The Principle of Special Relativity

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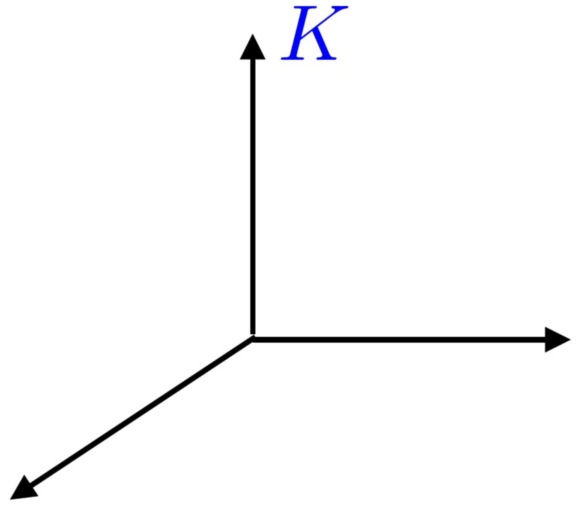

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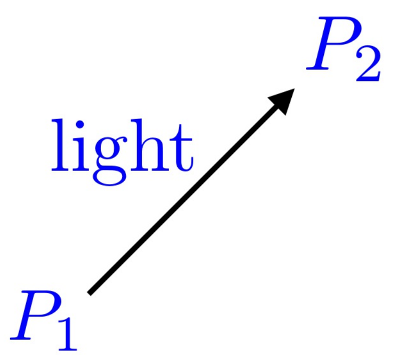

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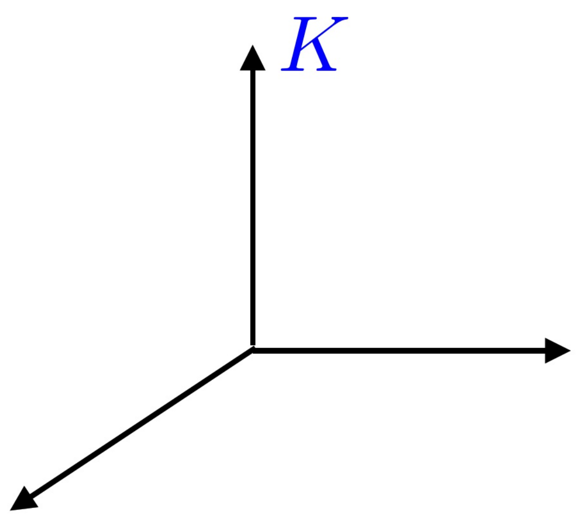

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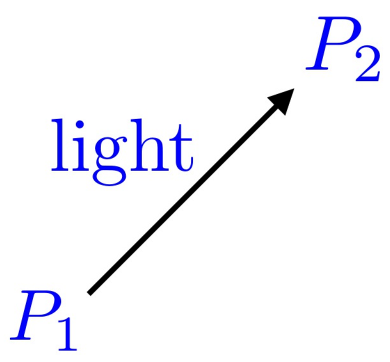

