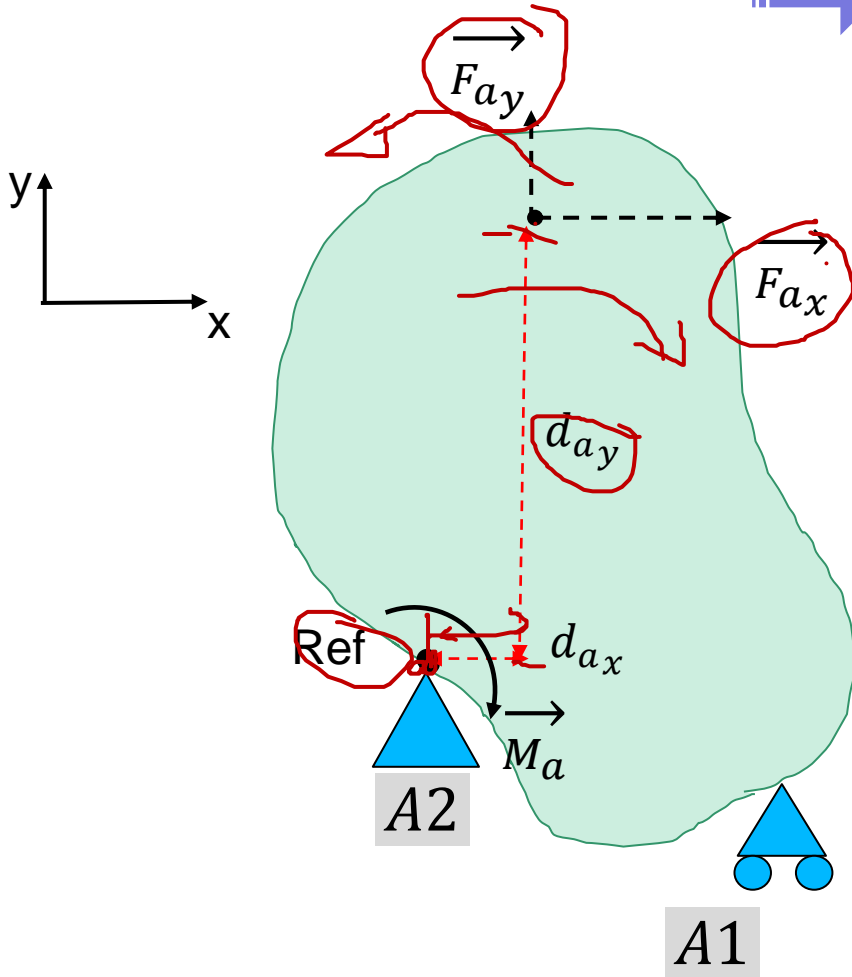
$$\vec{F}_a \cdot \sin \theta_a + \vec{F}_c \cdot \sin \theta_c - \vec{F}_b \cdot \sin \theta_b + \vec{F}_{A2\ y} + \vec{F}_{A1\ y} = 0$$



## Apoios e reações no equilíbrio

$\Rightarrow \sum_{Ref} \vec{M}_n = 0$

(+)



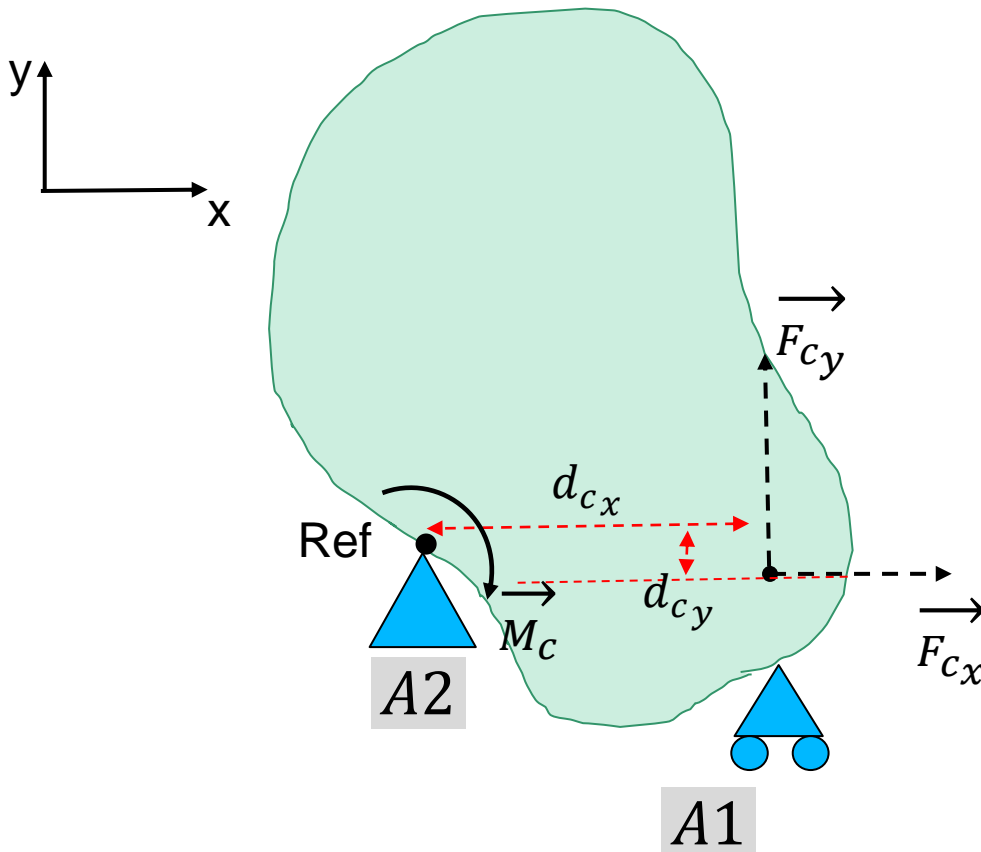
$\vec{M}_a = F_{ax} * d_{ay} - F_{ay} * d_{ax}$

