Integral por Partes

Sejam $f(x)$ e $g(x)$ funções deriváveis. Derivando o produto das funções tem-se

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x)f'(x)$$

Então tem-se:

$$\underbrace{f(x)}_{u}.\underbrace{g(x)}_{v} = \int \underbrace{f(x)}_{u}.\underbrace{g'(x)dx}_{dv} + \int \underbrace{g(x)}.\underbrace{f'(x)dx}_{d.}$$

$u = f(x)$ $\quad$ $v = g(x)$

$du = f'(x)dx$ $\quad$ $dv = g'(x)dx$

Onde,

$$\boxed{\int u\,dv = u.v - \int v\,du}$$

Essa fórmula expressa a integral $\int u\,dv$ em termos de uma outra integral $\int v\,du$. Considere os seguintes exemplos:

a) $\int \sec^3 3x\,dx \Rightarrow \int \sec 3x.\sec^2 3x\,dx$

Do formulário tem-se: $\int \sec^2 u\,du = tg\,u + C$

$u = \sec 3x$

$\frac{du}{dx} = \sec 3x.tg3x.3$

$du = \sec 3x.tg3x.3dx$

$\int dv = \frac{1}{3}\int \sec^2 \underbrace{3x}_{u} 3dx$

$v = \frac{1}{3} tg\,3x$

$u = 3x$

$\frac{du}{dx} = 3 \Rightarrow du = 3dx$

$$\boxed{\int u\,dv = u\,v - \int v\,du}$$

$$\int \sec^3 3x\,dx = \frac{\sec 3x.tg3x}{3} - \int \frac{1}{\cancel{3}} tg3x.\sec 3x.tg3x.\cancel{3}dx$$

$$\int \sec^3 3x\,dx = \frac{\sec 3x\, tg 3x}{3} - \int tg^2 3x \cdot \sec 3x\,dx$$

$tg^2 3x = \sec^2 3x - 1$

$$\int \sec^3 3x\,dx = \frac{\sec 3x \cdot tg 3x}{3} - \int (\sec^2 3x - 1) \cdot \sec 3x\,dx$$

$$\int \sec^3 3x\,dx^{*} = \frac{\sec 3x\, tg 3x}{3} - \int \sec^3 3x\,dx^{*} + \frac{1}{3}\int \sec \underbrace{3x}_{u} 3\,dx$$

$u = 3x$
$du = 3dx$

Do formulário
$\int \sec u\,du = \ln(\sec u + tg u) + C$

$$2\int \sec^3 3x\,dx = \frac{\sec 3x \cdot tg 3x}{3} + \frac{1}{3}\ln|\sec 3x + tg 3x| \quad (\div 2)$$

$$\int \sec^3 3x\,dx = \frac{\sec 3x\, tg 3x}{6} + \frac{\ln|\sec 3x + tg 3x|}{6} + C$$

b) $\int \underbrace{\ln(x^2+3)}_{u} \underbrace{dx}_{dv}$

$u = \ln(x^2+3)$
$\frac{du}{dx} = \frac{1}{x^2+3} \cdot 2x$
$du = \frac{2x}{x^2+3}dx$

$\int dv = \int dx$
$v = x$

$$\boxed{\int v\,du = u \cdot v - \int v\,du}$$

$$\int \ln(x^2+3)\,dx = x\ln(x^2+3) - \int \frac{2x^2}{x^2+3}dx$$

$$\begin{array}{r|l} 2x^2 + 0x & x^2+3 \\ \cancel{-2x^2} - 6 & 2 \\ \hline -6 & \end{array}$$

$$\int \ln(x^2+3)\,dx = x\ln(x^2+3) - \int \left(2 - \frac{6}{x^2+3}\right)dx$$

$$= x\ln(x^2+3) - 2\int dx + \underbrace{6\int \frac{dx}{x^2+3}}_{I_1}$$

$\Delta < 0$

$$I_1 = 6\int \frac{dx}{x^2+3} = \frac{1}{\cancel{3}}\cancel{6}\int \frac{dx}{\frac{x^2}{3} + \frac{3}{3}} = 2\sqrt{3}\int \frac{\frac{1}{\sqrt{3}}dx}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1}$$