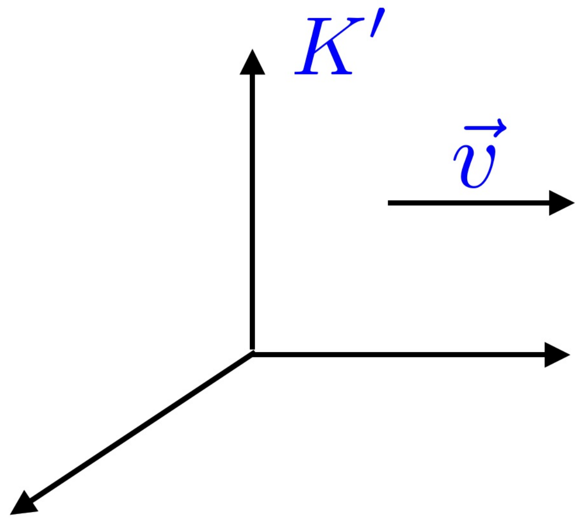

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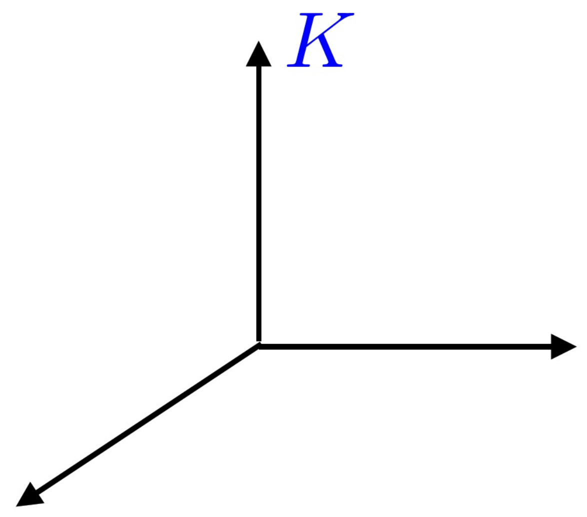

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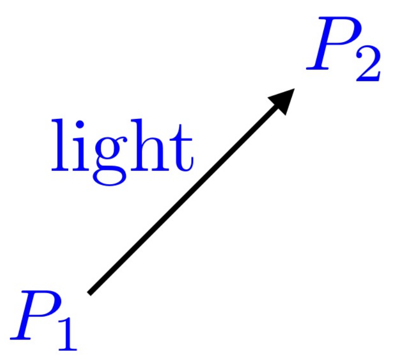

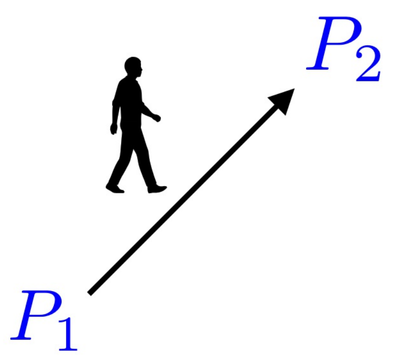

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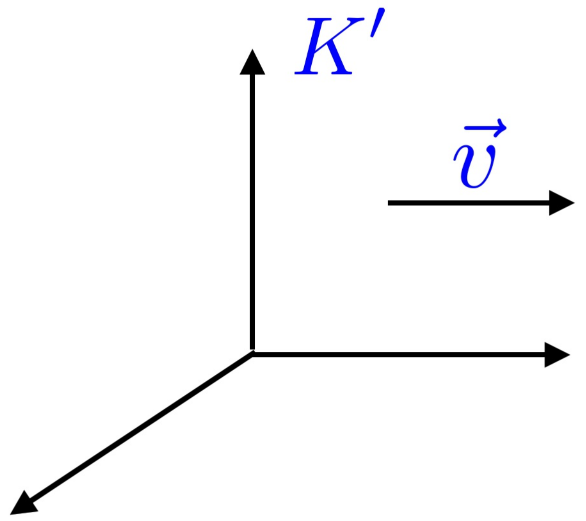

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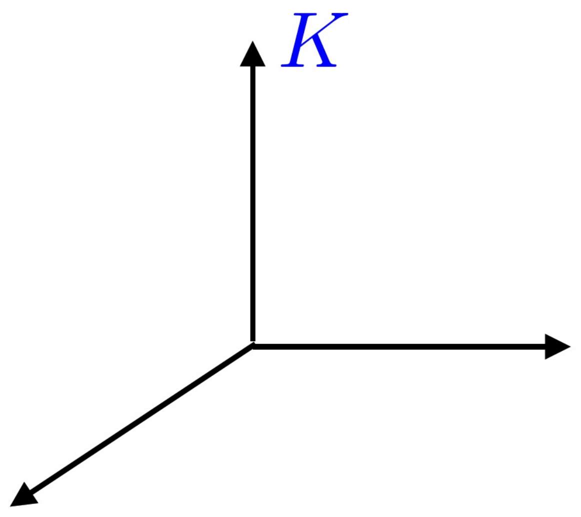

