Propagação de incertezas

A grandeza física $w = w(x, y, z, \dots)$

incerteza padrão : $x \to \sigma_x$

$y \to \sigma_y$

$z \to \sigma_z$

Para erros completamente independentes entre si:

$$\boxed{\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial w}{\partial z}\right)^2 \sigma_z^2 + \dots}$$

No caso de uma única variável x, tem-se:

$$\sigma_w^2 = \left(\frac{dw}{dx}\right)^2 \sigma_x^2 \quad \Rightarrow \quad \sigma_w = \left|\frac{dw}{dx}\right| \sigma_x$$

As incertezas σ_x e σ_w são positivas, por definição.

Exemplo: Incerteza no volume de um cilindro

O volume de um cilindro pode ser calculado medindo-se o comprimento L e o raio R. O volume V é calculado em função de L e R.

$$V = V(L, R) = \pi L R^2$$

Relação entre incertezas: $\sigma_V^2 = \left(\frac{\partial V}{\partial L}\right)^2 \sigma_L^2 + \left(\frac{\partial V}{\partial R}\right)^2 \sigma_R^2$

$$\sigma_V^2 = \left[\frac{\partial}{\partial L}(\pi L R^2)\right]^2 \sigma_L^2 + \left[\frac{\partial}{\partial R}(\pi L R^2)\right]^2 \sigma_R^2$$

$$\sigma_V^2 = \left[\pi R^2\right]^2 \sigma_L^2 + \left[2 \pi L R\right]^2 \sigma_R^2$$

Dividindo por $\pi L R^2$; tem-se:

$$\frac{\sigma_V^2}{(\pi L R^2)^2} = \frac{\pi^2 R^4 \sigma_L^2}{(\pi L R^2)^2} + \frac{4\pi^2 L^2 R^2 \sigma_R^2}{(\pi L R^2)^2}$$

$$\frac{\sigma_V^2}{V^2} = \frac{\sigma_L^2}{L^2} + 4\frac{\sigma_R^2}{R^2}$$

Em termos de incerteza relativa, tem-se:

$$\varepsilon_V^2 = \varepsilon_L^2 + 4\varepsilon_R^2$$

$$\therefore \boxed{\varepsilon_V = +\sqrt{\varepsilon_L^2 + 4\varepsilon_R^2}}$$

<u>Algumas fórmulas de propagação</u>:

- <u>Soma ou subtração de variáveis</u>: $w = x \pm y \pm z \pm \cdots$

$$\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial w}{\partial z}\right)^2 \sigma_z^2 + \cdots$$

$$\sigma_w^2 = \left[\frac{\partial}{\partial x}(x \pm y \pm z)\right]^2 \sigma_x^2 + \left[\frac{\partial}{\partial y}(x \pm y \pm z)\right]^2 \sigma_y^2 + \left[\frac{\partial}{\partial z}(x \pm y \pm z)\right]^2 \sigma_z^2 + \cdots$$

$$\sigma_w^2 = [\pm 1]^2 \sigma_x^2 + [\pm 1]^2 \sigma_y^2 + [\pm 1]^2 \sigma_z^2 + \cdots$$

$$\therefore \boxed{\sigma_w^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + \cdots}$$

Relação linear: $w = ax + b \Rightarrow w(x) = ax + b$

$$\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2$$

$$\sigma_w^2 = \left[\frac{\partial}{\partial x}(ax+b)\right]^2 \sigma_x^2$$

$$\sigma_w^2 = a^2 \sigma_x^2$$

ou $\sigma_w = +\sqrt{a^2 \sigma_x^2}$

$$\boxed{\sigma_w = |a|\,\sigma_x}$$

pois, σ_w sempre deve ser positivo, por definição.

$w = ax \rightarrow \boxed{\sigma_w = |a|\,\sigma_x}$

Produto ou razão de variáveis: $w = axy$

ou

$w = a\dfrac{x}{y}$

Produto: $w(x,y) = axy$

$$\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2$$

$$\sigma_w^2 = \left[\frac{\partial}{\partial x}(axy)\right]^2 \sigma_x^2 + \left[\frac{\partial}{\partial y}(axy)\right]^2 \sigma_y^2$$

$$\sigma_w^2 = (ay)^2 \sigma_x^2 + (ax)^2 \sigma_y^2$$

Dividindo tudo por $(axy)^2$, tem-se:

$$\frac{\sigma_w^2}{(axy)^2} = \frac{a^2 y^2 \sigma_x^2}{a^2 x^2 y^2} + \frac{a^2 x^2 \sigma_y^2}{a^2 x^2 y^2}$$

$$\frac{\sigma_w^2}{w^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}$$

$$\therefore \boxed{\left(\frac{\sigma_w}{w}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

Divisão: $w(x,y) = a\,\frac{x}{y}$

$$\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2$$

$$\sigma_w^2 = \left[\frac{\partial}{\partial x}\left(a\,\frac{x}{y}\right)\right]^2 \sigma_x^2 + \left[\frac{\partial}{\partial y}\left(a\,\frac{x}{y}\right)\right]^2 \sigma_y^2$$

$$\sigma_w^2 = \left(\frac{a}{y}\right)^2 \sigma_x^2 + \left(-a x y^{-2}\right)^2 \sigma_y^2$$

$$\sigma_w^2 = \left(\frac{a}{y}\right)^2 \sigma_x^2 + \left(-\frac{a x}{y^2}\right)^2 \sigma_y^2$$

Dividindo tudo $\left(a\,\frac{x}{y}\right)^2$, tem-se:

$$\frac{\sigma_w^2}{w^2} = \frac{a^2}{y^2}\,\frac{y^2}{a^2 x^2}\,\sigma_x^2 + \frac{a^2 x^2}{y^4}\,\frac{y^2}{a^2 x^2}\,\sigma_y^2$$

$$\left(\frac{\sigma_w}{w}\right)^2 = \frac{\sigma_x^2}{x^2} - \frac{\sigma_y^2}{y^2}$$

$$\therefore \boxed{\left(\frac{\sigma_w}{w}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 - \left(\frac{\sigma_y}{y}\right)^2}$$

Exemplos de fórmulas de propagação de incertezas.

$$w = \pm x \pm y \pm z \pm \dots \rightarrow \sigma_w = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + \dots}$$

$$w = x^m \longrightarrow \sigma_w = \left| m\, x^{m-1} \right| \sigma_x$$

$$w = ax \longrightarrow \sigma_w = |a|\, \sigma_x$$

$$w = ax + b \longrightarrow \sigma_w = |a|\, \sigma_x$$

$$w = axy \longrightarrow \sigma_w = w \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

$$w = a\frac{x}{y} \longrightarrow \sigma_w = w \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

$$w = x^p y^q \longrightarrow \sigma_w = w \sqrt{\left(p\,\frac{\sigma_x}{x}\right)^2 + \left(q\,\frac{\sigma_y}{y}\right)^2}$$

$$w = \operatorname{sen} x \longrightarrow \sigma_w = |\cos x|\, \sigma_x$$

* $w = \log_a x \rightarrow$