$u = \frac{x}{\sqrt{3}}$
$du = \frac{dx}{\sqrt{3}}$

$$= x\ln(x^2+3) - 2x + 2\sqrt{3}\,arctg\left(\frac{x}{\sqrt{3}}\right) + C$$

c) $\int \frac{x^3 dx}{\sqrt{1-x^2}} = \int \frac{\overbrace{x^2}^{u} \cdot \overbrace{x\,dx}^{dv}}{\sqrt{1-x^2}}$

$u = x^2$
$du = 2x\,dx$

$dv = \frac{x\,dx}{\sqrt{1-x^2}}$

$\int dv = \int (1-x^2)^{-1/2} \cdot x\,dx$ $\quad u = 1-x^2$, $du = -2x\,dx$

$v = -\frac{1}{2}\int (1-x^2)^{-1/2} -2x\,dx$

$v = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}} = -(1-x^2)^{1/2}$

$\boxed{\int u\,dv = u \cdot v - \int v\,du}$

$\int \frac{x^3 dx}{1-x^2} = -x^2(1-x^2)^{1/2} + (-)\int (\underbrace{1-x^2}_{u})^{1/2} -2x\,dx$ $\quad u = 1-x^2$, $du = -2x\,dx$

$= -x^2(1-x^2)^{1/2} - \frac{(1-x^2)^{3/2}}{\frac{3}{2}}$

$= -x^2(1-x^2)^{1/2} - \frac{2}{3}(1-x^2)^{3/2}$

$= (1-x^2)^{1/2}\left[-x^2 - \frac{2}{3}(1-x^2)\right] + C$

d) $\int e^{-\theta} \cos 2\theta\, d\theta$

$u = e^{-\theta}$
$du = -e^{-\theta} d\theta$

$d(e^u) = e^u \cdot du$

$\int \cos u\,du = \operatorname{sen} u + C$

$\int dv = \frac{1}{2}\int \cos \underbrace{2\theta}_{u} 2d\theta$ $\quad u = 2\theta$, $du = 2d\theta$

$v = \frac{1}{2}\operatorname{sen} 2\theta$

$\int u\,dv = u\,v - \int v\,du$

$\int e^{-\theta} \cos 2\theta\, d\theta = \frac{1}{2}e^{-\theta}\operatorname{sen} 2\theta + \frac{1}{2}\underbrace{\int e^{-\theta}\operatorname{sen} 2\theta\, d\theta}_{I_1}$ (1)

$\int \operatorname{sen} u\,du = -\cos u + C$

$I_1 = \int e^{-\theta}\operatorname{sen} 2\theta\, d\theta$ $\quad u = e^{-\theta}$, $du = -e^{-\theta}d\theta$ $\quad \int dv = \frac{1}{2}\int \operatorname{sen} 2\theta\, 2d\theta$, $v = -\frac{1}{2}\cos 2\theta$

$\int e^{-\theta}\operatorname{sen} 2\theta\, d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\, d\theta$

Retornando a expressão (1) tem-se.

$\int e^{-\theta}\cos 2\theta\, d\theta^{*} = \frac{1}{2}e^{-\theta}\operatorname{sen} 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}\int e^{-\theta}\cos 2\theta\, d\theta^{*}$

$\frac{5}{4}\int e^{-\theta}\cos 2\theta\, d\theta = \frac{1}{2}e^{-\theta}\operatorname{sen} 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta \quad \left(\times \frac{4}{5}\right)$

$\int e^{-\theta}\cos 2\theta\, d\theta = \frac{2}{5}e^{-\theta}\operatorname{sen} 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$