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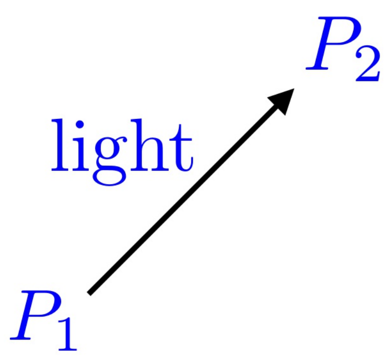

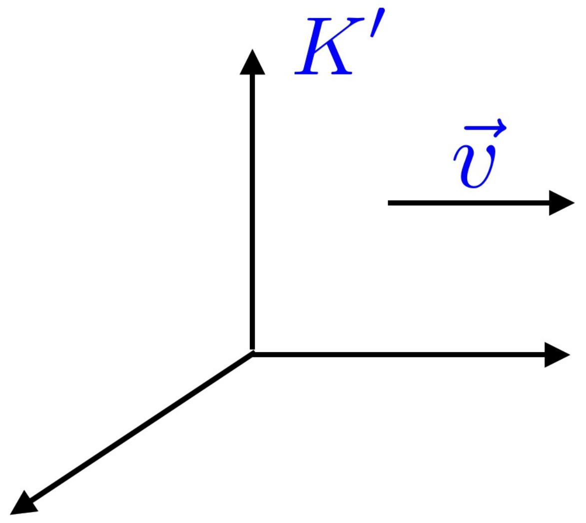

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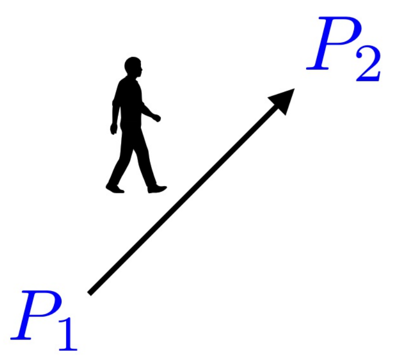

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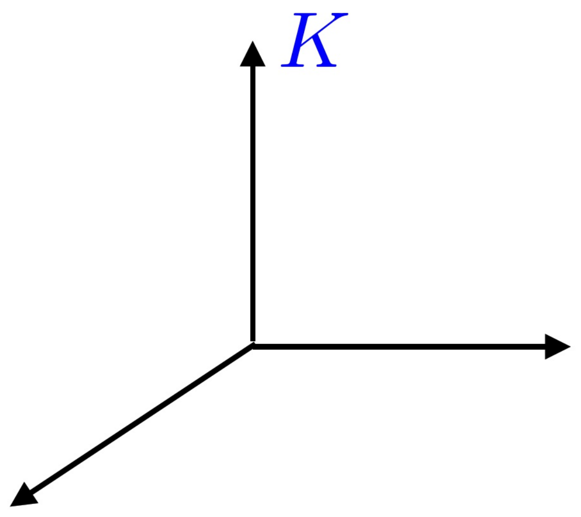

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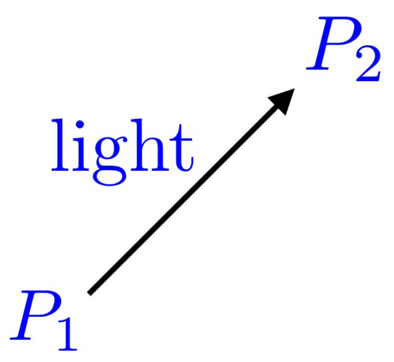

