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Relativistic Invariance and Quantum Phenomena*

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INTRODUCTION

THE principal theme of this discourse is the great difference between the relation of special relativity and quantum theory on the one hand, and general relativity and quantum theory on the other. Most of the conclusions which will be reported on in connection with the general theory have been arrived at in collaboration with Dr. H. Salecker,[1] who has spent a year in Princeton to investigate this question.

The difference between the two relations is, briefly, that while there are no conceptual problems to separate the theory of special relativity from quantum theory, there is hardly any common ground between the general theory of relativity and quantum mechanics. The statement, that there are no conceptual conflicts between quantum mechanics and the special theory, should not mean that the mathematical formulations of the two theories naturally mesh. This is not the case, and it required the very ingenious work of Tomonaga, Schwinger, Feynman, and Dyson[2] to adjust quantum mechanics to the postulates of the special theory and this was so far successful only on the working level. What is meant is, rather, that the concepts which are used in quantum mechanics, measurements of positions, momenta, and the like, are the same concepts in terms of which the special relativistic postulate is formulated. Hence, it is at least possible to formulate the requirement of special relativistic invariance for quantum theories and to ascertain whether these requirements are met. The fact that the answer is more nearly *no* than *yes*, that quantum mechanics has not yet been fully adjusted to the postulates of the special theory, is perhaps irritating. It does not alter the fact that the question of the consistency of the two theories can at least be formulated, that the question of the special relativistic invariance of quantum mechanics by now has more nearly the aspect of a puzzle than that of a problem.

This is not so with the general theory of relativity. The basic premise of this theory is that coordinates are only auxiliary quantities which can be given arbitrary values for every event. Hence, the measurement of position, that is, of the space coordinates, is certainly not a significant measurement if the postulates of the general theory are adopted: the coordinates can be given any value one wants. The same holds for momenta. Most of us have struggled with the problem of how, under these premises, the general theory of relativity can make meaningful statements and predictions at all. Evidently, the usual statements about future positions of particles, as specified by their coordinates, are not meaningful statements in general relativity. This is a point which cannot be emphasized strongly enough and is the basis of a much deeper dilemma than the more technical question of the Lorentz invariance of the quantum field equations. It pervades all the general theory, and to some degree we mislead both our students and ourselves when we calculate, for instance, the mercury perihelion motion without explaining how our coordinate system is fixed in space, what defines it in such a way that it cannot be rotated, by a few seconds a year, to follow the perihelion's apparent motion. Surely the x axis of our coordinate system could be defined in such a way that it pass through all successive perihelions. There must be some assumption on the nature of the coordinate system which keeps it from following the perihelion. This is not difficult to exhibit in the case of the motion of the perihelion, and it would be useful to exhibit it. Neither is this, in general, an academic point, even

* Address of retiring president of the American Physical Society, January 31, 1957.

[1] This will be reported jointly with H. Salecker in more detail in another journal.

[2] See, e.g., J. M. Jauch and F. Rohrlich, *The Theory of Protons and Electrons* (Addison-Wesley Press, Cambridge, Massachusetts, 1955).

though it may be academic in the case of the mercury perihelion. A difference in the tacit assumptions which fix the coordinate system is increasingly recognized to be at the bottom of many conflicting results arrived at in calculations based on the general theory of relativity. Expressing our results in terms of the values of coordinates became a habit with us to such a degree that we adhere to this habit also in general relativity where values of coordinates are not *per se* meaningful. In order to make them meaningful, the mollusk-like coordinate system must be somehow anchored to space-time events and this anchoring is often done with little explicitness. If we want to put general relativity on speaking terms with quantum mechanics, our first task has to be to bring the statements of the general theory of relativity into such form that they conform with the basic principles of the general relativity theory itself. It will be shown below how this may be attempted.

RELATIVISTIC QUANTUM THEORY OF ELEMENTARY SYSTEMS

The relation between special theory and quantum mechanics is most simple for single particles. The equations and properties of these, in the absence of interactions, can be deduced already from relativistic invariance. Two cases have to be distinguished: the particle either can, or cannot, be transformed to rest. If it can, it will behave, in that coordinate system, as any other particle, such as an atom. It will have an intrinsic angular momentum called J in the case of atoms and spin S in the case of elementary particles. This leads to the various possibilities with which we are familiar from spectroscopy, that is spins 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\cdots$ each corresponding to a type of particle. If the particle cannot be transformed to rest, its velocity must always be equal to the velocity of light. Every other velocity can be transformed to rest. The rest-mass of these particles is zero because a nonzero rest-mass would entail an infinite energy if moving with light velocity.

Particles with zero rest-mass have only two directions of polarization, no matter how large their spin is. This contrasts with the $2S+1$ directions of polarization for particles with nonzero rest-mass and spin S. Electromagnetic radiation, that is, light, is the most familiar example for this phenomenon. The "spin" of light is 1, but it has only two directions of polarization, instead of $2S+1=3$. The number of polarizations seems to jump discontinuously to two when the rest-mass decreases and reaches the value 0. Bass and Schrödinger[3] followed this out in detail for electromagnetic radiation, that is, for $S=1$. It is good to realize, however, that this decrease in the number of possible polarizations is purely a property of the Lorentz transformation and holds for any value of the spin.

There is nothing fundamentally new that can be said about the number of polarizations of a particle and the principal purpose of the following paragraphs is to illuminate it from a different point of view.[4] Instead of the question: "Why do particles with zero rest-mass have only two directions of polarization?" the slightly different question, "Why do particles with a finite rest-mass have more than two directions of polarization?" is proposed.

The intrinsic angular momentum of a particle with zero rest-mass is parallel to its direction of motion, that is, parallel to its velocity. Thus, if we connect any internal motion with the spin, this is perpendicular to the velocity. In case of light, we speak of transverse polarization. Furthermore, and this is the salient point, the statement that the spin is parallel to the velocity is a relativistically invariant statement: it holds as well if the particle is viewed from a moving coordinate system. If the problem of polarization is regarded from this point of view, it results in the question, "Why can't the angular momentum of a particle with finite rest-mass be parallel to its velocity?" or "Why can't a plane wave represent transverse polarization unless it propagates with light velocity?" The answer is that the angular momentum *can* very well be parallel to the direction of motion and the wave *can* have transverse polarization, but these are not Lorentz invariant statements. In other words, even if velocity and spin are parallel in one coordinate system, they do not appear to be parallel in other coordinate systems. This is most evident if, in this other coordinate system, the particle is at rest: in this coordinate system the

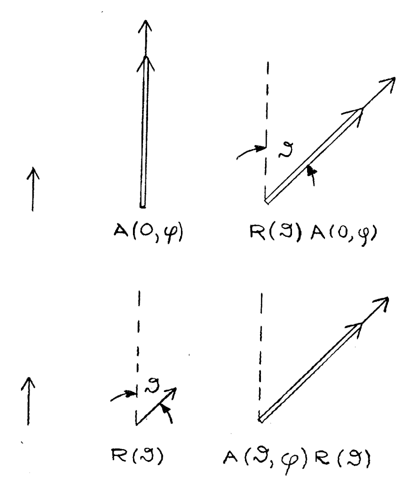

FIG. 1. The short simple arrows illustrate the spin, the double arrows the velocity of the particle. One obtains the same state, no matter whether one first imparts to it a velocity in the direction of the spin, then rotates it $(R(\vartheta)A(0,\varphi))$, or whether one first rotates it, then gives a velocity in the direction of the spin $(A(\vartheta,\varphi)R(\vartheta))$. See Eq. (1.3).

[3] L. Bass and E. Schrödinger, Proc. Roy. Soc. (London) **A232**, 1 (1955).

[4] The essential point of the argument which follows is contained in the present writer's paper, Ann. Math. **40**, 149 (1939) and more explicitly in his address at the Jubilee of Relativity Theory, Bern, 1955 (Birkhauser Verlag, Basel, 1956), A. Mercier and M. Kervaire, editors, p. 210.