$\vec{M}_b = -F_{bx} * d_{ay} - F_{by} * d_{ax}$

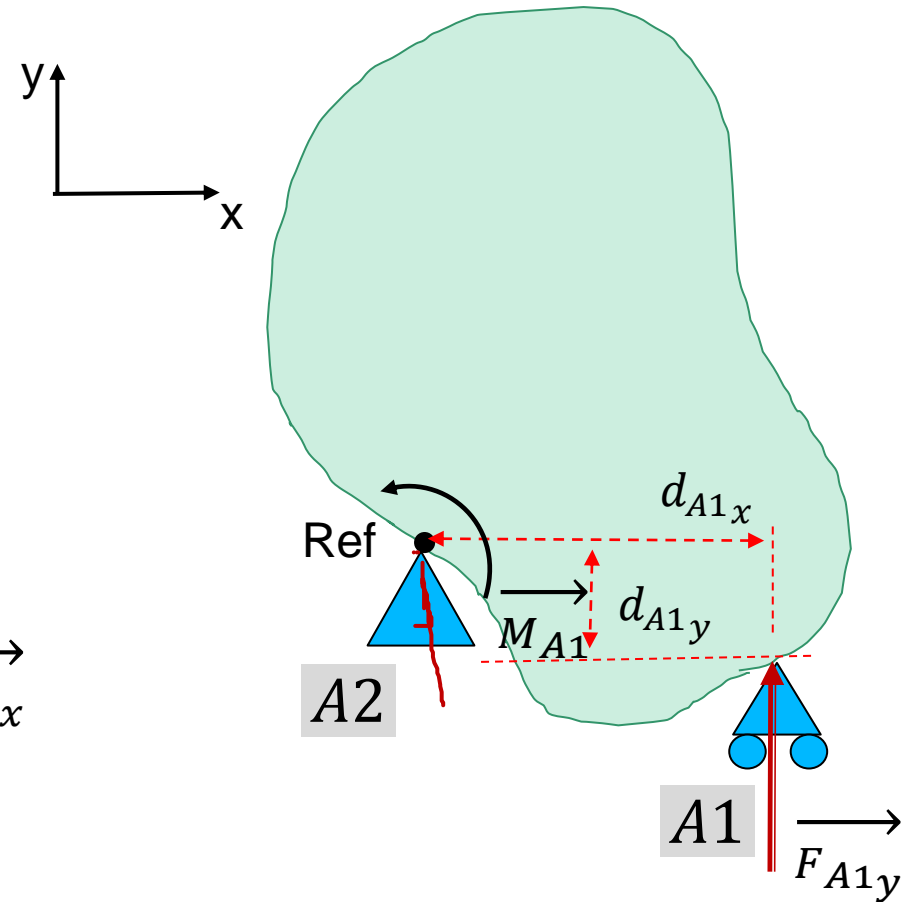


## Apoios e reações no equilíbrio

$$\Rightarrow \sum_{Ref} \vec{M}_n = 0 \quad (+)$$



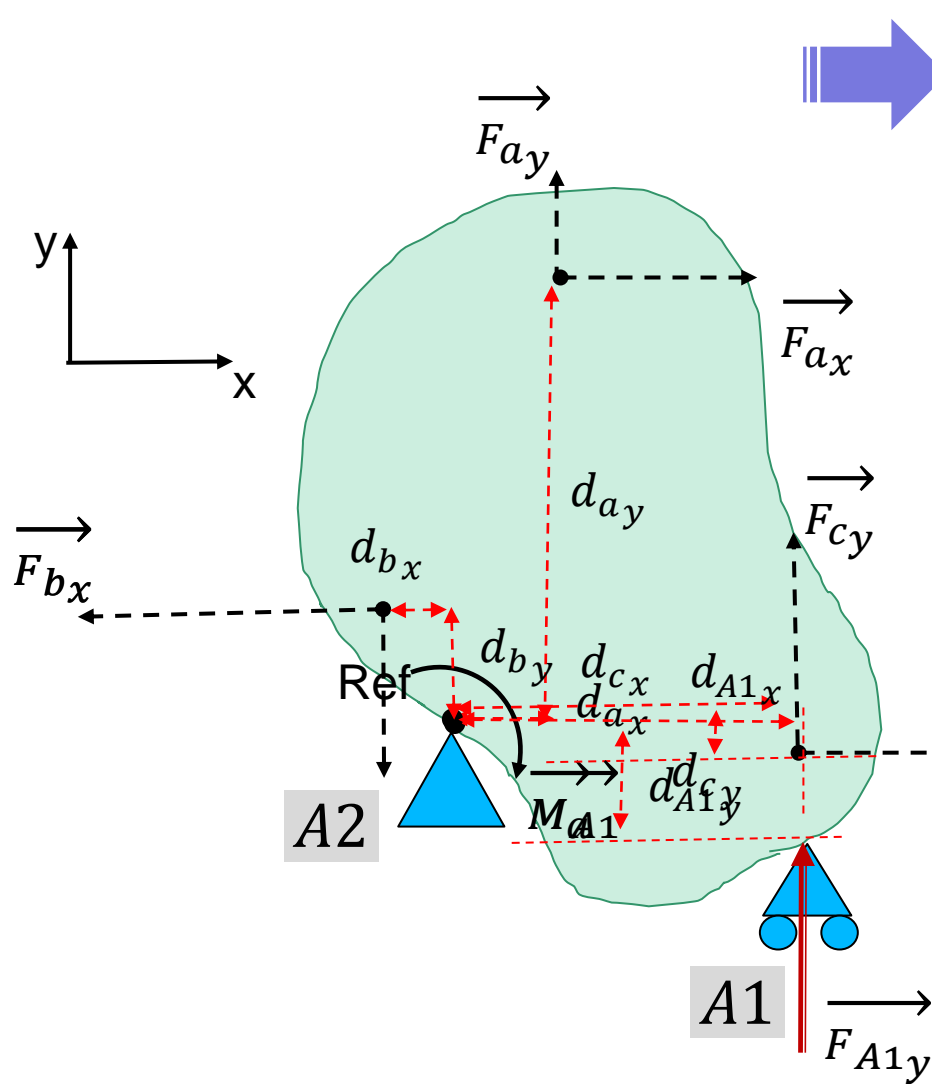
$$\vec{M}_c = -F_{cx} * d_{cy} + F_{cy} * d_{cx}$$



$$\vec{M}_{A1} = F_{A1y} * d_{A1x}$$



## Apoios e reações no equilíbrio



$$\sum_{Ref} \vec{M}_n = 0 \quad +$$

$$\vec{M}_a = \vec{F}_{ax} * d_{ay} - \vec{F}_{ay} * d_{ax}$$

$$\vec{M}_b = -\vec{F}_{bx} * d_{ay} - \vec{F}_{by} * d_{ax}$$

$$\vec{M}_c = -\vec{F}_{cx} * d_{cy} + \vec{F}_{cy} * d_{cx}$$

$$\vec{M}_{A1} = \vec{F}_{A1y} * d_{A1x}$$

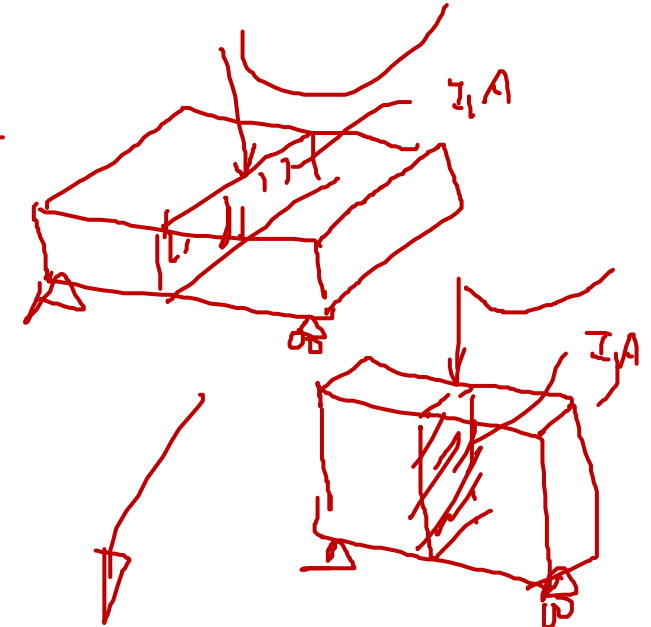
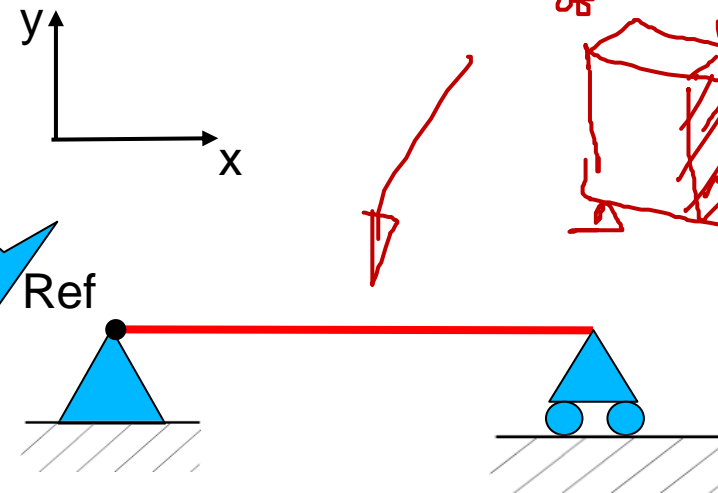
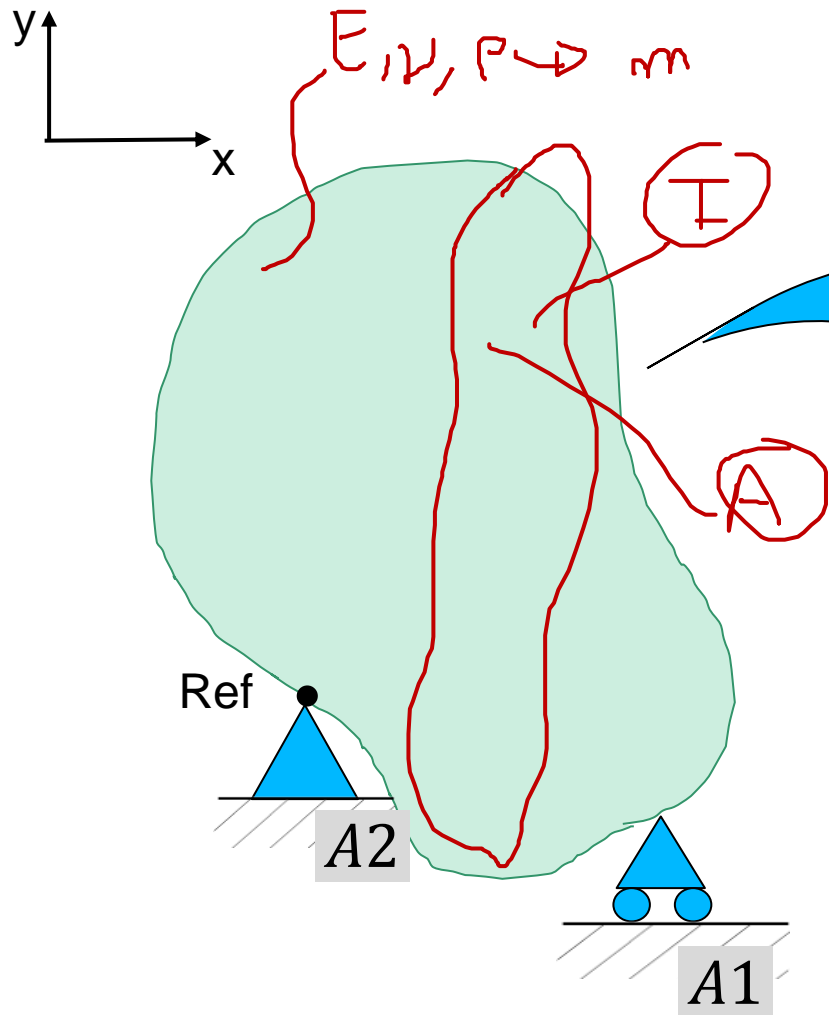
Eq 3 eq

$$\begin{aligned}
 & \vec{F}_{ax} * d_{ay} - \vec{F}_{ay} * d_{ax} - \vec{F}_{bx} \\
 & * d_{ay} - \vec{F}_{by} * d_{ax} - \vec{F}_{cx} * d_{cy} \\
 & + \vec{F}_{cy} * d_{cx} + \vec{F}_{A1y} * d_{A1x} = 0
 \end{aligned}$$

$\left\{ \begin{array}{l} Eq1 \\ Eq2 \\ Eq3 \end{array} \right\} \rightarrow R_{A1}, R_{A2} \Rightarrow$