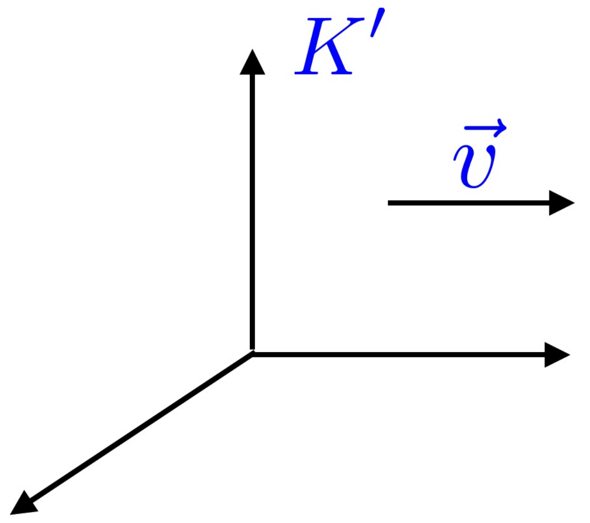

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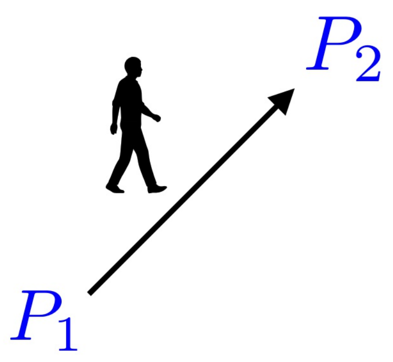

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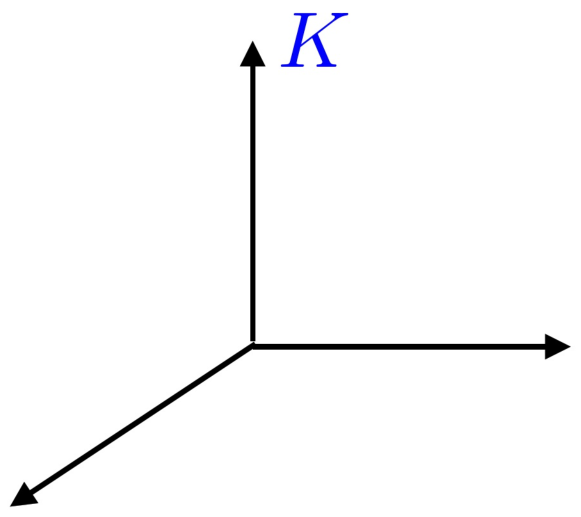

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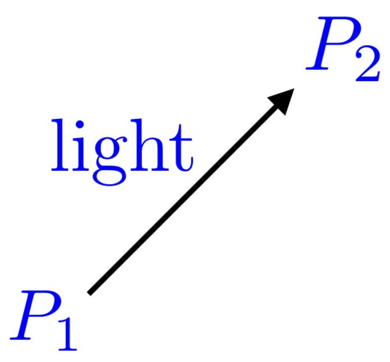

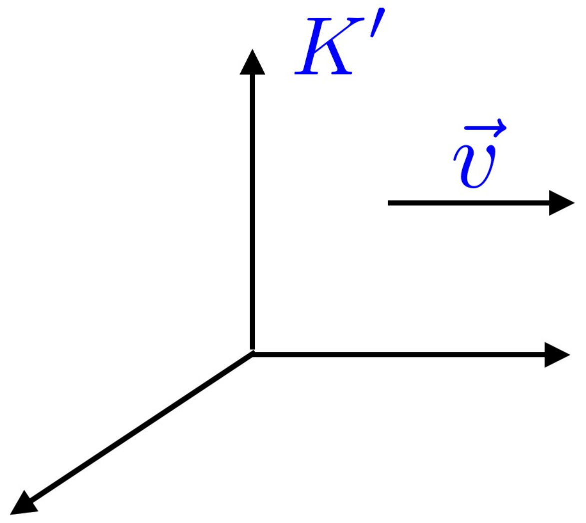

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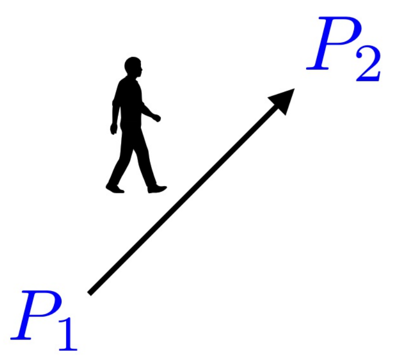