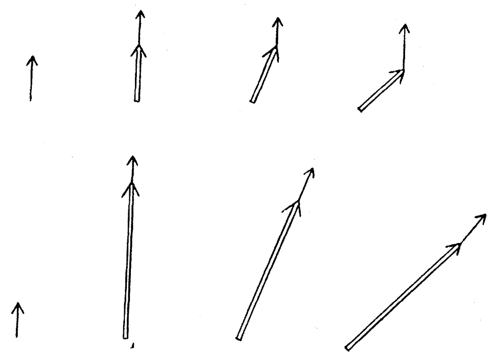

FIG. 2. The particle is first given a small velocity in the direction of its spin, then increasing velocities in a prependicular direction (upper part of the figure). The direction of the spin remains essentially unchanged; it includes an increasingly large angle with the velocity as the velocity in the perpendicular direction increases. If the velocity imparted to the particle is large (lower part of the figure), the direction of the spin seems to follow the direction of the velocity. See Eqs. (1.8) and (1.7).

angular momentum should be parallel to nothing. However, every particle, unless it moves with light velocity, can be viewed from a coordinate system in which it is at rest. In this coordinate system its angular momentum is surely not parallel to its velocity. Hence, the statement that spin and velocity are parallel cannot be universally valid for the particle with finite rest-mass and such a particle must have other states of polarization also.

It may be worthwhile to illustrate this point somewhat more in detail. Let us consider a particle at rest with a given direction of polarization, say the direction of the z axis. Let us consider this particle now from a coordinate system which is moving in the $-z$ direction. The particle will then appear to have a velocity in the z direction and its polarization will be parallel to its velocity (Fig. 1). It will now be shown that this last statement is nearly invariant if the velocity is high. It is evident that the statement is entirely invariant with respect to rotations and with respect to a further increase of the velocity in the z direction. This is illustrated at the bottom of the figure. The coordinate system is first turned to the left and then given a velocity in the direction opposite to the old z axis. The state of the system appears to be exactly the same as if the coordinate system had been first given a velocity in the $-z$ direction and then turned, which is the operation illustrated at the top of the figure. The state of the system appears to be the same not for any physical reason but because the two coordinate systems are identical and they view the same particle (see Appendix I).

Let us now take our particle with a high velocity in the z direction and view it from a coordinate system which moves in the $-y$ direction. The particle now will appear to have a momentum also in the y direction, its velocity will have a direction between the y and z axes (Fig. 2). Its spin, however, will not be in the direction of its motion any more. In the nonrelativistic case, that is, if all velocities are small as compared with the velocity of light, the spin will still be parallel to z and it will, therefore, enclose an angle with the particle's direction of motion. This shows that the statement that the spin is parallel to the direction of motion is not invariant in the nonrelativisitic region. However, if the original velocity of the particle is close to the light velocity, the Lorentz contraction works out in such a way that the angle between spin and velocity is given by

$$\tan\,(\text{angle between spin and velocity}) = (1-v^2/c^2)^{\frac{1}{2}}\sin\vartheta, \quad (1)$$

where ϑ is the angle between the velocity v in the moving coordinate system and the velocity in the coordinate system at rest. This last situation is illustrated at the bottom of the figure. If the velocity of the particle is small as compared with the velocity of light, the direction of the spin remains fixed and is the same in the moving coordinate system as in the coordinate system at rest. On the other hand, if the particle's velocity is close to light velocity, the velocity carries the spin with itself and the angle between direction of motion and spin direction becomes very small in the moving coordinate system. Finally, if the particle has light velocity, the statement "spin and velocity are parallel" remains true in every coordinate system. Again, this is not a consequence of any physical property of the spin, but is a consequence of the properties of Lorentz transformations: it is a kind of Lorentz contraction. It is the reason for the different behavior of particles with finite, and particles with zero, rest-mass, as far as the number of states of polarization is concerned. (Details of the calculation are in Appendix I.)

The preceding consideration proves more than was intended: it shows that the statement "spin and velocity are parallel for zero mass particles" is invariant and that, for relativistic reasons, one needs only *one* state of polarization, rather than *two*. This is true as far as proper Lorentz transformations are concerned. The second state of polarization, in which spin and velocity are antiparallel, is a result of the reflection symmetry. Again, this can be illustrated on the example of light: right circularly polarized light appears as right circularly polarized light in all Lorentz frames of reference which can be continuously transformed into each other. Only if one looks at the right circularly polarized light in a mirror does it appear as left circularly polarized light. The postulate of reflection symmetry allows us to infer the existence of left circularly polarized light from the existence of right circularly polarized light—if there were no such reflection symmetry in the real world, the existence of *two* modes of polarization of light, with virtually identical properties, would appear to be a miracle. The situation is entirely different for particles with nonzero mass. For these, the $2S+1$ directions of polarization follow from the invariance of the theory with respect to proper Lorentz transformations. In particular, if the particle is at rest, the spin will have different orientations with respect to coordinate systems which have different orientations in space. Thus, the existence of all the states of polarization follow from the existence of one, if only the theory is invariant with respect to proper Lorentz transformations. For particles with zero rest-mass, there are only two states of polarization, and even the existence of the second one can be inferred only on the basis of reflection symmetry.

REFLECTION SYMMETRY

The problem and existence of reflection symmetry have been furthered in a brilliant way by recent theoretical and experimental research. There is nothing essential that can be added at present to the remarks and conjectures of Lee, Yang, and Oehme, and all that follows has been said, or at least implied, by Salam, Lee, Yang, and Oehme.[5] The sharpness of the break with past concepts is perhaps best illustrated by the cobalt experiment of Wu, Ambler, Hayward, Hoppes, and Hudson.

The ring current—this may be a permanent current in a superconductor—creates a magnetic field. The Co source is in the plane of the current and emits β particles (Fig. 3). The whole experimental arrangement, as shown in Fig. 3, has a symmetry plane and, if the principle of sufficient cause is valid, the symmetry plane should remain valid throughout the further fate of the system. In other words, since the right and left sides of the plane had originally identical properties, there is no sufficient reason for any difference in their properties at a later time. Nevertheless, the intensity of the β radiation is larger on one side of the plane than the other side. The situation is paradoxical no matter what the mechanism of the effect is—in fact, it is most paradoxical if one disregards its mechanism and theory entirely. If the experimental circumstances can be idealized as indicated, even the principle of sufficient cause seems to be violated.

It is natural to look for an interpretation of the experiment which avoids this very far-reaching conclusion and, indeed, there is such an interpretation.[5a] It is good to reiterate, however, that no matter what interpretation is adopted, we have to admit that the symmetry of the real world is smaller than we had thought. However, the symmetry may still include reflections.

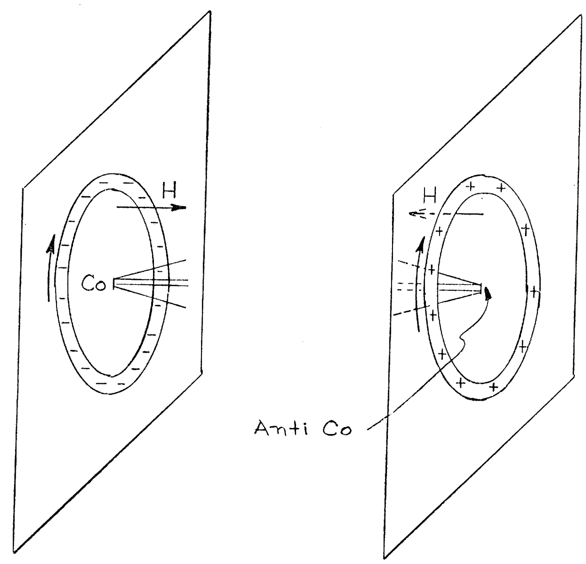

FIG. 3. The right side is the mirror image of the left side, according to the interpretation of the parity experiments[5a] which maintains the reflection as a symmetry element of all physical laws. It must be assumed that the reflection transforms matter into antimatter: the electronic ring current becomes a positronic ring current, the radioactive cobalt is replaced by radioactive anticobalt.