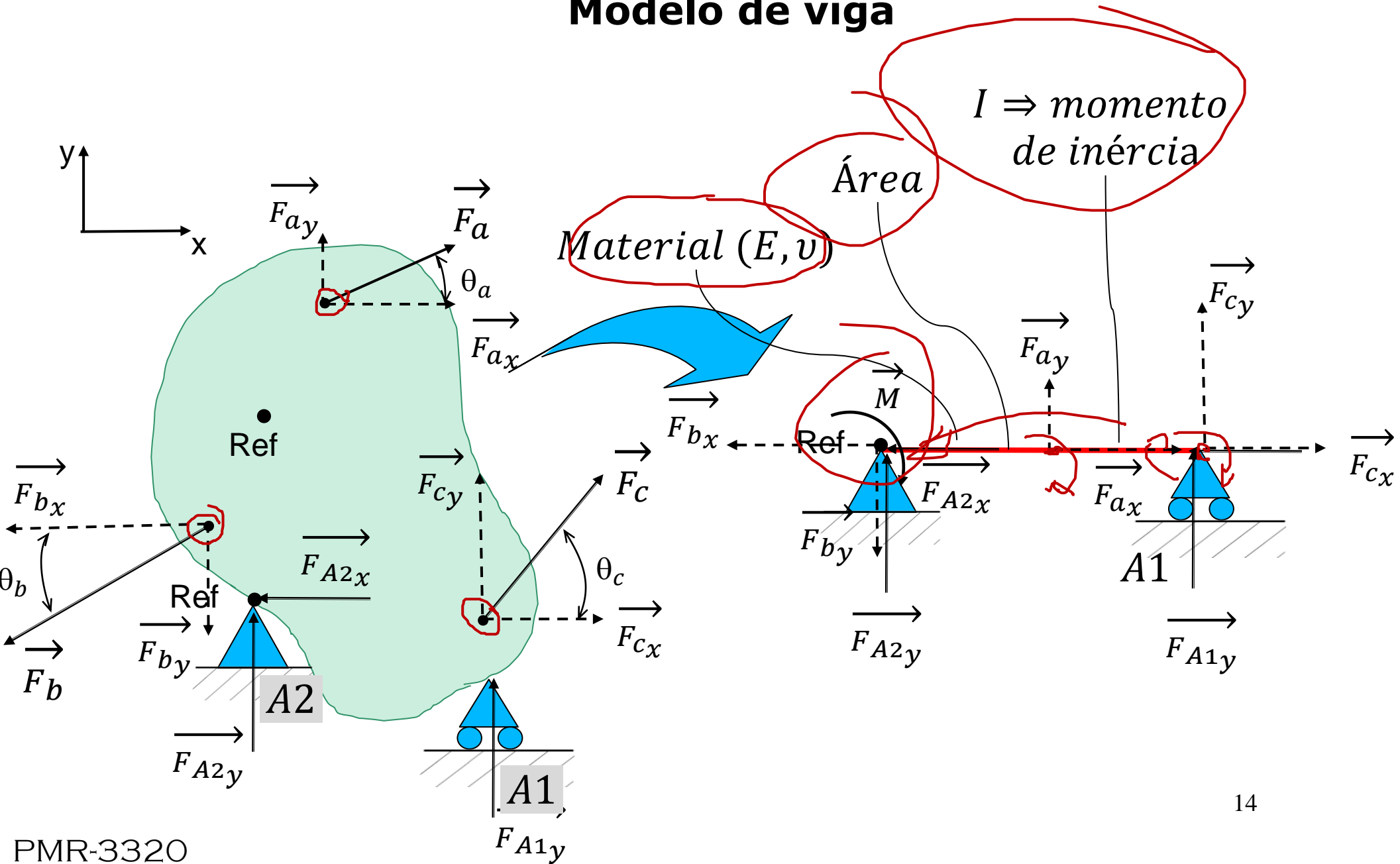


## Modelo de viga





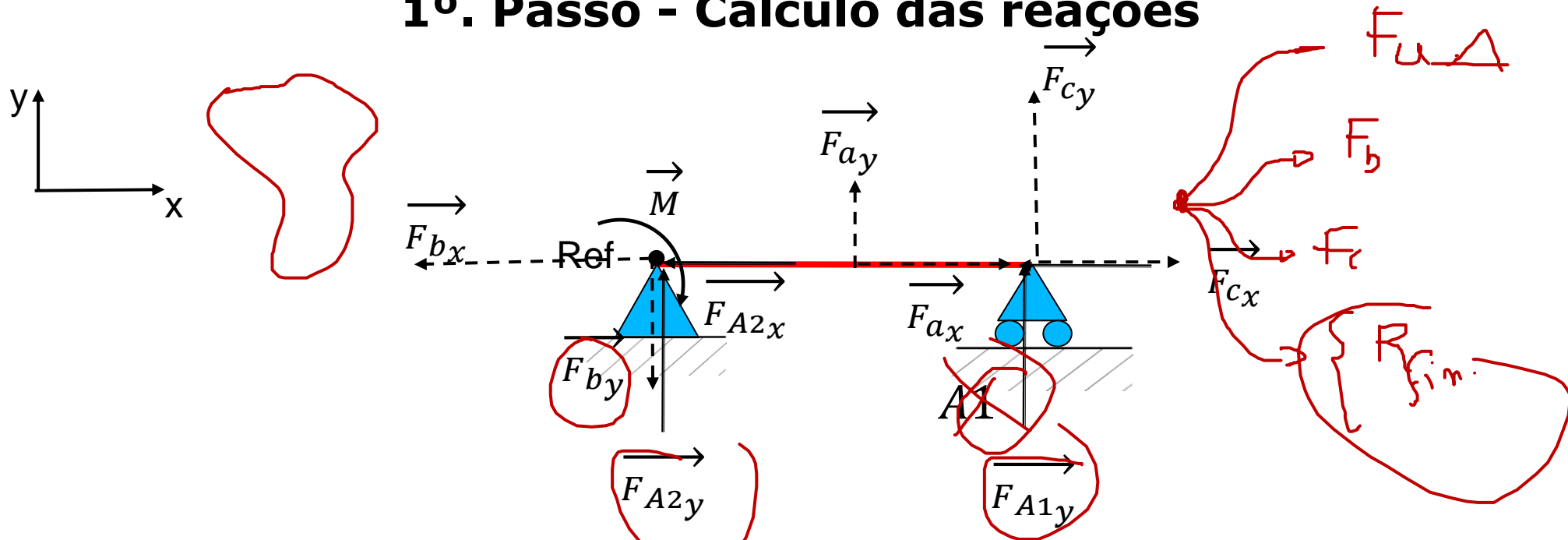
## Modelo de viga





## Método das seções

### 1º. Passo - Cálculo das reações



$$\sum \vec{F}_{ref_x} = 0 \quad \vec{F}_a \cdot \cos \theta_a + \vec{F}_c \cdot \cos \theta_c - \vec{F}_b \cdot \cos \theta_b + \vec{F}_{A2_x} = 0$$

$$\sum \vec{F}_{ref_y} = 0 \quad \vec{F}_a \cdot \sin \theta_a + \vec{F}_c \cdot \sin \theta_c - \vec{F}_b \cdot \sin \theta_b + \vec{F}_{A2_y} + \vec{F}_{A1_y} = 0$$

$$\sum_{Ref} \vec{M}_n = 0 \quad \vec{F}_{a_x} * d_{a_y} - \vec{F}_{a_y} * d_{a_x} - \vec{F}_{b_x} * d_{a_y} - \vec{F}_{b_y} * d_{a_x} - \vec{F}_{c_x} * d_{c_y} + \vec{F}_{c_y} * d_{c_x} + \vec{F}_{A1_y} * d_{A1_x} = 0$$

$$\vec{F}_a \cdot \cos \theta_a * d_{a_y} - \vec{F}_a \cdot \sin \theta_a * d_{a_x} - \dots + \vec{F}_{A1_y} * d_{A1_x} = 0$$



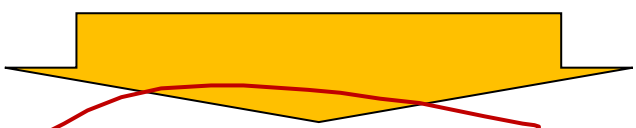
## Modelo de viga

$$\sum \vec{F}_{ref_x} = 0 \quad \vec{F}_a \cdot \cos \theta_a + \vec{F}_c \cdot \cos \theta_c - \vec{F}_b \cdot \cos \theta_b + \vec{F}_{A2_x} = 0$$

$$\sum \vec{F}_{ref_y} = 0 \quad \vec{F}_a \cdot \sin \theta_a + \vec{F}_c \cdot \sin \theta_c - \vec{F}_b \cdot \sin \theta_b + \vec{F}_{A2_y} + \vec{F}_{A1_y} = 0$$

$$\sum_{Ref} \vec{M}_n = 0 \quad \vec{F}_{a_x} * d_{a_y} - \vec{F}_{a_y} * d_{a_x} - \vec{F}_{b_x} * d_{a_y} - \vec{F}_{b_y} * d_{a_x} - \vec{F}_{c_x} * d_{c_y} + \vec{F}_{c_y} * d_{c_x} + \vec{F}_{A1_y} * d_{A1_x} = 0$$

