1 Some Facts on Symmetric Matrices

Definition: Matrix A is symmetric if $A = A^T$.

Theorem: Any symmetric matrix

1) has only real eigenvalues;

2) is always diagonalizable;

3) has orthogonal eigenvectors.

Corollary: If matrix A then there exists $Q^T Q = I$ such that $A = Q^T \Lambda Q$.

Proof:

1) Let $\lambda \in \mathbb{C}$ be an eigenvalue of the symmetric matrix A. Then $Av = \lambda v$, $v \neq 0$, and

$$v^* A v = \lambda v^* v, \qquad v^* = \bar{v}^T.$$

But since A is symmetric

$$\lambda v^* v = v^* A v = (v^* A v)^* = \bar{\lambda} v^* v.$$

Therefore, λ must be equal to $\bar{\lambda}$!

2) If the symmetric matrix A is not diagonalizable then it must have generalized eigenvalues of order 2 or higher. That is, for some repeated eigenvalue λ_i there exists $v \neq 0$ such that

$$(A - \lambda_i I)^2 v = 0, \quad (A - \lambda_i I) v \neq 0$$

But note that

$$0 = v^*(A - \lambda_i I)^2 v = v^*(A - \lambda_i I)(A - \lambda_i I) \neq 0,$$

which is contradiction. Therefore, as there exists no generalized eigenvectors of order 2 or higher, A must be diagonalizable.

3) As A must have no generalized eigenvector of order 2 or higher

$$AT = A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \Lambda = T\Lambda, \quad |T| \neq 0.$$

That is $A = T^{-1} \Lambda T$. But since A is symmetric

$$T^{-1} \Lambda T = A = A^T = (T^{-1} \Lambda T)^T = T^T \Lambda T^{-T} \qquad \Rightarrow \quad T^T = T^{-1}$$

or

$$T^T T = I \qquad \Rightarrow \quad v_i^T v_i = 1, \quad v_i^T v_j = 0, \forall i \neq j.$$

1.1 Positive definite matrices

Definition: The symmetric matrix A is said positive definite $(A > 0)$ if all its eigenvalues are positive.
Definition: The symmetric matrix A is said positive semidefinite $(A \geq 0)$ if all its eigenvalues are non negative.

Theorem: If A is positive definite (semidefinite) there exists a matrix $A^{1/2} > 0$ $(A^{1/2} \geq 0)$ such that $A^{1/2}A^{1/2} = A$.

Proof: As A is positive definite (semidefinite)

$$
\begin{aligned}
A &= Q^T \Lambda Q, \qquad Q^T Q = Q Q^T = I \\
&= Q^T \Lambda^{1/2} \Lambda^{1/2} Q, \qquad \Lambda_{ii}^{1/2} = \sqrt{\lambda_i} \\
&= \underbrace{Q^T \Lambda^{1/2} Q}_{A^{1/2}} \underbrace{Q^T \Lambda^{1/2} Q}_{A^{1/2}},
\end{aligned}
$$

Theorem: A is positive definite if and only if $x^T A x > 0, \quad \forall x \neq 0$.

Proof:
Assume there is $x \neq 0$ such that $x^T A x \leq 0$ and A is positive definite. Then there exists $Q^T Q = I$ such that $A = Q^T \Lambda Q$ with $\Lambda_{ii} = \lambda_i > 0$. Then for $y \neq 0$ such that $x = Q^T y$

$$
0 \geq x^T A x = y^T Q A Q y = y^T Q Q^T \Lambda Q Q^T y = y^T \Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2 > 0
$$

which is a contradiction.

2 Controllability Gramian

LTI system in state space

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

Problem: Given $x(0) = 0$ and *any* $\bar{x}$, compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$.

Solution: We know that

$$\bar{x} = x(\bar{t}) = \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau.$$

If we limit our search for solutions u in the form

$$u(t) = B^T e^{A^T(\bar{t}-t)} \bar{z}$$

we have

$$\begin{aligned} \bar{x} &= \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} BB^T e^{A^T(\bar{t}-\tau)} \bar{z} d\tau, \\ &= \left(\int_0^{\bar{t}} e^{A(\bar{t}-\tau)} BB^T e^{A^T(\bar{t}-\tau)} d\tau \right) \bar{z}, \qquad \xi = \bar{t} - \tau \\ &= \left(\int_0^{\bar{t}} e^{A\xi} BB^T e^{A^T\xi} d\xi \right) \bar{z}, \end{aligned}$$

and

$$\bar{z} = \left(\int_0^{\bar{t}} e^{A\xi} BB^T e^{A^T\xi} d\xi \right)^{-1} \bar{x},$$

$$\Rightarrow \qquad u(t) = B^T e^{A^T(\bar{t}-t)} \left(\int_0^{\bar{t}} e^{A\xi} BB^T e^{A^T\xi} d\xi \right)^{-1} \bar{x}$$

The symmetric matrix

$$X(t) := \int_0^t e^{A\xi} BB^T e^{A^T\xi} d\xi$$

is known as the *Controllability Gramian*.