If it is true that a symmetry plane always remains a symmetry plane, the initial state of the Co experiment could not have contained a symmetry plane. This would not be the case if the magnetic vector were polar—in which case the electric vector would be axial. The charge density, the divergence of the electric vector, would then become a pseudoscalar rather than a simple scalar as in current theory. The mirror image of a negative charge would be positive, the mirror image of an electron a positron, and conversely. The mirror image of matter would be antimatter. The Co experiment, viewed through a mirror, would not present a picture contrary to established fact: it would present an experiment carried out with antimatter. The right side of Fig. 3 shows the mirror image of the left side. Thus, the principle of sufficient cause, and the validity of symmetry planes, need not be abandoned if one is willing to admit that the mirror image of matter is antimatter.

The possibility just envisaged would be technically described as the elimination of the operations of reflection and charge conjugation, as presently defined, as true symmetry operations. Their product would still be assumed to be a symmetry operation and proposed to be named, simply, reflection. A few further technical remarks are contained in Appendix II. The proposition just made has two aspects: a very appealing one, and a very alarming one.

Let us look first at the appealing aspect. Dirac has said that the number of elementary particles shows an alarming tendency of increasing. One is tempted to add to this that the number of invariance properties

[5] Lee, Yang, and Oehme, Phys. Rev. **106**, 340 (1957).

[5a] The interpretation referred to has been proposed independently by numerous authors, including A. Salam, Nuovo cimento **5**, 229 (1957); L. Landau, Nuclear Phys. **3**, 127 (1957); H. D. Smyth and L. Biedenharn (personal communication). Dr. S. Deser has pointed out that the "perturbing possibility" was raised already by Wick, Wightman, and Wigner [Phys. Rev. **88**, 101 (1952)] but was held "remote at that time." Naturally, the apparent unanimity of opinion does not prove its correctness.

also showed a similar tendency. It is not equally alarming because, while the increase in the number of elementary particles complicates our picture of nature, that of the symmetry properties on the whole simplifies it. Nevertheless the clear correspondence between the invariance properties of the laws of nature, and the symmetry properties of space-time, was most clearly breached by the operation of charge conjugation. This postulated that the laws of nature remain the same if all positive charges are replaced by negative charges and vice versa, or more generally, if all particles are replaced by antiparticles. Reasonable as this postulate appears to us, it corresponds to no symmetry of the space-time continuum. If the preceding interpretation of the Co experiments should be sustained, the correspondence between the natural symmetry elements of space-time, and the invariance properties of the laws of nature, would be restored. It is true that the role of the planes of reflection would not be that to which we are accustomed—the mirror image of an electron would become a positron—but the mirror image of a sequence of events would still be a possible sequence of events. This possible sequence of events would be more difficult to realize in the actual physical world than what we had thought, but it would still be possible.

The restoration of the correspondence between the natural symmetry properties of space-time on one hand, and the laws of nature on the other hand, is the appealing feature of the proposition. It has, actually, two alarming features. The first of these is that a symmetry operation is, physically, so complicated. If it should turn out that the operation of time inversion, as we now conceive it, is not a valid symmetry operation (e.g., if one of the experiments proposed by Treiman and Wyld gave a positive result) we could still maintain the validity of this symmetry operation by reinterpreting it. We could postulate, for instance, that time inversion transforms matter into *meta*-matter which will be discovered later when higher energy accelerators will become available. Thus, maintaining the validity of symmetry planes forces us to a more artificial view of the concept of symmetry and of the invariance of the laws of physics.

The other alarming feature of our new knowledge is that we have been misled for such a long time to believe in more symmetry elements than actually exist. There was ample reason for this and there was ample experimental evidence to believe that the mirror image of a possible event is again a possible event with electrons being the mirror images of electrons and not of positrons. Let us recall in this connection first how the concept of parity, resulting from the beautiful though almost forgotten experiments of Laporte,[6] appeared to be a perfectly valid concept in spectroscopy and in nuclear physics. This concept could be explained very naturally as a result of the reflection symmetry of space-time, the mirror image of electrons being electrons and not positrons. We are now forced to believe that this symmetry is only approximate and the concept of parity, as used in spectroscopy and nuclear physics, is also only approximate. Even more fundamentally, there is a vast body of experimental information in the chemistry of optically active substances which are mirror images of each other and which have optical activities of opposite direction but exactly equal strength. There is the fact that molecules which have symmetry planes are optically inactive; there is the fact of symmetry planes in crystals.[7] All these facts relate properties of right-handed matter to left-handed *matter*, not of right-handed matter to left-handed *antimatter*. The new experiments leave no doubt that the symmetry plane in this sense is not valid for all phenomena, in particular not valid for β decay, that if the concept of symmetry plane is at all valid for all phenomena, it can be valid only in the sense of converting matter into antimatter.

Furthermore, the old-fashioned type of symmetry plane is not the only symmetry concept that is only approximately valid. Charge conjugation was mentioned before, and we are remainded also of isotopic spin, of the exchange character, that is multiplet system, for electrons and also of nuclei which latter holds so accurately that, in practice, parahydrogen molecules can be converted into orthohydrogen molecules only by first destroying them.[8] This approximate validity of laws of symmetry is, therefore, a very general phenomenon—it may be *the* general phenomenon. We are reminded of Mach's axiom that the laws of nature depend on the physical content of the universe, and the physical content of the universe certainly shows no symmetry. This suggests—and this may also be the spirit of the ideas of Yang and Lee—that all symmetry properties are only approximate. The weakest interaction, the gravitational force, is the basis of the distinction between inertial and accelerated coordinate systems, the second weakest known interaction, that leading to β decay, leads to the distinction between matter and antimatter. Let me conclude this subject by expressing the conviction that the discoveries of Wu, Ambler, Hayward, Hoppes, and Hudson,[9] and of Garwin, Lederman, and Weinreich[10] will not remain isolated discoveries. More likely, they herald a revision of our concept of invariance and possibly

[6] O. Laporte, Z. Physik **23**, 135 (1924). For the interpretation of Laporte's rule in terms of the quantum-mechanical operation of inversion, see the writer's *Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atmospektren* (Friedrich Vieweg und Sohn, Braunschweig, 1931), Chap. XVIII.

[7] For the role of the space and time inversion operators in classical theory, see H. Zocher and C. Török, Proc. Natl. Acad. Sci. U.S. **39**, 681 (1953) and literature quoted there.

[8] See A. Farkas, *Orthohydrogen, Parahydrogen and Heavy Hydrogen* (Cambridge University Press, New York, 1935).

[9] Wu, Ambler, Hayward, Hoppes, and Hudson, Phys. Rev. **105**, 1413(L) (1957).

[10] Garwin, Lederman, and Weinreich, Phys. Rev. **105**, 1415(L) (1957); also, J. L. Friedman and V. L. Telegdi, *ibid.* **105**, 1681(L) (1957).

of other concepts which are even more taken for granted.

QUANTUM LIMITATIONS OF THE CONCEPTS OF GENERAL RELATIVITY

The last remarks naturally bring us to a discussion of the general theory of relativity. The main premise of this theory is that coordinates are only labels to specify space-time points. Their values have no particular significance unless the coordinate system is somehow anchored to events in space-time.

Let us look at the question of how the equations of the general theory of relativity could be verified. The purpose of these equations, as of all equations of physics, is to calculate, from the knowledge of the present, the state of affairs that will prevail in the future. The quantities describing the present state are called initial conditions; the ways these quantities change are called the equations of motion. In relativity theory, the state is described by the metric which consists of a network of points in space-time, that is a network of events, and the distances between these events. If we wish to translate these general statements into something concrete, we must decide what events are, and how we measure distances between *events*. The metric in the general theory of relativity is a metric in space-time, its elements are distances between space-time points, not between points in ordinary space.

The events of the general theory of relativity are coincidences, that is, collisions between particles. The founder of the theory, when he created this concept, had evidently macroscopic bodies in mind. Coincidences, that is, collisions between such bodies, are immediately observable. This is not the case for elementary particles; a collision between these is something much more evanescent. In fact, the point of a collision between two elementary particles can be closely localized in space-time only in case of high-energy collisions. (See

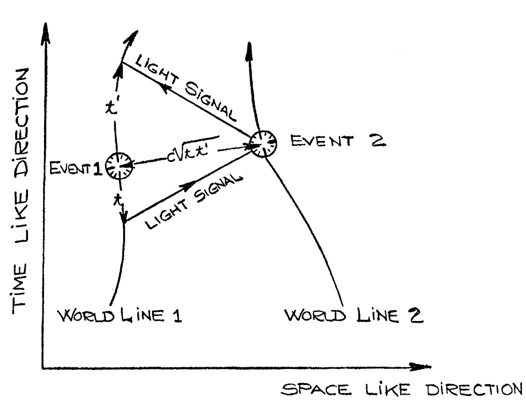

FIG. 4. Measurement of space-like distances by means of a clock. It is assumed that the metric tensor is essentially constant within the space-time region contained in the figure. The space-like distance between events 1 and 2 is measured by means of the light signals which pass through event 2 and a geodesic which goes through event 1. Explanation in Appendix IV.