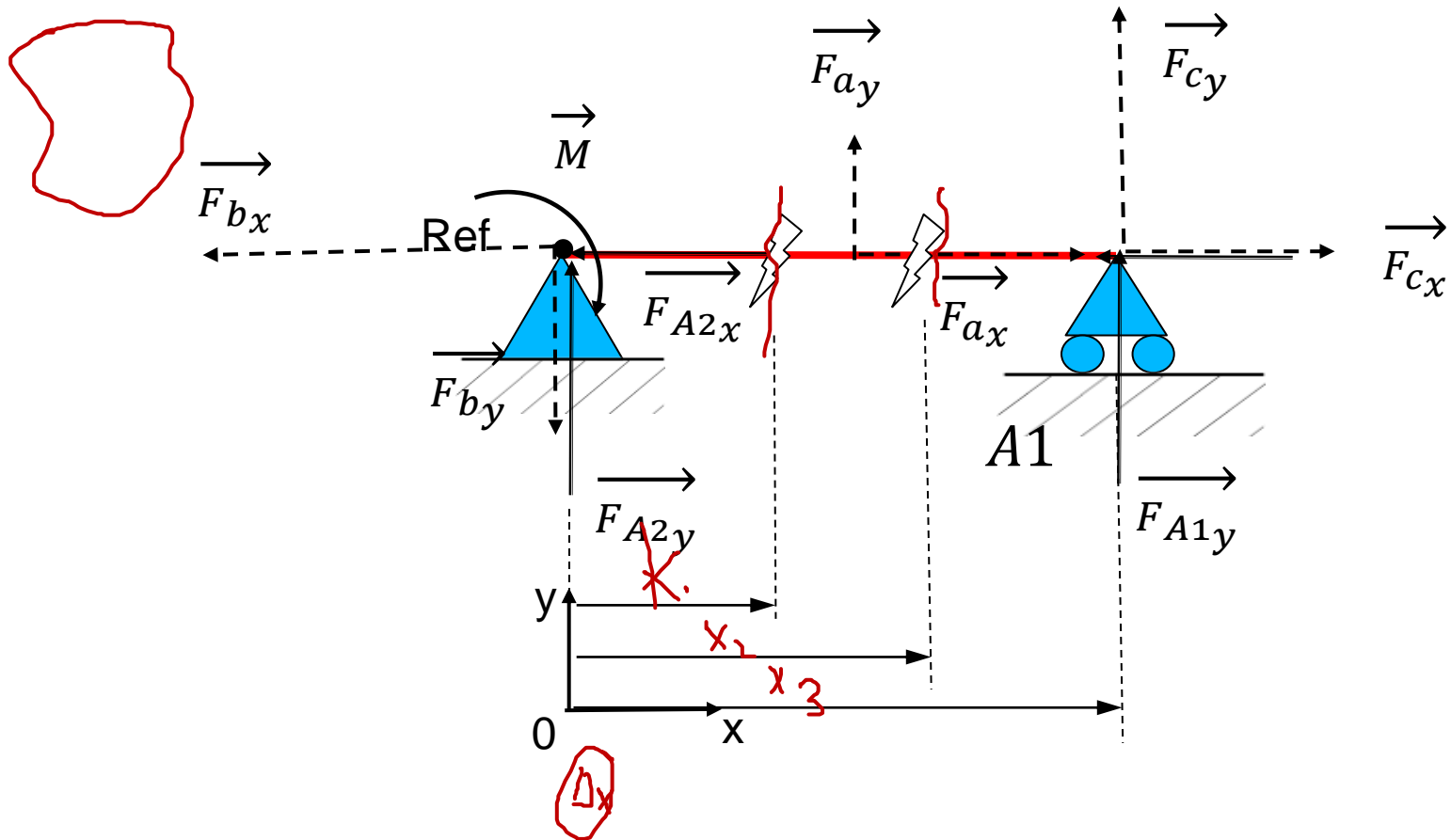
$$\vec{F}_a \cdot \cos \theta_a * d_{a_y} - \vec{F}_a \cdot \sin \theta_a * d_{a_x} - \dots + \vec{F}_{A1_y} * d_{A1_x} = 0$$


$$F_{A1_y}, F_{A2_y}, F_{A2_x}$$



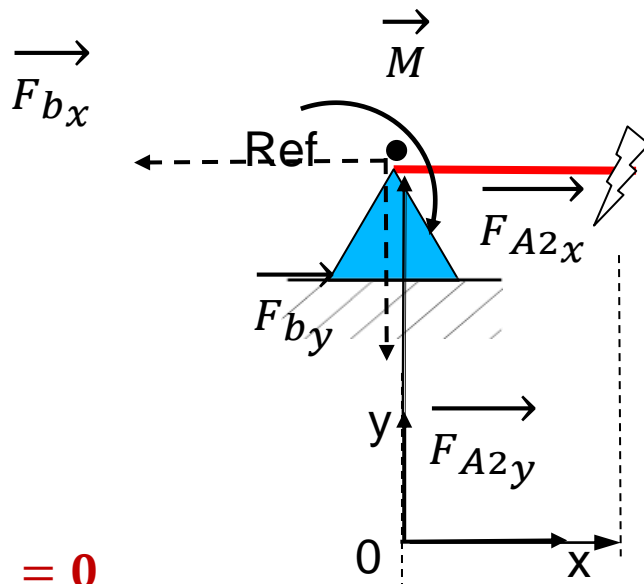


## Método das seções

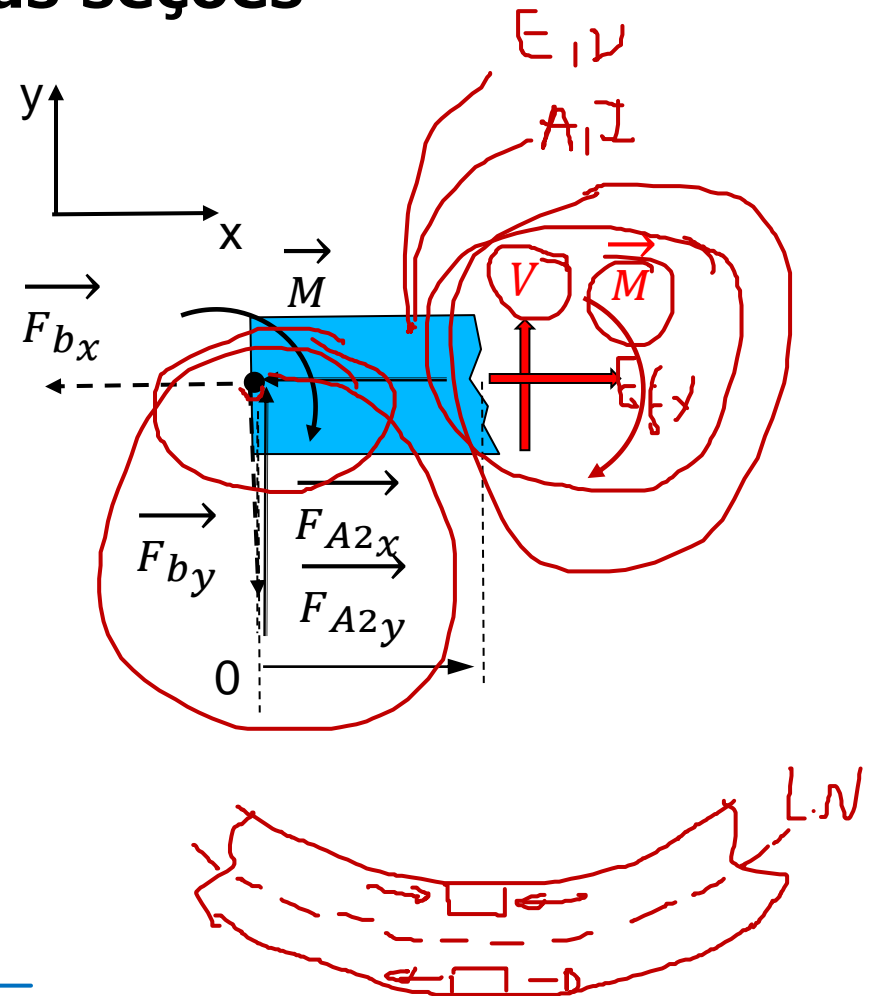
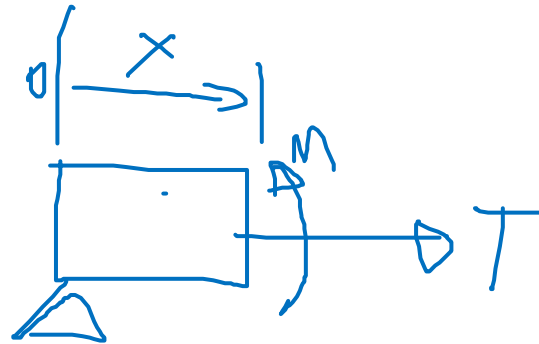




## Método das seções

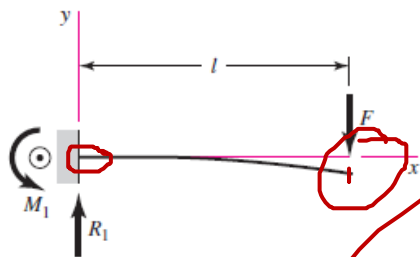


$$\left\{ \begin{aligned} \sum \vec{F}_{ref\ x} &= 0 \\ \sum \vec{F}_{ref\ y} &= 0 \\ \sum_{Ref} \vec{M}_n &= 0 \end{aligned} \right.$$





1 Cantilever—end load

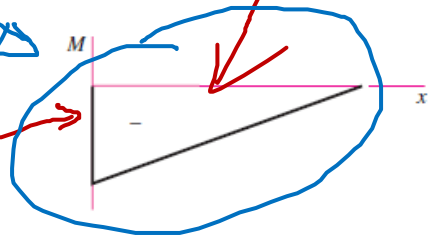
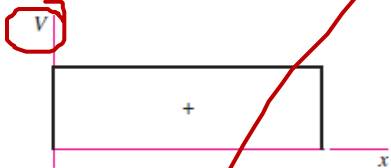