2.1 Properties of the Controllability Gramian

Theorem: The Controllability Gramian

$$X(t) = \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi,$$

is the solution to the differential equation

$$\frac{d}{dt} X(t) = A X(t) + X(t) A^T + B B^T.$$

If $X = \lim_{t \to \infty} X(t)$ exists then

$$A X + X A^T + B B^T = 0.$$

Proof: For the first part, compute

$$\begin{aligned}
\frac{d}{dt} X(t) &= \frac{d}{dt} \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi = \frac{d}{dt} \int_0^t e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} d\tau, \\
&= \int_0^t \frac{d}{dt} e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} + e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} \Big|_{\tau = t}, \\
&= A \left(\int_0^t e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} d\tau \right) \\
&\qquad + \left(\int_0^t e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} d\tau \right) A^T + B B^T, \\
&= A X(t) + X(t) A^T + B B^T.
\end{aligned}$$

For the second part, use the fact that $X(t)$ is smooth and therefore

$$\lim_{t \to \infty} X(t) = X \quad \Rightarrow \quad \lim_{t \to \infty} \frac{d}{dt} X(t) = 0.$$

2.2 Summary on Controllability

Theorem: The following are equivalent

1) The pair (A, B) is controllable;

2) The Controllability Matrix $\mathcal{C}(A, B)$ has full-row rank;

3) There exists no $z \neq 0$ such that $z^* A = \lambda z, \quad z^* B = 0$;

4) The Controllability Gramian $X(t)$ is positive definite for some $t \geq 0$.

Proof:
Everything has already been proved except the equivalence of 4).
Sufficiency: Immediate from the construction of $u(t)$.
Necessity: First part:

$$X(t) = \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi \geq 0$$

by construction. We have to prove that when (A, B) is controllable then $X(t) > 0$. To prove this assume that (A, B) is controllable but $X(t)$ is not positive definite. So there exists $z \neq 0$ such that

$$z^* e^{A\tau} B \equiv 0, \quad \forall\, 0 \leq \tau \leq t.$$

But this implies

$$\frac{d^i}{d\tau^i}(i!\, z^* e^{A\tau} B)\bigg|_{\tau=0} = z^* A^i e^{A\tau} B\big|_{\tau=0} = z^* A^i B = 0, \quad i = 0, \ldots, n-1$$

which implies $\mathcal{C}(A, B)$ does not have full-row rank (see proof of the Popov-Belevitch-Hautus Test).

3 Observability Gramian

LTI system in state space

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

Problem: Given $u(t) = 0$ and $y(t)$ compute $x(0)$.

Solution: We know that

$$y(t) = Ce^{At}x(0).$$

Multiplying on the left by $e^{A^T t}C^T$ and integrating from 0 to t we have

$$\int_0^t e^{A^T\xi}C^T y(\xi)d\xi = \left(\int_0^t e^{A^T\xi}C^T C e^{A\xi}d\xi\right)x(0)$$

from which

$$x(0) = \left(\int_0^t e^{A^T\xi}C^T C e^{A\xi}d\xi\right)^{-1}\int_0^t e^{A^T\xi}C^T y(\xi)d\xi.$$

The symmetric matrix

$$Y(t) := \int_0^t e^{A^T\xi}C^T C e^{A\xi}d\xi$$

is known as the *Observability Gramian*.

3.1 Properties of the Observability Gramian

Theorem: The Observability Gramian

$$Y(t) = \int_0^t e^{A^T\xi} C^T C e^{A\xi} d\xi,$$

is the solution to the differential equation

$$\frac{d}{dt} Y(t) = A^T Y(t) + Y(t) A + C^T C.$$

If $Y = \lim_{t\to\infty} X(t)$ exists then

$$A^T Y + Y A + C^T C = 0.$$

3.2 Summary on Observability

Theorem: The following are equivalent

1) The pair (A, C) is observable;

2) The Observability Matrix $\mathcal{O}(A, C)$ has full-column rank;

3) There exists no $x \neq 0$ such that $Ax = \lambda x, \quad Cx = 0$;

4) The Observability Gramian $Y = Y(t)$ is positive definite for some $t \geq 0$.

Lemma: Consider the Lyapunov Equation

$$A^T X + XA + C^T C = 0$$

where $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$.

1. A solution $X \in \mathbb{C}^{n\times n}$ exists and is unique if and only if $\lambda_j(A) + \lambda_i^*(A) \neq 0$ for all $i, j = 1, \ldots, n$. Furthermore X is symmetric.
2. If A is Hurwitz then X is positive semidefinite.
3. If (A, C) is detectable and X is positive semidefinite then A is Hurwitz.
4. If (A, C) is observable and A is Hurwitz then X is positive definite.

Proof:
Item 1. The Lyapunov Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Lyapunov equation admits only the trivial solution. Assume it does not, that is, there $\bar{X} \neq 0$ such that

$$A^T \bar{X} + \bar{X} A = 0$$

Then, multiplication of the above on the right by $x_i^* \neq 0$, the ith eigenvector of A and on the right by $x_j^* \neq 0$ yields

$$0 = x_i^* A^T \bar{X} x_j + x_i^* \bar{X} A x_j = \left[\lambda_j(A) + \lambda_i^*(A)\right] x_i^* \bar{X} x_j.$$

Since $\lambda_i(A) + \lambda_j(B) \neq 0$ by hypothesis we must have $x_i^* \bar{X} x_j = 0$ for all i, j. One can show that this indeed implies $\bar{X} = 0$, establishing a contradiction.
That X is symmetric follows from uniqueness since

$$\begin{aligned} 0 &= (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) \\ &= A^T (X^T - X) + (X^T - X) A \end{aligned}$$

so that $X^T - X = 0$.

Item 2. If A is Hurwitz then $\lim_{t\to\infty} e^{At} = 0$. But

$$X = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt \succeq 0$$

and

$$A^T X + XA = \lim_{t\to\infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At}\, dt = e^{A^T t} C^T C e^{At}\Big|_0^\infty = -C^T C.$$

4 Controllability, Observability and Duality

Primal LTI system in state space

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned}$$

Dual LTI system in state space

$$\begin{aligned}\dot{x}(t) &= A^T x(t) + C^T u(t),\\ y(t) &= B^T x(t).\end{aligned}$$

The primal system is observable if and only if the dual system in controllable.
The primal system is controllable if and only if the dual system in observable.