Appendix III.) This shows that the establishment of a close network of points in space-time requires a reasonable energy density, a dense forest of world lines wherever the network is to be established. However, it is not necessary to discuss this in detail because the measurement of the distances between the points of the network gives more stringent requirements than the establishment of the network.

It is often said that the distances between events must be measured by yardsticks and rods. We found that measurements with a yardstick are rather difficult to describe and that their use would involve a great deal of unnecessary complications. The yardstick gives the distance between events correctly only if its marks coincide with the two events simultaneously from the point of view of the rest-system of the yardstick. Furthermore, it is hard to image yardsticks as anything but macroscopic objects. It is desirable, therefore, to reduce all measurements in space-time to measurements by clocks. Naturally, one can measure by clocks directly only the distances of points which are in time-like relation to each other. The distances of events which are in space-like relation, and which would be measured more naturally by yardsticks, will have to be measured, therefore, indirectly.

It appears, thus, that the simplest framework in space-time, and the one which is most nearly microscopic, is a set of clocks, which are only slowly moving with respect to each other, that is, with world lines which are approximately parallel. These clocks tick off periods and these ticks form the network of events which we wanted to establish. This, at the same time, establishes the distance of those adjacent points which are on the same world line.

Figure 4 shows two world lines and also shows an event, that is, a tick of the clock, on each. The figure shows an artifice which enables one to measure the distance of space-like events: a light signal is sent out from the first clock which strikes the second clock at event 2. This clock, in turn, sends out a light signal which strikes the first clock at time t' after the event 1. If the first light signal had to be sent out at time t before the first event, the calculation given in Appendix IV shows that the space-like distance of events 1 and 2 is the geometric average of the two measured time-like distances t and t'. This is then a way to measure distances between space-like events by clocks instead of yardsticks.

It is interesting to consider the quantum limitations on the accuracy of the conversion of time-like measurements into space-like measurements, which is illustrated in Fig. 4. Naturally, the times t and t' will be well defined only if the light signal is a short pulse. This implies that it is composed of many frequencies and, hence, that its energy spectrum has a corresponding width. As a result, it will give an indeterminate recoil to the second clock, thus further increasing the uncertainty of its momentum. All this is closely related to Heisenberg's uncertainty principle. A more detailed calculation[1] shows that the added uncertainty is of the same order of magnitude as the uncertainty inherent in the nature of the best clock that we could think of, so that the conversion of time-like measurements into space-like measurements is essentially free.

We finally come to the discussion of one of the principal problems—the limitations on the accuracy of the clock. It led us to the conclusion that the inherent limitations on the accuracy of a clock of given weight and size, which should run for a period of a certain length, are quite severe. In fact, the result in summary is that a clock is an essentially nonmicroscopic object. In particular, what we vaguely call an atomic clock, a single atom which ticks off its periods, is surely an idealization which is in conflict with fundamental concepts of measurability. This part of our conclusions can be considered to be well established. On the other hand, the actual formula which will be given for the limitation of the accuracy of time measurement, a sort of uncertainty principle, should be considered as the best present estimate.

Let us state the requirements as follows. The watch shall run T seconds, shall measure time with an accuracy of $T/n=t$, its linear extension shall not exceed l, its mass shall be below m. Since the pointer of the watch must be able to assume n different positions, the system will have to run, in the course of the time T, over at least n orthogonal states. Its state must, therefore, be the superposition of at least n stationary states. It is clear, furthermore, that unless its total energy is at least $\hbar/t$, it cannot measure a time interval which is smaller than t. This is equivalent to the usual uncertainty principle. These two requirements follow directly from the basic principles of quantum theory; they are also the requirements which could well have been anticipated. A clock which conforms with these postulates is, for instance, an oscillator, with a period which is equal to the running time of the clock, if it is with equal probabilty in any of the first n quantum states. Its energy is about n times the energy of the first excited state. This corresponds to the uncertainty principle with the accuracy t as time uncertainty. Broadly speaking, the clock is a very soft oscillator, the oscillating particle moving very slowly and with a rather large amplitude. The pointer of the clock is the position of the oscillating particle.

The clock of the preceding paragraph is still very light. Let us consider, however, the requirement that the linear dimensions of the clock be limited. Since there is little point in dealing with the question in great generality, it may as well be assumed here that the linear dimension shall correspond to the accuracy in time. The requirement $l\approx ct$ increases the mass of the clock by n^3 which may be a very large factor indeed:

$$m>n^3\hbar t/l^2\approx n^3\hbar/c^2t. \tag{2}$$

For example, a clock, with a running time of a day and an accuracy of 10^{-8} second, must weigh almost a gram—for reasons stemming solely from uncertainty principles and similar considerations.

So far, we have paid attention only to the physical dimension of the clock and the requirement that it be able to distinguish between events which are only a distance t apart on the time scale. In order to make it usable as part of the framework which was described before, it is necessary to *read* the clock and to start it. As part of the framework to map out the metric of space-time, it must either register the readings at which it receives impulses, or transmit these readings to a part of space outside the region to be mapped out. This point was already noted by Schrödinger.[11] However, we found it reassuring that, in the most interesting case in which $l=ct$, that is, if space and time inaccuracies are about equal, the reading requirement introduces only an insignificant numerical factor but does not change the form of the expression for the minimum mass of the clock.

The arrangement to map the metric might consist, therefore, of a lattice of clocks, all more or less at rest with respect to each other. All these clocks can emit light signals and receive them. They can also transmit their reading at the time of the receipt of the light signal to the outside. The clocks may resemble oscillators, well in the nonrelativistic region. In fact, the velocity of the oscillating particle is about n times smaller than the velocity of light where n is the ratio of the *error* in the time measurement, to the *duration* of the whole interval to be measured. This last quantity is the spacing of the events on the time axis, it is also the distance of the clocks from each other, divided by the light velocity. The world lines of the clocks from the dense forest which was mentioned before. Its branches suffuse the region of space-time in which the metric is to be mapped out.

We are not absolutely convinced that our clocks are the best possible. Our principal concern is that we have considered only one space-like dimension. One consequence of this was that the oscillator had to be a one-dimensional oscillator. It is possible that the size limitation does not increase the necessary mass of the clock to the same extent if use is made of all three spatial dimensions.

The curvature tensor can be obtained from the metric in the conventional way, if the metric is measured with sufficient accuracy. It may be of interest, nevertheless, the describe a more direct method for measuring the curvature of space. It involves an arrangement, illustrated in Fig. 5, which is similar to that used for obtaining the metric. There is a clock, and a mirror, at such a distance from each other that the curvature of space can be assumed to be constant in the interven-

[11] E. Schrödinger, Ber. Preuss. Akad. Wiss. phys.-math. Kl. **1931**, 238.