$R_1 = V = F$        $M_1 = Fl$

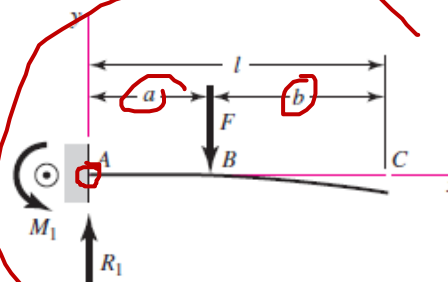
$M = F(x - l)$

$y = \frac{Fx^2}{6EI}(x - 3l)$

$y_{\max} = -\frac{Fl^3}{3EI}$



2 Cantilever—intermediate load



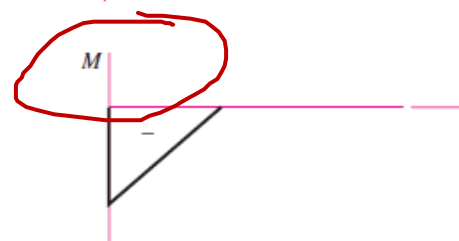
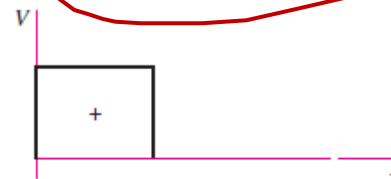
$R_1 = V = F$        $M_1 = Fa$

$M_{AB} = F(x - a)$        $M_{BC} = 0$

$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$

$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$

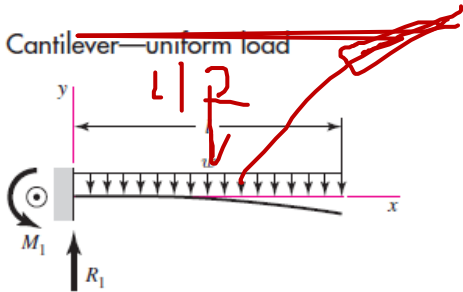
$y_{\max} = \frac{Fa^2}{6EI}(a - 3l)$





$$\downarrow F = vA$$

3 Cantilever—uniform load



$$R_1 = wl \quad M_1 = \frac{wl^2}{2}$$

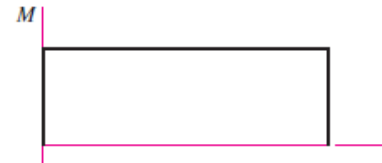
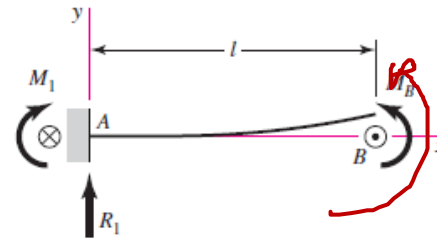
$$V = w(l - x) \quad M = -\frac{w}{2}(l - x)^2$$

$$y = \frac{wx^2}{24EI}(4lx - x^2 - 6l^2)$$

$$y_{\max} = -\frac{wl^4}{8EI}$$



4 Cantilever—moment load



$$R_1 = V = 0$$

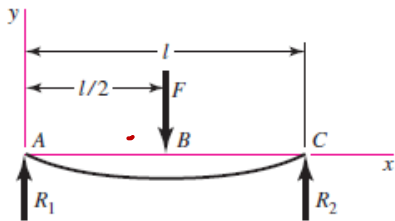
$$M_1 = M = M_B$$

$$y = \frac{M_B x^2}{2EI}$$

$$y_{\max} = \frac{M_B l^2}{2EI}$$



5 Simple supports—center load



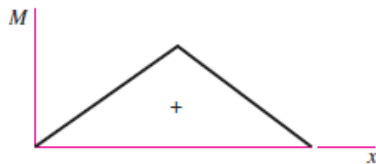
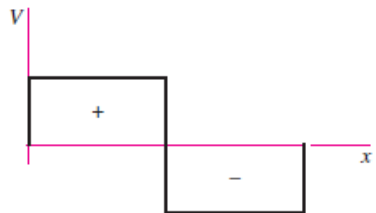
$$R_1 = R_2 = \frac{F}{2}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

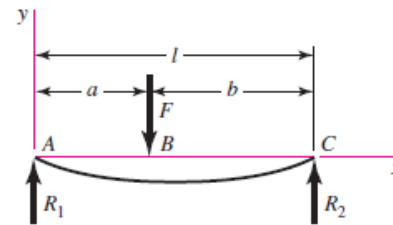
$$M_{AB} = \frac{Fx}{2} \quad M_{BC} = \frac{F}{2}(l-x)$$

$$y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)$$

$$y_{\max} = -\frac{Fl^3}{48EI}$$



6 Simple supports—intermediate load



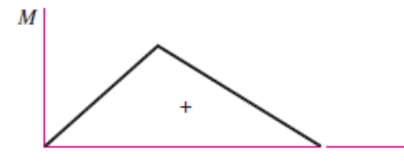
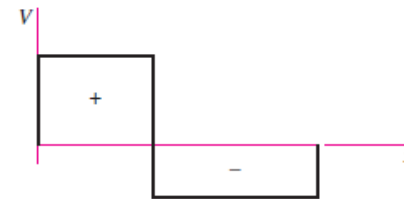
$$R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l-x)$$

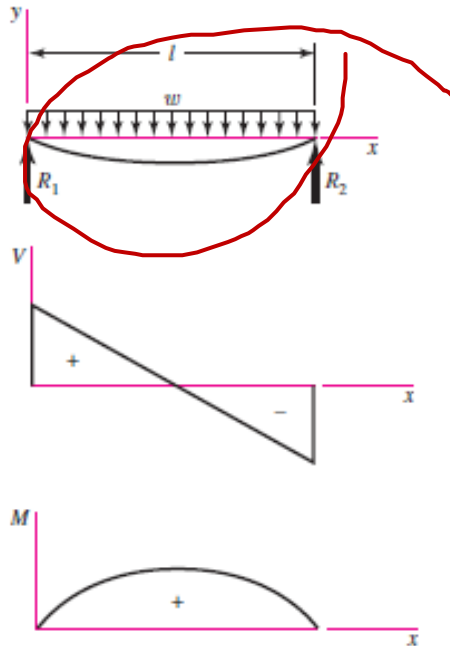
$$y_{AB} = \frac{Fbx}{6EI}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l-x)}{6EI}(x^2 + a^2 - 2lx)$$





7 Simple supports—uniform load



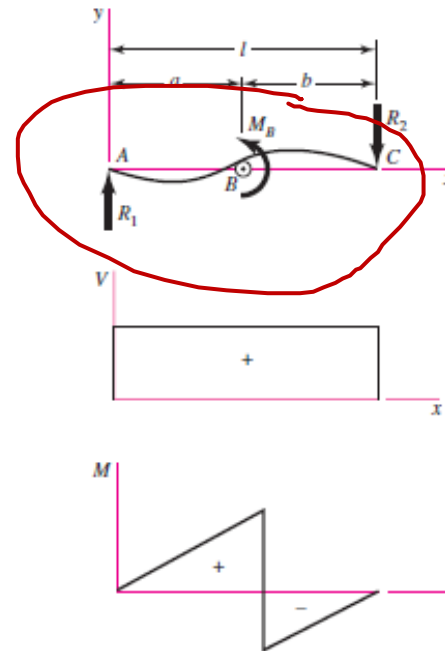
$$R_1 = R_2 = \frac{wl}{2} \quad V = \frac{wl}{2} - wx$$

$$M = \frac{wx}{2}(l - x)$$

$$y = \frac{wx}{24EI}(2lx^2 - x^3 - l^3)$$

$$y_{\max} = -\frac{5wl^4}{384EI}$$

8 Simple supports—moment load



$$R_1 = R_2 = \frac{M_B}{l} \quad V = \frac{M_B}{l}$$

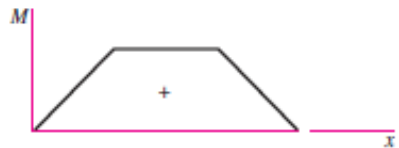
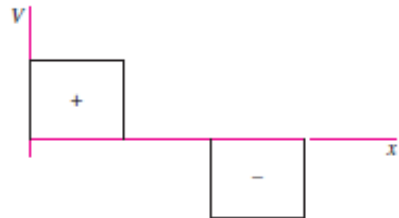
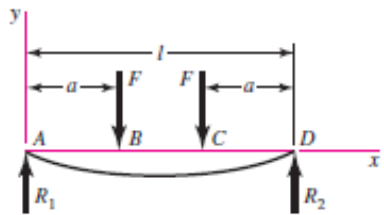
$$M_{AB} = \frac{M_B x}{l} \quad M_{BC} = \frac{M_B}{l}(x - l)$$

$$y_{AB} = \frac{M_B x}{6EI}(x^2 + 3a^2 - 6al + 2l^2)$$

$$y_{BC} = \frac{M_B}{6EI}[x^3 - 3lx^2 + x(2l^2 + 3a^2) - 3a^2l]$$



9 Simple supports—twin loads



$$R_1 = R_2 = F \quad V_{AB} = F \quad V_{BC} = 0$$

$$V_{CD} = -F$$

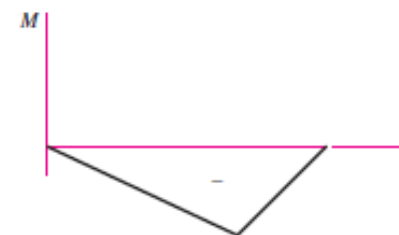
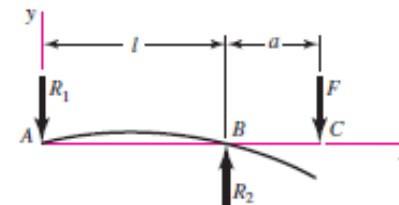
$$M_{AB} = Fx \quad M_{BC} = Fa \quad M_{CD} = F(l-x)$$

