A History of the Finite Element Method: From $-\infty$ to 1970

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Quiz

- How many papers have been written on the Finite Element Method and its applications?
- How many books on FEM are available to buy on Amazon?

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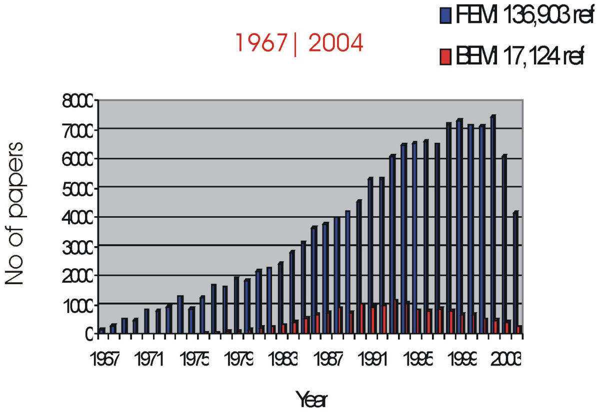


Outline

1. Writing history is not an easy task.
2. The finite element method and its interpretation
3. The mathematical development of finite elements up to 1943
4. Richard Courant and his element
5. Engineering development up to 1943
6. Engineering finite elements from 1943 to 1970
7. Mathematical finite elements from 1943 to 1970
8. The state of the art in 1970

Why it is not easy to write history

- Comprehension of the literature
- Designing a criteria to select relevant papers
- Controversy is impossible to avoid

Who should write history of science?

- History of the science (written by historians)
- Science history (written by scientists)

A historians opinion:

- Scientists are emotionally attached to their science
- This disqualifies them from writing the history of the science

Question: should the results be assessed by today's knowledge or the knowledge in the time when the paper was written?

The Finite Element Method and its intepreration

- The term Finite Element Method was introduced in 1960 by R W Clough
- Previously the Direct Stiffness Method

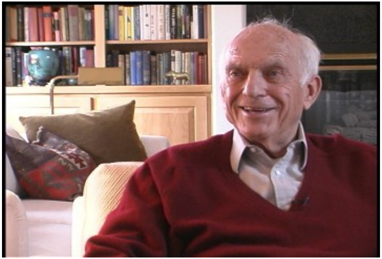


Figure: Ray W Clough, professor emeritus of civil and environmental engineering at the University of California, Berkeley

Interpretations of FEM

There are two interpretations of the approach: the physical and the mathematical. Example: deformation of a cable under load f

- The physical/structural approach
  - Replace the cable by a chain with links of length δ
  - Use structural mechanics
- The mathematical approach
  - Approximation
  - Differential equation for deformation $u'' = f$
  - Discretize the operator by finite differences or finite elements of size δ

Both will obtain the same results.

Meaning of the error

Two interpretations of the finite element method

- Finite element modeling/finite element method as a numerical method for solving differential equations
- Finite element modeling, connection of the elements as plates, shells, etc.

What is the meaning of the error in each approach?

Mathematical development up to 1943

- Why 1943? Because Courant introduced what is today called the Courant element.
- XVII and XVIII century : Development of variational methods, Euler equation

Figure: Leonhard Euler, 1707-1783. Most prolific mathematician ever; collected work more than 80 volumes

The Brachistochrone problem

Brachistochrone minimization problem $\Rightarrow$ Euler nonlinear differential equation. In 1696 Leibniz wrote a letter to Bernoulli about solving the Brachistochrone problem numerically

(a) Gottfried Wilhelm Leibniz, 1646- 1716

(b) Johann Bernoulli, 1667-1748

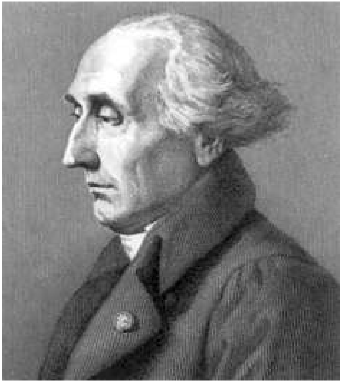
Leibniz's method

- What was Leibniz's method? The solution of the equation was the solution of a 1D minimization problem. He divided the interval in 4 pieces, used piecewise linear functions and minimized the functional.
- What is today's finite element method? Solving the minimization problem over the set of piecewise linear functions.

Hence Leibniz was the first FEM man.

The Plateau Problem and FE Triangles

- In 1851, K.H Schellbach (1806-1892) solved the Plateau Problem, the minimal surface problem formulated by Lagrange.
- Schellbach used approximation by piecewise linear functions on triangles (he had difficulties with squares). These triangles were called elements.



(c) Joseph-Louis Lagrange, 1736-1813

Variational problem for Poisson problem $-\Delta u = f$

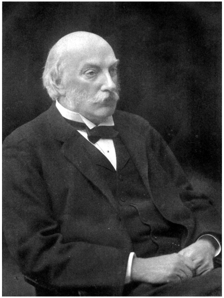
- Riemann proved existence using the variational approach, minimization of energy.
- Weirstrass showed the proof was wrong - the minimizer doesn't necessarily satisfy the equation.
- Only in 1901 did Hilbert solve this problem, essentially using what we call today Hilbert spaces.