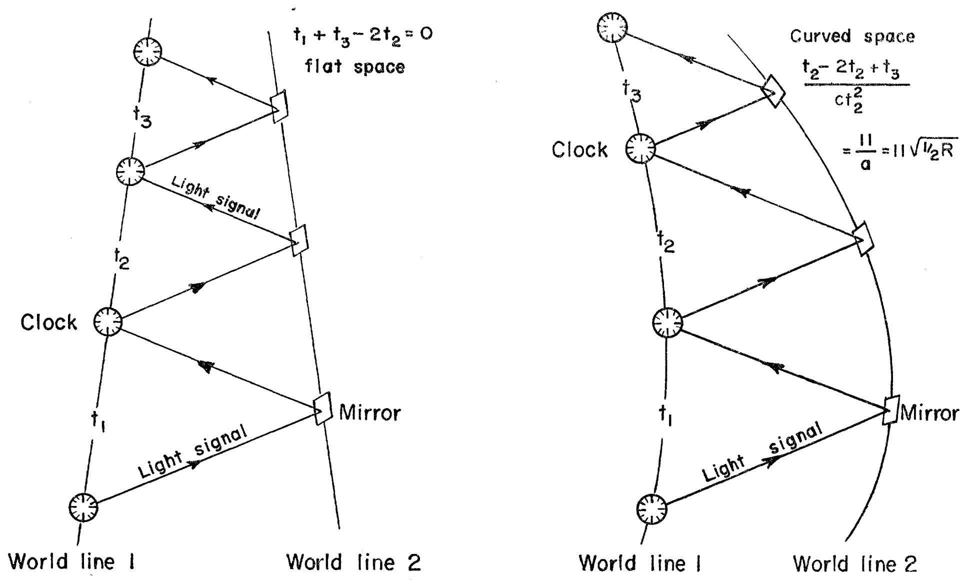

FIG. 5. Direct measurement of the curvature by means of a clock and mirror. Only one space-like dimension is considered and the curvature assumed to be constant within the space-time region contained in the figure. The explanation is given in Appendix V.

ing region. The two clocks need not be at rest with respect to each other, in fact, such a requirement would involve additional measurements to verify it. If the space is flat, the world lines of the clocks can be drawn straight. In order to measure the curvature, a light signal is emitted by the clock, and this is reflected by the mirror. The time of return is read on the clock—it is t_1—and the light signal returned to the mirror. The time which the light signal takes on its second trip to return to the clock is denoted by t_2. The process is repeated a third time, the duration of the last roundtrip denoted by t_3. As shown in Appendix V, the radius of curvature, a, and the relevant component R_{0101} of the Riemann tensor are given by

$$\frac{t_1-2t_2+t_3}{t_2^2}=\frac{11}{a}=11\left(\tfrac{1}{2}R_{0101}\right)^{\frac{1}{2}}. \qquad (2)$$

If classical theory would be valid also in the microscopic domain, there would be no limit on the accuracy of the measurement indicated in Fig. 5. If $\hbar$ is infinitely small, the time intervals t_1, t_2, t_3 can all be measured with arbitrary accuracy with an infinitely light clock. Similarly, the light signals between clock and mirror, however short, need carry only an infinitesimal amount of momentum and thus deflect clock and mirror arbitrarily little from their geodesic paths. The quantum phenomena considered before force us, however, to use a clock with a minimum mass if the measurement of the time intervals is to have a given accuracy. In the present case, this accuracy must be relatively high unless the time intervals t_1, t_2, t_3 are of the same order of magnitude as the curvature of space. Similarly, the deflection of clock and mirror from their geodesic paths must be very small if the result of the measurement is to be meaningful. This gives an effective limit for the accuracy with which the curvature can be measured. The result is, as could be anticipated, that the curvature at a *point* in space-time cannot be measured at all; only the average curvature over a finite region of space-time can be obtained. The error of the measurement[1] is inversely proportional to the two-thirds power of the area available in space-time, that is, the area around which a vector is carried, always parallel to itself, in the customary definition of the curvature. The error is also proportional to the cube root of the Compton wavelength of the clock. Our principal hesitation in considering this result as definitive is again its being based on the consideration of only one space-like dimension. The possibilities of measuring devices, as well as the problems, may be substantially different in three-dimensional space.

Whether or not this is the case, the essentially nonmicroscopic nature of the general relativistic concepts seems to us inescapable. If we look at this first from a practical point of view, the situation is rather reassuring. We can note first, that the measurement of electric and magnetic fields, as discussed by Bohr and Rosenfeld,[12] also requires macroscopic, in fact *very* macroscopic, equipment and that this does not render the electromagnetic field concepts useless for the purposes of quantum electrodynamics. It is true that the measurement of space-time curvature requires a finite region of space and there is a minimum for the mass, and even the mass uncertainty, of the measuring equipment. However, numerically, the situation is by no means alarming. Even in interstellar space, it should be possible to measure the curvature

[12] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **12**, No. 8 (1933). See also further literature quoted in L. Rosenfeld's article in *Niels Bohr and the Development of Physics* (Pergamon Press, London, 1955).

in a volume of a light second or so. Furthermore, the mass of the clocks which one will wish to employ for such a measurement is of the order of several micrograms so that the finite mass of elementary particles does not cause any difficulty. The clocks will contain many particles and there is no need, and there is not even an incentive, to employ clocks which are lighter than the elementary particles. This is hardly surprising since the mass which can be derived from the gravitational constant, light velocity, and Planck's constant, is about 20 micrograms.

It is well to repeat, however, that the situation is less satisfactory from a more fundamental point of view. It remains true that we consider, in ordinary quantum theory, position operators as observables without specifying what the coordinates mean. The concepts of quantum field theories are even more weird from the point of view of the basic observation that only coincidences are meaningful. This again is hardly surprising because even a 20-microgram clock is too large for the measurement of atomic times or distances. If we analyze the way in which we "get away" with the use of an absolute space concept, we simply find that we do not. In our experiments we surround the microscopic objects with a very macroscopic framework and observe *coincidences* between the particles emanating from the microscopic system, and parts of the framework. This gives the collision matrix, which is observable, and observable in terms of macroscopic coincidences. However, the so-called observables of the microscopic system are not only not observed, they do not even appear to be meaningful. There is, therefore, a boundary in our experiments between the region in which we use the quantum concepts without worrying about their meaning in face of the fundamental observation of the general theory of relativity, and the surrounding region in which we use concepts which are meaningful also in the face of the basic observation of the general theory of relativity but which cannot be described by means of quantum theory. This appears most unsatisfactory from a strictly logical standpoint.

APPENDIX I

It will be necessary, in this appendix, to compare various states of the same physical system. These states will be generated by looking at the same state—the standard state—from various coordinate systems. Hence every Lorentz frame of reference will define a state of the system—the state as which the standard state appears from the point of view of this coordinate system. In order to define the standard state, we choose an arbitrary but fixed Lorentz frame of reference and stipulate that, in this frame of reference, the particle in the standard state be at rest and its spin (if any) have the direction of the z axis. Thus, if we wish to have a particle moving with a velocity v in the z direction and with a spin also directed along this axis, we look at the particle in the standard state from a coordinate system moving with the velocity v in the $-z$ direction. If we wish to have a particle at rest but with its spin in the yz plane, including an angle α with the z axis, we look at the standard state from a coordinate system the y and z axes of which include an angle α with the y and z axes of the coordinate system in which the standard state was defined. In order to obtain a state in which both velocity and spin have the aforementioned direction (i.e., a direction in the yz plane, including the angles α and $\frac{1}{2}\pi-\alpha$ with the y and z axes), we look at the standard state from the point of view of a coordinate system in which the spin of the standard state is described as this direction and which is moving in the opposite direction.

Two states of the system will be identical only if the Lorentz frames of reference which define them are identical. Under this definition, the relations which will be obtained will be valid independently of the properties of the particle, such as spin or mass (as long as the mass in nonzero so that the standard state exists). Two states will be approximately the same if the two Lorentz frames of reference which define them can be obtained from each other by a very small Lorentz transformation, that is, one which is near the identity. Naturally, all states of a particle which can be compared in this way are related to each other inasmuch as they represent the same standard state viewed from various coordinate systems. However, we shall have to compare only these states.

Let us denote by $A(0,\varphi)$ the matrix of the transformation in which the transformed coordinate system moves with the velocity $-v$ in the z direction where $v=c\tanh\varphi$

$$A(0,\varphi)=\left\|\begin{matrix}1 & 0 & 0\\ 0 & \cosh\varphi & \sinh\varphi\\ 0 & \sinh\varphi & \cosh\varphi\end{matrix}\right\|. \tag{1.1}$$

Since the x axis will play no role in the following consideration, it is suppressed in (1.1) and the three rows and the three columns of this matrix refer to the y', z', ct' and to the y, z, ct axes, respectively. The matrix (1.1) characterizes the state in which the particle moves with a velocity v in the direction of the z axis and its spin is parallel to this axis.