$$y_{AB} = \frac{Fx}{6EI}(x^2 + 3a^2 - 3la)$$

$$y_{BC} = \frac{Fa}{6EI}(3x^2 + a^2 - 3lx)$$

$$y_{\max} = \frac{Fa}{24EI}(4a^2 - 3l^2)$$

10 Simple supports—overhanging load



$$R_1 = \frac{Fa}{l} \quad R_2 = \frac{F}{l}(l+a)$$

$$V_{AB} = -\frac{Fa}{l} \quad V_{BC} = F$$

$$M_{AB} = -\frac{Fax}{l} \quad M_{BC} = F(x-l-a)$$

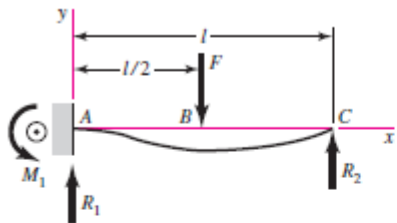
$$y_{AB} = \frac{Fax}{6EI}(l^2 - x^2)$$

$$y_{BC} = \frac{F(x-l)}{6EI}[(x-l)^2 - a(3x-l)]$$

$$y_c = -\frac{Fa^2}{3EI}(l+a)$$



11 One fixed and one simple support—center load



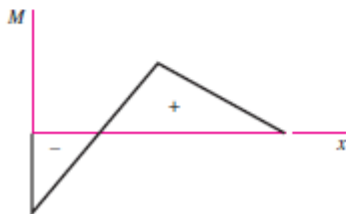
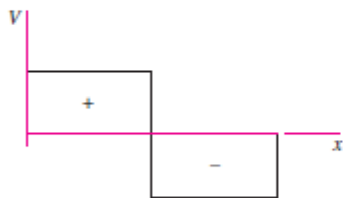
$$R_1 = \frac{11F}{16} \quad R_2 = \frac{5F}{16} \quad M_1 = \frac{3Fl}{16}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

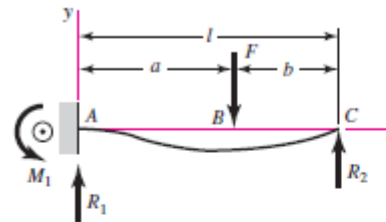
$$M_{AB} = \frac{F}{16}(11x - 3l) \quad M_{BC} = \frac{5F}{16}(l - x)$$

$$y_{AB} = \frac{Fx^2}{96EI}(11x - 9l)$$

$$y_{BC} = \frac{F(l-x)}{96EI}(5x^2 + 2l^2 - 10lx)$$



12 One fixed and one simple support—intermediate load



$$R_1 = \frac{Fb}{2l^3}(3l^2 - b^2) \quad R_2 = \frac{Fa^2}{2l^3}(3l - a)$$

$$M_1 = \frac{Fb}{2l^2}(l^2 - b^2)$$

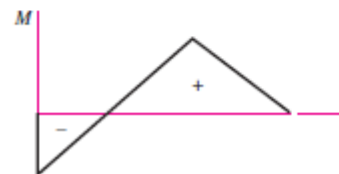
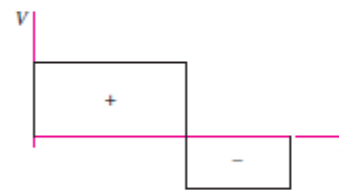
$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb}{2l^3}[b^2l - l^3 + x(3l^2 - b^2)]$$

$$M_{BC} = \frac{Fa^2}{2l^3}(3l^2 - 3lx - al + ax)$$

$$y_{AB} = \frac{Fbx^2}{12EI^3}[3l(b^2 - l^2) + x(3l^2 - b^2)]$$

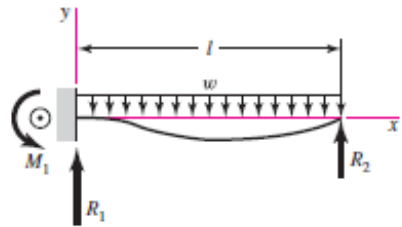
$$y_{BC} = y_{AB} - \frac{F(x-a)^3}{6EI}$$







13 One fixed and one simple support—uniform load

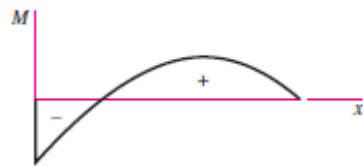
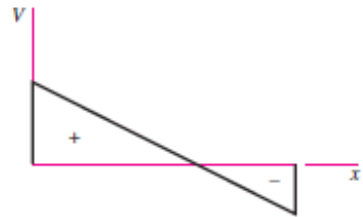


$$R_1 = \frac{5wl}{8} \quad R_2 = \frac{3wl}{8} \quad M_1 = \frac{wl^2}{8}$$

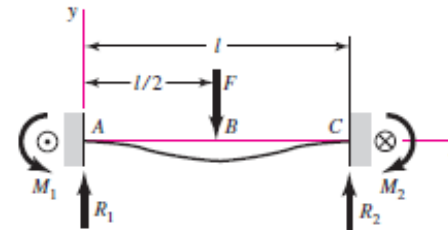
$$V = \frac{5wl}{8} - wx$$

$$M = -\frac{w}{8}(4x^2 - 5lx + l^2)$$

$$y = \frac{wx^2}{48EI}(l-x)(2x-3l)$$



14 Fixed supports—center load



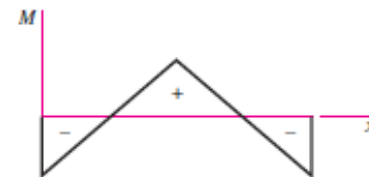
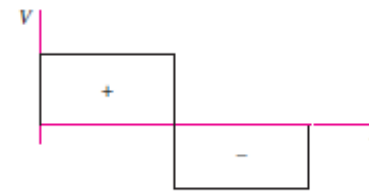
$$R_1 = R_2 = \frac{F}{2} \quad M_1 = M_2 = \frac{Fl}{8}$$

$$V_{AB} = -V_{BC} = \frac{F}{2}$$

$$M_{AB} = \frac{F}{8}(4x-l) \quad M_{BC} = \frac{F}{8}(3l-4x)$$

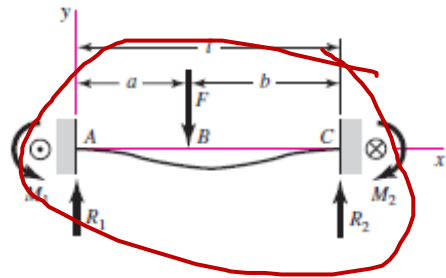
$$y_{AB} = \frac{Fx^2}{48EI}(4x-3l)$$

$$y_{\max} = -\frac{Fl^3}{192EI}$$





15 Fixed supports—intermediate load



$$R_1 = \frac{Fb^2}{l^3}(3a + b) \quad R_2 = \frac{Fa^2}{l^3}(3b + a)$$

$$M_1 = \frac{Fab^2}{l^2} \quad M_2 = \frac{Fa^2b}{l^2}$$

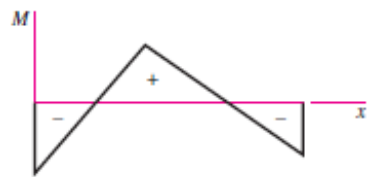
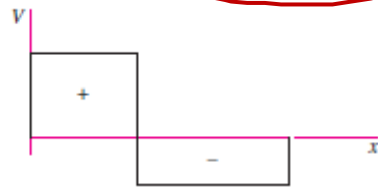
$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fb^2}{l^3}[x(3a + b) - al]$$

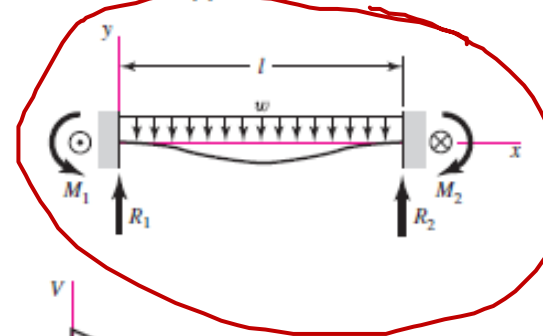
$$M_{BC} = M_{AB} - F(x - a)$$

$$y_{AB} = \frac{Fb^2x^2}{6EI l^3}[x(3a + b) - 3al]$$

$$y_{BC} = \frac{Fa^2(l - x)^2}{6EI l^3}[(l - x)(3b + a) - 3bl]$$



16 Fixed supports—uniform load



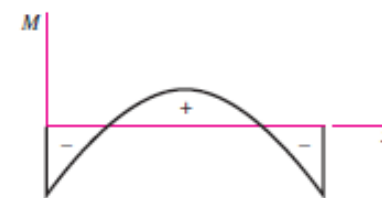
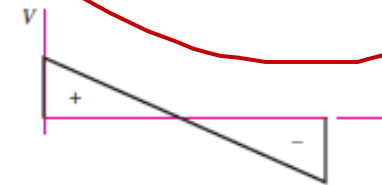
$$R_1 = R_2 = \frac{wl}{2} \quad M_1 = M_2 = \frac{wl^2}{12}$$