(d) Bernhard Riemann, 1826-1866

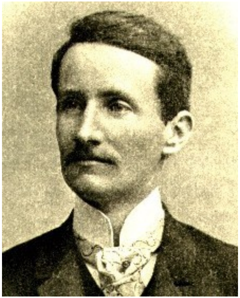
(e) Karl Weierstrass, 1815-1897

(f) David Hilbert, 1862-1943

- Ritz in his 1909 paper introduces his "new method", which uses the minimization principle with global functions. He does not prove convergence or cite Schellbach.
- In 1910, Rayleigh complains that Ritz's method is not "new" because he has a paper on eigenfunctions and is using the minimization principle.
- This method is now often called the Rayleigh-Ritz method.



(g) Lord Rayleigh (J. W. Strutt), 1842-1919



(h) Walther Ritz, 1878-1909

1915 - The Galerkin method

- Introduced the weak formulation with global trial and test functions in an engineering paper.
- Designed this method before 1915.

Figure: Boris Grigoyevich Galerkin, 1871- 1945. He was also a political revolutionary.

Courant's work

- In 1943, Courant proposes explicitly piecewise linears on triangles for solving a torsion problem related to the Laplace equation, using the minimization of energy.
- Courant had suggested this method in 1922 in a footnote in his book with Hurwitz, but in the second edition, this footnote was deleted, as the authors didn't think it was interesting or important.

Figure: Richard Courant, 1888-1972. The triangular linear element is often called the "Courant element".

Engineering development up to 1943

- Structural mechanics introduces the Force method
- Mueller-Breslau introduce the Graphical method
- 1866, Mueller-Breslau method Force method
- Similar method suggested by Maxwell in 1864
- In 1943, a complete schism between practical engineering analysis and mathematical numerical methods

Finite elements in engineering from 1943-1970

During World War II, aircraft engineering began to move ahead of civil engineering in the analysis of complex structures

- In 1946, Analysis Laboratory at Caltech built an analog computer to solve engineering mechanics problems
- In 1952, CEA (Computer Engineering Associates) was established to solve problems in aeroengineering with hundreds of degrees of freedom by using an analog computer of the similar to the one at Caltech
- In 1967, in France, two dimensional analysis of a Concorde wing with approximately 2000 dof was done on an analog computer.
- Formalization and standardization was based on the recognition that the global behavior could be obtained through local assembly. The first progress was made by Langefors in 1952.

Boeing's contribution

- In 1952, there was a great effort at Boeing to analyze aircraft structures . Not a great success
- Late in 1952, N. J. Turner, head of Structural Dynamics Boeing Unit, proposed a new idea: the deformation of any plane stress element could be approximated by a combination of simple strain fields. In 1953, this idea was successfully tested in the computation of a wing structure composed of a combination of plate spars and rib elements.
- The results were submitted in a paper appearing in 1956. This paper is heralded as the origin of modern FEM. Nevertheless, it was NOT based explicitly on the variational method.

- In 1957, Clough began to develop Matrix Algebra problem for IBM 701 and IBM 704.
- In 1963, E.L.Wilson, a student of Clough's, wrote a code which has many features of today''s finite element codes. Program had about 500 Fortran lines.

Figure: Ed Wilson, PhD at UC Berkeley in 1963

NASA's contribution

- In July 1965, NASA issued a RFP for writing a general FE code. Specification called for both force and displacement approaches, although force was recognized as inferior for practical use.
- Based on this RFP and further NASA funding NASTRAN was written over the next 5 years (cca $20 millions) by MacNeal-Schwendler corp.
- At this time NASTRAN had 150,000 Fortran source statements. NASTRAN eventually became commercial code due to legal problems.
- During this period other FE codes appeared on the market. Agyris in the 1960 developed ASKA which also used variational principles

- In 1965, the basic development of engineering FEM was in place, but without mathematical theory. It was believed that approaches based on variational principles would converge.
- Various principles and principles based on saddle points (hybrid, Lagrangian multipliers, etc) were used. It is known today that the saddle point principle does not directly lead to convergence.
- During this period Zienkiewicz made significant progress. His 1967 book had a large influence.

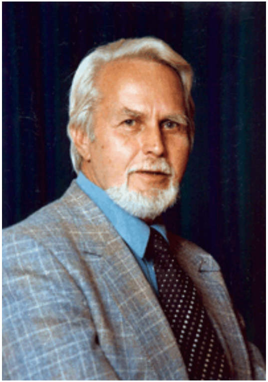


Figure: Olgierd Zienkiewicz, 1921-2009

Mathematical theory of FE from 1943-1970

FEM, as a numerical method of discretization using a variational principle, was independently studied in USA, USSR and China.

- 1962 K.O Friedrichs uses the Dirichlet Principle and linear functions on triangles to solve Laplace with Dirichlet and Neumann boundary conditions on a general domain.
- He proved convergence in $H^1(\Omega)$ and $L_2(\Omega)$ norms in today's standard mathematical way, but does not give a rate of convergence or cite Courant.

Figure: Kurt Otto Friedrichs, 1901-1982

Mathematical theory of FE from 1943-1970, cont.

- In 1963 Ogenesejan (USSR) proves, for a triangular mesh and linear elements, the classical error estimate $\|u - u_{FE}\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}$ for Laplace and more general elliptic equations. Courant is not cited.
- In 1965, Feng Keng proposed FEM for second order Laplace and elasticity equations. He use various finite element spaces, including quadrilaterals with hanging nodes. He proves convergence, but not the rate. Again, Courant is not cited.
- 1967 M Zlamal proved the rate of convergence for quadratic elements.

The state of the art in 1970

- The engineering theory without mathematical theory is in very good shape.
- Implementation on computers, basic problems resolved.
- Mathematical theory is at the very beginning. Very rapid development.
- In 1972, 1973 the first mathematical books appeared.

Thank you!

Questions?