Let us further denote the matrix of the rotation by an angle φ in the yz plane by

$$R(\vartheta)=\left\|\begin{matrix}\cos\vartheta & \sin\vartheta & 0\\ -\sin\vartheta & \cos\vartheta & 0\\ 0 & 0 & 1\end{matrix}\right\|. \tag{1.2}$$

We refer to the direction in the yz plane which lies between the y and z axes and includes an angle ϑ with the z axis as the direction ϑ. The coordinate system which moves with the velocity $-v$ in the ϑ direction is obtained by the transformation

$$A(\vartheta,\varphi)=R(\vartheta)A(0,\varphi)R(-\vartheta). \tag{1.3}$$

In order to obtain a particle which moves in the direction ϑ and is polarized in this direction, we first rotate the coordinate system counterclockwise by ϑ (to have the particle polarized in the proper direction) and impart it then a velocity $-v$ in the ϑ direction. Hence, it is the transformation

$$T(\vartheta,\varphi)=A(\vartheta,\varphi)R(\vartheta) = \begin{Vmatrix} \cos\vartheta & \sin\vartheta\cosh\varphi & \sin\vartheta\sinh\varphi \\ -\sin\vartheta & \cos\vartheta\cosh\varphi & \cos\vartheta\sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{Vmatrix} \tag{1.4}$$

which characterizes the aforementioned state of the particle. It follows from (1.3) that

$$T(\vartheta,\varphi)=R(\vartheta)A(0,\varphi)=R(\vartheta)T(0,\varphi) \tag{1.5}$$

so that the same state can be obtained also by viewing the state characterized by (1.1) from a coordinate system that is rotated by ϑ. It follows that the statement "velocity and spin are parallel" is invariant under rotations. This had to be expected.

If the state generated by $A(0,\varphi)=T(0,\varphi)$ is viewed from a coordinate system which is moving with the velocity u in the direction of the z axis, the particle will still appear to move in the z direction and its spin will remain parallel to its direction of motion, unless $u>v$ in which case the two directions will become antiparallel, or unless $u=v$ in which case the statement becomes meaningless, the particle appearing to be at rest. Similarly, the other states in which spin and velocity are parallel, i.e., the states generated by the transformations $T(\vartheta,\varphi)$, remain such states if viewed from a coordinate system moving in the direction of the particle's velocity, as long as the coordinate system is not moving faster than the particle. This also had to be expected. However, if the state generated by $T(0,\varphi)$ is viewed from a coordinate system moving with velocity $v'=c\tanh\varphi'$ in the $-y$ direction, spin and velocity will *not* appear parallel any more, *provided the velocity v of the particle is not close to light velocity.* This last proviso is the essential one; it means that the high velocity states of a particle for which spin and velocity are parallel (i.e., the states generated by (1.4) with a large φ) are states of this same nature if viewed from a coordinate system which is not moving too fast in the direction of motion of the particle itself. In the limiting case of the particle moving with light velocity, the aforementioned states become invariant under *all* Lorentz transformations.

Let us first convince ourselves that if the state (1.1) is viewed from a coordinate system moving in the $-y$ direction, its spin and velocity no longer appear parallel. The state in question is generated from the normal state by the transformation

$$A(\tfrac{1}{2}\pi,\varphi')A(0,\varphi) = \begin{Vmatrix} \cosh\varphi' & \sinh\varphi\sinh\varphi' & \cosh\varphi\sinh\varphi' \\ 0 & \cosh\varphi & \sinh\varphi \\ \sinh\varphi' & \sinh\varphi\cosh\varphi' & \cosh\varphi\cosh\varphi' \end{Vmatrix}. \tag{1.6}$$

This transformation does not have the form (1.4). In order to bring it into that form, it has to be multiplied on the right by $R(\epsilon)$, i.e., one has to rotate the spin ahead of time. The angle ϵ is given by the equation

$$\tan\epsilon=\frac{\tanh\varphi'}{\sinh\varphi}=\frac{v'}{v}(1-v^2/c^2)^{\frac{1}{2}} \tag{1.7}$$

and is called the angle between spin and velocity. For $v\ll c$, it becomes equal to the angle which the ordinary resultant of two perpendicular velocities, v and v', includes with the first of these. However, ϵ becomes very small if v is close to c; in this case it is hardly necessary to rotate the spin away from the z axis before giving it a velocity in the z direction. These statements express the identity

$$A(\tfrac{1}{2}\pi,\varphi')A(0,\varphi)R(\epsilon)=T(\vartheta,\varphi'') \tag{1.8}$$

which can be verified by direct calculation. The right side represents a particle with parallel spin and velocity, the magnitude and direction of the latter being given by the well-known equations

$$v''=c\tanh\varphi''=(v^2+v'^2-v^2v'^2/c^2)^{\frac{1}{2}} \tag{1.8a}$$

and

$$\tan\vartheta=\frac{\sinh\varphi'}{\tanh\varphi}=\frac{v'}{v(1-v'^2/c^2)^{\frac{1}{2}}}. \tag{1.8b}$$

Equation (1) given in the text follows from (1.7) and (1.8b) for $v\sim c$.

The fact that the states $T(\vartheta,\varphi)\psi_0$ (where ψ_0 is the standard state and $\varphi\gg1$) are approximately invariant under all Lorentz transformations is expressed mathematically by the equations,

$$R(\vartheta)\cdot T(0,\varphi)\psi_0=T(\vartheta,\varphi)\psi_0, \tag{1.5a}$$

$$A(0,\varphi')\cdot T(0,\varphi)\psi_0=T(0,\varphi'+\varphi)\psi_0, \tag{1.9a}$$

and

$$A(\tfrac{1}{2}\pi,\varphi')\cdot T(0,\varphi)\psi_0\rightarrow T(\vartheta,\varphi'')\psi_0, \tag{1.9b}$$

which give the wave function of the state $T(0,\varphi)\psi_0$, as viewed from other Lorentz frames of reference. Naturally, similar equations apply to all $T(\alpha,\varphi)\psi_0$. In particular, (1.5a) shows that the states in question are invariant under rotations of the coordinate system, (1.9a) that they are invariant with respect to Lorentz transformations with a velocity not too high *in* the direction of motion (so that $\varphi'+\varphi\gg0$, i.e., φ' not too large a negative number). Finally, in order to prove (1.9b), we calculate the transition probability between the states $A(\frac{1}{2}\pi,\varphi')\cdot T(0,\varphi)\psi_0$ and $T(\vartheta,\varphi'')\psi_0$ where ϑ and φ'' are given by (1.8a) and (1.8b). For this, (1.8) gives

$$\begin{aligned}(A(\tfrac{1}{2}\pi,\varphi')\cdot T(0,\varphi)\psi_0,\ T(\vartheta,\varphi'')\psi_0) &= (T(\vartheta,\varphi'')R(\epsilon)^{-1}\psi_0,\ T(\vartheta,\varphi'')\psi_0) \\ &= (R(\epsilon)^{-1}\psi_0,\psi_0)\rightarrow(\psi_0,\psi_0).\end{aligned}$$

The second line follows because $T(\vartheta,\varphi'')$ represents a coordinate transformation and is, therefore, unitary. The last member follows because $\epsilon\rightarrow 0$ as $\varphi\rightarrow\infty$ as can be seen from (1.7) and $R(0)=1$.

The preceding consideration is not fundamentally new. It is an elaboration of the facts (a) that the subgroup of the Lorentz group which leaves a null-vector invariant is different from the subgroup which leaves a time like vector invariant[4] and (b) that the representations of the latter subgroup decompose into one dimensional representations if this subgroup is "contracted" into the subgroup which leaves a null-vector invariant.[13]

APPENDIX II

Before the hypothesis of Lee and Yang[14] was put forward, it was commonly assumed that there are, in addition to the symmetry operations of the proper Poincaré group, three further independent symmetry operations. The proper Poincaré group consists of all Lorentz transformations which can be continuously obtained from unity and all translations in space-like and time-like directions, as well as the products of all these transformations. It is a continuous group; the Lorentz transformations contained in it do not change the direction of the time axis and their determinant is 1. The three independent further operations which were considered to be rigorously valid, were

Space inversion I, that is, the transformation $x, y, z\rightarrow -x, -y, -z$, without changing particles into antiparticles.