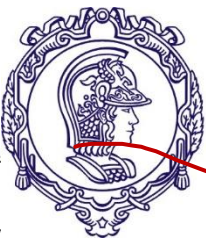
$$V = \frac{w}{2}(l - 2x)$$

$$M = \frac{w}{12}(6lx - 6x^2 - l^2)$$

$$y = -\frac{wx^2}{24EI}(l - x)^2$$

$$y_{\max} = -\frac{wl^4}{384EI}$$





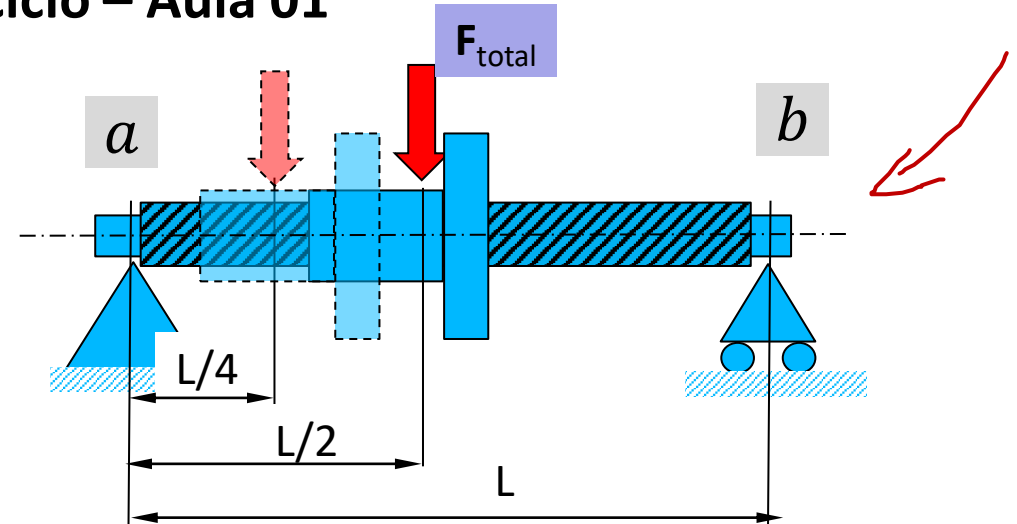
# ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

	$v = \frac{P}{6EI} (2L^3 - 3L^2x + x^3)$ $v_{\max} = v(0) = -\frac{PL^3}{3EI}$	$\theta(0) = -\frac{PL^2}{2EI}$		$M_{ab} = -M_{ba} = -\frac{q_0 L^2}{12}$ $R_a = R_b = -\frac{q_0 L}{2}$
	$v = \frac{q_0}{24EI} (x^4 - 4L^3x + 3L^4)$ $v_{\max} = v(0) = \frac{q_0 L^4}{8EI}$	$\theta(0) = -\frac{q_0 L^3}{6EI}$		$M_{cd} = -\frac{Pab^2}{L^2}$ $M_{dc} = \frac{Pba^2}{L^2}$ $R_c = -\frac{Pb^2}{L^3} (3a + b)$ $R_d = -\frac{Pa^2}{L^3} (a + 3b)$
	$v = \frac{q_0 x}{24EI} (L^3 - 2Lx^2 + x^3)$ $v_{\max} = v(L/2) = \frac{5q_0 L^4}{384EI}$	<p>See Example 10-3.</p> $\theta(0) = -\theta(L) = \frac{q_0 L^3}{24EI}$		$M_{ab} = -\frac{q_0 L^2}{30}$ $M_{ba} = \frac{q_0 L^2}{20}$ $R_a = -\frac{3q_0 L}{20}$ $R_b = -\frac{7q_0 L}{20}$
	<p>When <math>0 \leq x \leq a</math>, then</p> $v = \frac{Pbx}{6EIL} (L^2 - b^2 - x^2)$ <p>When <math>a = b = \frac{L}{2}</math>, then</p> $v = \frac{Px}{48EI} (3L^2 - 4x^2)$ $v_{\max} = v(L/2) = \frac{PL^3}{48EI}$	<p>See Example 10-6.</p> $\left(0 \leq x \leq \frac{L}{2}\right)$ $\theta(0) = -\theta(L) = \frac{PL^2}{16EI}$		$M_{ab} = \frac{2EI}{L} \theta_b$ $M_{ba} = \frac{4EI}{L} \theta_b$ $R_a = \frac{6EI}{L^2} \theta_b$ $R_b = -\frac{6EI}{L^2} \theta_b$
	$v = -\frac{M_0 x}{6EIL} (L^2 - x^2)$ $v_{\max} = v(L/\sqrt{3}) = -\frac{M_0 L^2}{9\sqrt{3}EI}$	<p>See Example 13-1.</p> $\theta(0) = -\frac{\theta(L)}{2} = -\frac{M_0 L}{6EI}$		$M_{ab} = -\frac{6EI}{L^2} \Delta$ $M_{ba} = -\frac{6EI}{L^2} \Delta$ $R_a = -\frac{12EI}{L^3} \Delta$ $R_b = \frac{12EI}{L^3} \Delta$
	$v_a = v(a) = \frac{Pa^2}{6EI} (3L - 4a)$ $v_{\max} = v(L/2) = \frac{Pa}{24EI} (3L^2 - 4a^2)$	$\theta(0) = \frac{Pa}{2EI} (L - a)$		$M_{bc} = M_0 \left(-1 + 4\frac{a}{L} - \frac{3a^2}{L^2}\right)$ $M_{cb} = \frac{M_0 a}{L} \left(2 - 3\frac{a}{L}\right)$ $R_b = \frac{6M_0 a}{L^2} \left(1 - \frac{a}{L}\right)$ $R_c = -\frac{6M_0 a}{L^2} \left(1 - \frac{a}{L}\right)$

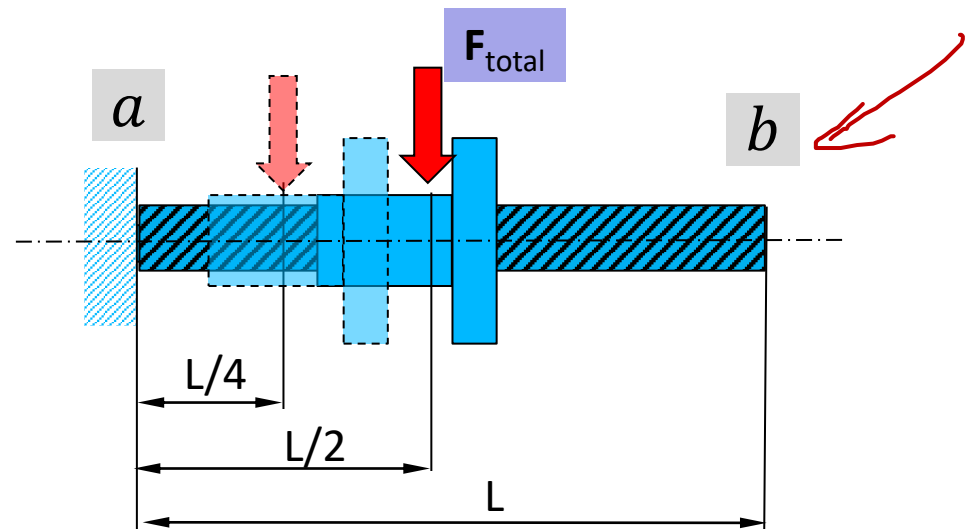
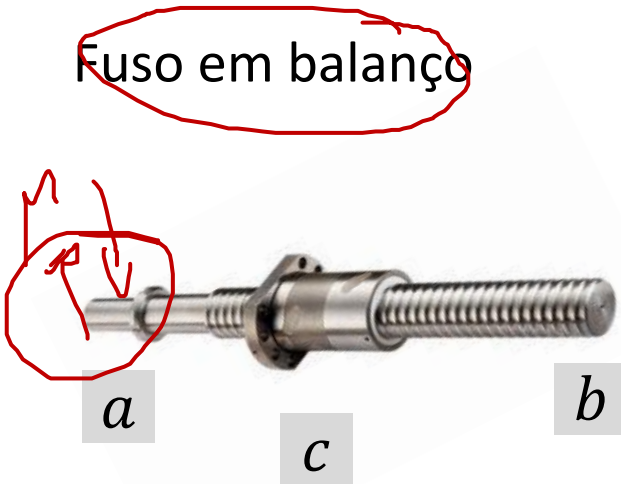