Time inversion T, more appropriately described by Lüders[15] as *Umkehr der Bewegungsrichtung*, which replaces every velocity by the opposite velocity so that the position of the particles at $+t$ becomes the same as it was, without time inversion, at $-t$. The time inversion T (also called time inversion of the first kind by Lüders[16]) does not convert particles into antiparticles either.

Charge conjugation C, that is, the replacement of positive charges by negative charges and more generally of particles by antiparticles, without changing either the position or the velocity of these particles.[17] The quantum-mechanical expressions for the symmetry operations I and C are unitary, that for T is antiunitary.

The three operations I, T, C, together with their products TC (Lüders' time inversion of the second kind), IC, IT, ITC and the unit operation form a group and the products of the elements of this group with those of the proper Poincaré group were considered to be the symmetry operations of all laws of physics. The suggestion given in the text amounts to eliminating the operations I and C separately while continuing to postulate their product IC as symmetry operation. The discrete symmetry group then reduces to the unit operation plus

$$IC,\ T, \text{ and } ICT, \tag{2.1}$$

and the total symmetry group of the laws of physics becomes the proper Poincaré group plus its products with the elements (2.1). This group is isomorphic (essentially identical) with the unrestricted Poincaré group, i.e., the product of *all* Lorentz transformations with all the displacements in space and time. The quantum mechanical expressions for the operations of the proper Lorentz group and its product with IC are unitary, those for T and ICT (as well as for their products with the elements of the proper Poincaré group) antiunitary. Lüders[16] has pointed out that, under certain very natural conditions, ICT belongs to the symmetry group of every *local* field theory.

APPENDIX III

Let us consider, first, the collision of two particles of equal mass m in the coordinate system in which the average of the sum of their momenta is zero. Let us assume that, at a given time, the wave function of both particles is confined to a distance l in the direction of their average velocity with respect to each other. If we consider only this space-like direction, and the time axis, the area in space-time in which the two wave functions will substantially overlap is [see Fig. 6(a)]

$$a=l^2/2v_{\min}, \tag{3.1}$$

where $v_{\min}$ is the lowest velocity which occurs with substantial probability in the wave packets of the colliding particles. Denoting the average momentum by $\bar{p}$ (this has the same value for both particles) the half-width of the momentum distribution by δ, then $v_{\min}=(\bar{p}-\delta)(m^2+(\bar{p}-\delta)^2/c^2)^{-\frac{1}{2}}$. Since l cannot be below h/δ, the area (3.1) is at least

$$\frac{\hbar^2}{2\delta^2}\frac{(m^2+(\bar{p}-\delta)^2/c^2)^{\frac{1}{2}}}{\bar{p}-\delta}. \tag{3.1a}$$

(Note that the area becomes infinite if $\delta>\bar{p}$.) The

[13] E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **39**, 510 (1953).

[14] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956). See also E. M. Purcell and N. F. Ramsey, Phys. Rev. **78**, 807 (1950).

[15] G. Lüders, Z. Physik **133**, 325 (1952).

[16] G. Lüders, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **28**, No. 5 (1954).

[17] All three symmetry operations were first discussed in detail by J. Schwinger, Phys. Rev. **74**, 1439 (1948). See also H. A. Kramers, Proc. Acad. Sci. Amsterdam **40**, 814 (1937) and W. Pauli's article in *Niels Bohr and the Development of Physics* (Pergamon Press, London, 1955). The significance of the first two symmetry operations (and their connection with the concepts of parity and the Kramers degeneracy respectively), were first pointed out by the present writer, Z. Physik **43**, 624 (1927) and Nachr. Akad. Wiss. Göttingen, Math.-physik. **1932**, 546. See also T. D. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949); S. Watanabe, Revs. Modern Phys. **27**, 26 (1945). The concept of charge conjugation is based on the observation of W. Furry, Phys. Rev. **51**, 125 (1937).

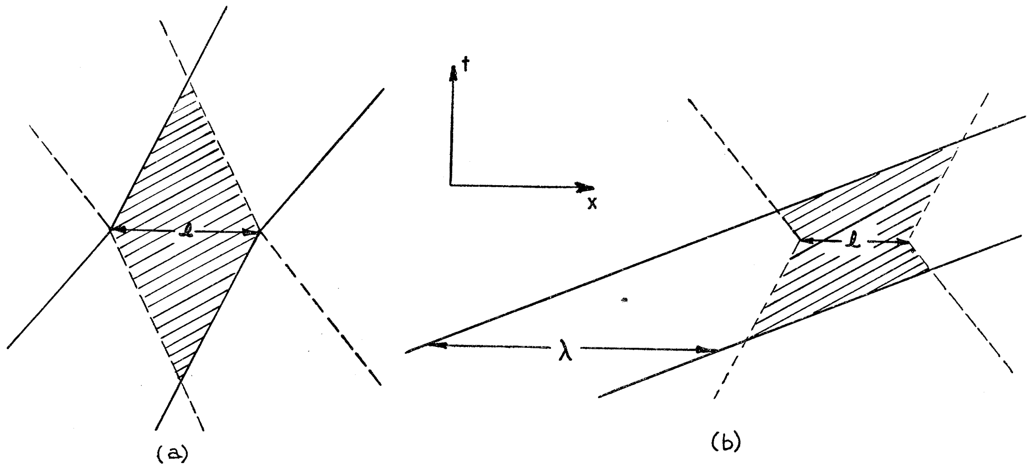

FIG. 6. (a) Localization of a collision of two particles of equal mass. The full lines indicate the effective boundaries of the wave packet of the particle traveling to the right, the broken lines the effective boundaries of the wave packet of the particle traveling to the left. The collision can take place in the shaded area of space-time. (b) Localization of a collision between a particle with finite mass and a particle with zero rest-mass. The full lines, at a distance λ apart in the x direction, indicate the boundary of the particle with zero rest-mass, the broken lines apply to the wave packet of the particle with nonzero rest-mass. The collision can take place in the shaded area.

minimum of (3.1a) is, apart from a numerical factor

$$a_{\min}\approx\frac{\hbar^2}{\bar{p}^3}(m^2+\bar{p}^2/c^2)^{\frac{1}{2}}\approx\frac{\hbar^2 c}{E^{\frac{3}{2}}(E+mc^2)^{\frac{1}{2}}}, \tag{3.2}$$

where E is the kinetic energy (total energy minus rest-energy) of the particles.

The kinetic energy E permits the contraction of the wave functions of the colliding particles also in directions perpendicular to the average relative velocity, to an area $\hbar^2c^2/E(E+2mc^2)$. Hence, again apart from a numerical factor, the volume to which the collision can be confined in four dimensional space-time becomes

$$V_{\min}=\frac{\hbar^4c^3}{E^{\frac{5}{2}}(E+mc^2)^{\frac{3}{2}}}. \tag{3.3}$$

E is the average kinetic energy of the particles in the coordinate system in which their center of mass is, on the average, at rest. Equation (3.3) is valid apart from a numerical constant of unit order of magnitude but this constant depends on E/mc^2.

Let us consider now the opposite limiting case, the collision of a particle with finite rest-mass m with a particle with zero rest-mass. The collision is viewed again in the coordinate system in which the average linear momentum is zero. In this case, one will wish to confine the wave function of the particle with finite rest-mass to a narrower region l than that of the particle with zero rest-mass. If the latter is confined to a region of thickness λ, [see Fig. 6(b)], its momentum and energy uncertainties will be at least $\hbar/\lambda$ and $\hbar c/\lambda$ and these expressions will also give, apart from a numerical factor, the average values of these quantities. Hence $\bar{p}\approx\hbar/\lambda$. The kinetic energy of the particle with finite restmass will be of the order of magnitude

$$\tfrac{1}{2}(m^2c^4+(\bar{p}+\hbar/l)^2c^2)^{\frac{1}{2}}+\tfrac{1}{2}(m^2c^4+(\bar{p}-\hbar/l)^2c^2)^{\frac{1}{2}}-mc^2, \tag{3.4}$$

since $\hbar/l$ is the momentum uncertainty. Since $l\leq\lambda$, one can neglect $\bar{p}$ in (3.4) if one is interested only in the order of magnitude. This gives for the total kinetic energy,

$$E\approx\hbar c/\lambda+(m^2c^4+\hbar^2c^2/l^2)^{\frac{1}{2}}-mc^2, \tag{3.5}$$

while the area in Fig. 6(b) is of the order of magnitude

$$a=(\lambda/c)(l+\Delta v\lambda/c), \tag{3.6}$$

where Δv is the uncertainty in the velocity of the second particle

$$\Delta v=\frac{\bar{p}+\hbar/l}{(m^2+(\bar{p}+\hbar/l)^2/c^2)^{\frac{1}{2}}}-\frac{\bar{p}-\hbar/l}{(m^2+(\bar{p}-\hbar/l)^2/c^2)^{\frac{1}{2}}}. \tag{3.6a}$$

This can again be replaced by $(\hbar/l)(m^2+\hbar^2/l^2c^2)^{-\frac{1}{2}}$.