### Exercício – Aula 01

Fuso bi – apoiado,



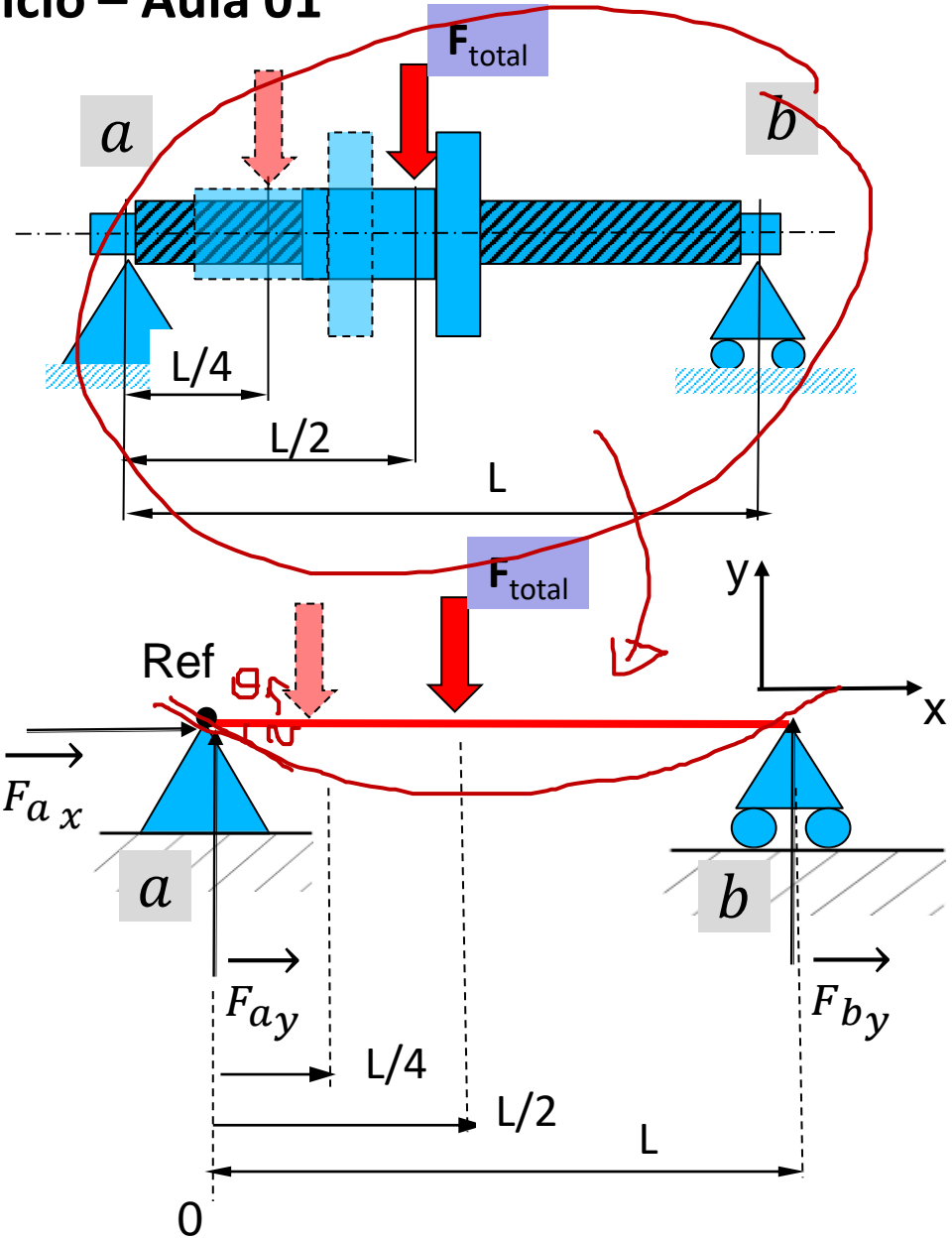
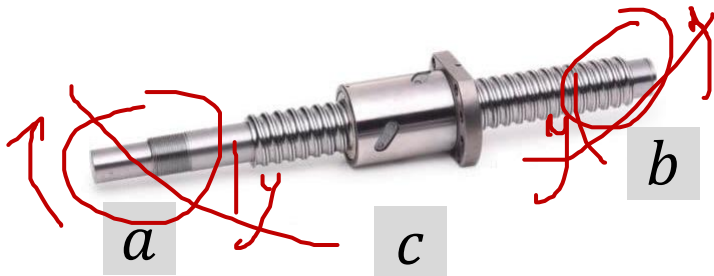
Fuso em balanço



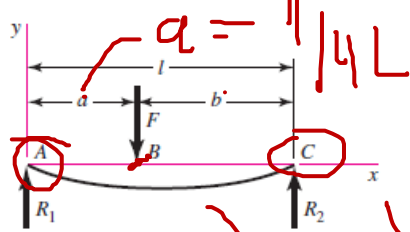


Exercício – Aula 01

Fuso bi – apoiado,



6 Simple supports—intermediate load



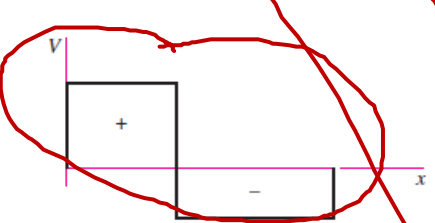
$$R_1 = \frac{Fb}{l} \quad R_2 = \frac{Fa}{l}$$

$$V_{AB} = R_1 \quad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fbx}{l} \quad M_{BC} = \frac{Fa}{l}(l-x)$$

$$y_{AB} = \frac{Fbx}{6EI}(x^2 + b^2 - l^2)$$

$$y_{BC} = \frac{Fa(l-x)}{6EI}(x^2 + a^2 - 2lx)$$

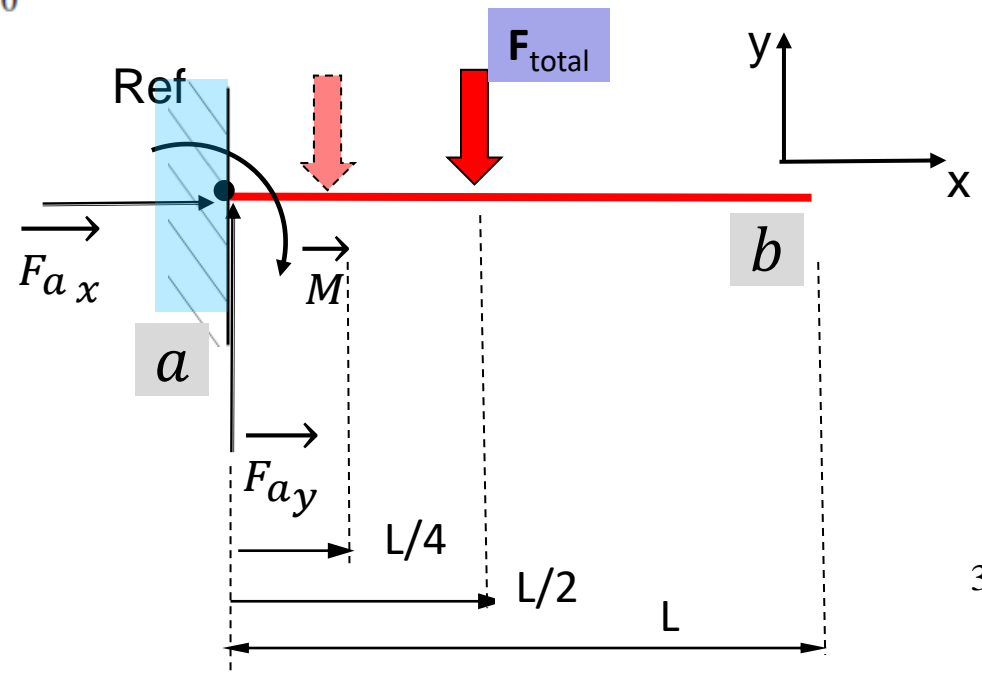
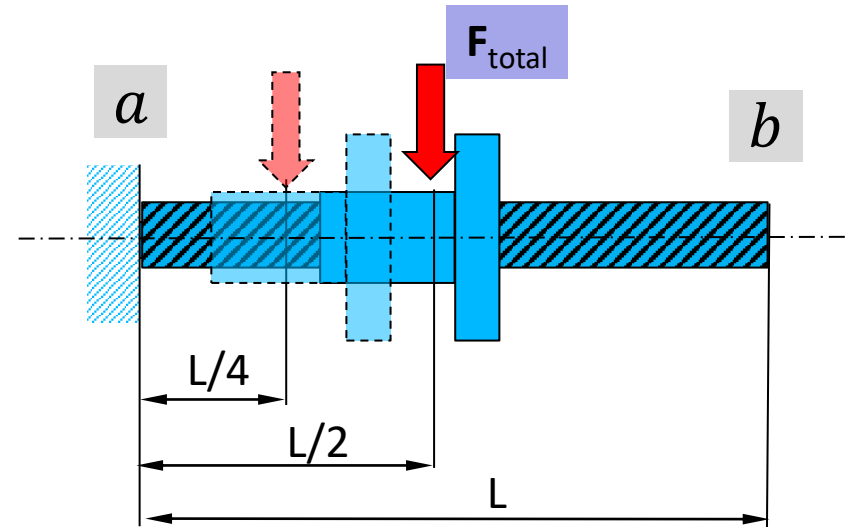
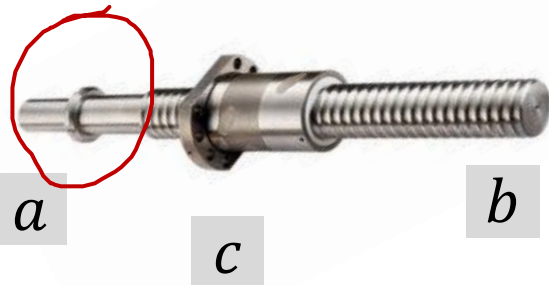


$a = b = \frac{1}{2}L$

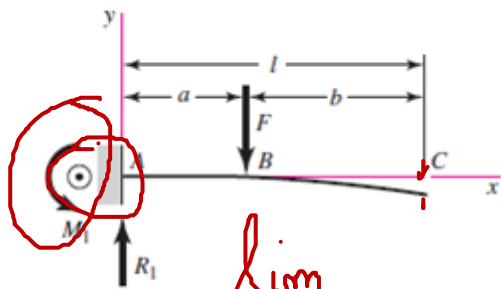


Exercício – Aula 01

Fuso em balanço



2 Cantilever—intermediate load



$$R_1 = V = F \quad M_1 = Fa$$

$$M_{AB} = F(x - a) \quad M_{BC} = 0$$

$$y_{AB} = \frac{Fx^2}{6EI}(x - 3a)$$

$$y_{BC} = \frac{Fa^2}{6EI}(a - 3x)$$

$$y_{max} = \frac{Fa^2}{6EI}(a - 3l)$$

lim  $a \rightarrow 0$

$a = 1/4 L$

$a = 1/2 L$





**FIM DA AULA**