For given E, the minimum value of a is assumed if the *kinetic energies* of the two particles are of the same order of magnitude. The two terms of (3.6) then become about equal and $l/\lambda\approx(E/(m+E))^{\frac{1}{2}}$. The minimum value of a, as far as order of magnitude is concerned, is again given by (3.2). Similarly, (3.3) also remains valid if one of the two particles has zero rest-mass.

The two-dimensional case becomes simplest if both particles have zero rest-mass. In this case the wave packets do not spread at all and (3.2) can be immediately seen to be valid. In the four-dimensional case, (3.3) again holds. However, its proof by means of explicitly constructed wave packets (rather than reference to the uncertainty relations) is by no means simple. It requires wave packets which are confined in every direction, do not spread too fast and progress essentially only into one half space (one particle going toward the right, the other toward the left). The construction of such wave packets will not be given in detail. They are necessary to prove (3.2) and (3.3) more rigorously also in the case of finite masses; the preceding proofs, based on the uncertainty relations show only that a and v cannot be *smaller* than the right sides of the corresponding equations. It is clear, in fact, that the limits given by (3.2) and (3.3) would be very difficult to realize, except in the two-dimensional case and for the collision of two particles with zero restmass. In all other cases, the relatively low values of $a_{\min}$ and $V_{\min}$ are predicated on the assumption that the wave packets of the colliding particles are so constituted that they assume a minimum size at the time of the collision. At any rate, (3.2) and (3.3) show that only collisions with a relatively high collision energy, and high energy uncertainty, can be closely localized in space-time.

APPENDIX IV

Let us denote the components of the vector from event 1 to event 2 by x_i, the components of the unit vector along the world line of the first clock at event 1 by e_i. The components of the first light signal are x_i+te_i, that of the second light signal $x_i-t'e_i$. Hence

$$g^{ik}(x_i+te_i)(x_k+te_k)=0 \tag{4.1}$$

$$g^{ik}(x_i-t'e_i)(x_k-t'e_k)=0. \tag{4.2}$$

Elimination of the linear terms in t and t' by multiplication of (4.1) with t' and (4.2) with t and addition gives

$$2g^{ik}x_ix_k+2tt'g^{ik}e_ie_k=0. \tag{4.3}$$

Since e is a unit vector $g^{ik}e_ie_k=1$ and (4.3) shows that the space-like distance between points 1 and 2 is $(tt')^{\frac{1}{2}}$.

APPENDIX V

Since the measurement of the curvature, described in the text, presupposes *constant curvature* over the space-time domain in which the measurement takes place, we use a space with constant curvature, or, rather, part of a space with constant curvature, to carry out the calculation. We consider only one spatial dimension, i.e., a two-dimensional deSitter space. This will be embedded, in the usual way, in a three-dimensional space[18] with coordinates x, y, τ. The points of the deSitter space then form the hyperboloid

$$x^2+y^2-\tau^2=a^2, \tag{5.1}$$

where a is the "radius of the universe." As coordinates of a point we use x and y, or rather the corresponding

[18] See, e.g., H. P. Robertson, Revs. Modern Phys. 5, 62 (1933).

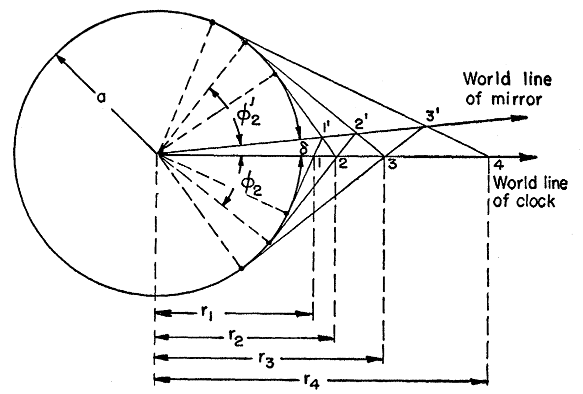

FIG. 7. Analysis of the experiment of Fig. 5. The figure represents a view of the hyperboloid of deSitter space, viewed along its axis. Every point of the plane which is outside the circle corresponds to two points of the deSitter world with one spatial dimension, those with oppositely equal times. The first light signal is emitted at 1, reaches the mirror at 1′, and returns to the clock at 2. The paths of the second and third light signals are 22′3 and 33′4.

polar angles r, ϕ. The metric form in terms of these is

$$ds^2=\frac{a^2}{r^2-a^2}dr^2-r^2d\phi^2. \tag{5.2}$$

Two points of deSitter space correspond to every pair r, ϕ (except $r=a$): those with positive and negative $\tau=(r^2-a^2)^{\frac{1}{2}}$. This will not lead to any confusion as all events take place at positive τ. The null lines (paths of light signals) are the tangents to the $r=a$ circle.

The experiment described in the text can be analyzed by means of Fig. 7. For the sake of simplicity, the clock and mirror are assumed to be "at rest," i.e., their world lines have constant polar angles which will be assumed as 0 and δ, respectively. The first light signal travels from 1 to 1′ and back to 2, the second from 2 to 2′ and back to 3, the third from 3 to 3′ and back to 4. The polar angle of the radius vector which is perpendicular to the first part 22′ of the world line of the second light signal is denoted by ϕ_2. The construction of Fig. 7 shows that angle ϕ_2' which the world line of the mirror includes with the radius vector perpendicular to the second part 2′3 of the second light signal's world line is

$$\phi_2'=\phi_2+\delta. \tag{5.3}$$

The angles ϕ_1, ϕ_1', ϕ_3, ϕ_3' have similar meanings; they are not indicated in the figure in order to avoid overcrowding. For reasons similar to those leading to (5.3), we have

$$\phi_3=\phi_2'+\delta=\phi_2+2\delta \tag{5.3a}$$

$$\phi_1=\phi_2-2\delta \tag{5.3b}$$

$$\phi_4=\phi_3+2\delta=\phi_2+4\delta. \tag{5.3c}$$

The radial coordinates of the points 1, 2, 3, 4 are denoted by r_1, r_2, r_3, r_4

$$r_i = a/\cos\phi_i. \tag{5.4}$$

The proper time t, registered by the clock, can be obtained by integrating the metric form (5.2) along the world line $\phi=0$ of the clock

$$t = a \ln[r+(r^2-a^2)^{\frac{1}{2}}]. \tag{5.5}$$

Hence, the traveling time t_2 of the second light signal becomes

$$t_2 = a \ln\frac{r_3+(r_3^2-a^2)^{\frac{1}{2}}}{r_2+(r_2^2-a^2)^{\frac{1}{2}}} = a \ln\frac{\cos\phi_2(1+\sin\phi_3)}{\cos\phi_3(1+\sin\phi_3)}. \tag{5.6}$$

Similar expressions apply for the traveling times of the first and third light signals; all ϕ can be expressed by means of (5.3a), (5.3b), (5.3c) in terms of ϕ_2 and δ. This allows the calculation of the expression (3). For small δ, one obtains

$$\frac{t_1-2t_2+t_3}{t_2^2} \approx \frac{11}{a}, \tag{5.7}$$

and Riemann's invariant $R=2/a^2$ is proportional to the square of (5.7). In particular, it vanishes if the expression (3) is zero.