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$$dx'_0 = \frac{dx_0 + (v/c)\,dx_1}{\sqrt{1 - v^2/c^2}} \qquad dx'_1 = \frac{dx_1 + (v/c)\,dx_0}{\sqrt{1 - v^2/c^2}}$$

1 Lorentz transformations for classical fields

The Lorentz transformations are the linear global transformations ($\mu\,,\,\nu = 0, 1, 2, 3$)

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu} \qquad \text{or} \qquad x' = \Lambda \cdot x \tag{1.1}$$

that leave invariant the quadratic form

$$ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2} = \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \qquad \eta_{\mu\nu} = \text{diag.}\,(1, -1, -1, -1) \tag{1.2}$$

It then follows that Λ has to satisfy

$$\eta_{\rho\sigma}\, \Lambda^{\rho}_{\mu}\, \Lambda^{\sigma}_{\nu} = \eta_{\mu\nu} \qquad \text{or} \qquad \Lambda^T\, \eta\, \Lambda = \eta \tag{1.3}$$

We also have that

$$\frac{\partial}{\partial\, x'^{\mu}} = \frac{\partial\, x^{\nu}}{\partial\, x'^{\mu}}\, \frac{\partial}{\partial\, x^{\nu}} = \Lambda^{-1\nu}{}_{\mu}\, \frac{\partial}{\partial\, x^{\nu}} \qquad \text{or} \qquad \frac{\partial}{\partial\, x'} = \Lambda^{-1}\, \frac{\partial}{\partial\, x} \tag{1.4}$$

Among the Lorentz transformations we have the spatial rotations and the Lorentz boosts. The rotations on the planes $(1\,,\,2)$, $(2\,,\,3)$, and $(3\,,\,1)$, are respectively ($0 \leq \theta \leq 2\,\pi$)

$$\Lambda\,(R_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

$$\Lambda\,(R_{23}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{1.6}$$

$$\Lambda\,(R_{31}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 - \sin\theta & \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.7}$$

The Lorentz boosts on the planes $(0\,,\,i)$, $i = 1, 2, 3$, are ($-\infty < \alpha < \infty$)

$$\Lambda\,(B_{01}) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.8}$$

$$\Lambda\,(B_{02}) = \begin{pmatrix} \cosh\alpha & 0 & -\sinh\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\alpha & 0 & \cosh\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.9}$$

$$\Lambda\left(B_{03}\right)=\left(\begin{array}{cccc}\cosh\alpha & 0 & 0 & -\sinh\alpha\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sinh\alpha & 0 & 0 & \cosh\alpha\end{array}\right) \tag{1.10}$$

with α being the rapidity

$$\tanh\alpha=\frac{v}{c} \qquad \cosh\alpha=\frac{1}{\sqrt{1-v^2/c^2}} \qquad \sinh\alpha=\frac{v/c}{\sqrt{1-v^2/c^2}} \tag{1.11}$$

Note that the matrices $\Lambda\left(R_{ij}\right)$ e $\Lambda\left(B_{0i}\right)$ are real, and that $\Lambda\left(R_{ij}\right)$ are orthogonal ($\Lambda^T\left(R_{ij}\right)=\Lambda^{-1}\left(R_{ij}\right)$) and therefore are unitary ($\Lambda^{\dagger}\left(R_{ij}\right)=\Lambda^{-1}\left(R_{ij}\right)$). However, the matrices $\Lambda\left(B_{0i}\right)$ are symmetric and so are neither orthogonal or unitary. Consequently, such a vector representation, of dimension 4, of the Lorentz group is not unitary. That is a particular case of a more general fact: any finite dimensional representation of a non-compact (infinite volume) Lie group is necessarily non unitary. The unitary representations of the Lorentz group (and also of the Poincaré group) are infinite dimensional.

For an infinitesimal rotation we shall write

$$R_{ij}=1+i\,\theta\,\varepsilon_{ijk}\,J_k+O\left(\theta^2\right) \tag{1.12}$$

and for an infinitesimal boost

$$B_{0i}=1+i\,\alpha\,K_i+O\left(\alpha^2\right) \tag{1.13}$$

We then have

$$\Lambda\left(J_1\right)=i\left(\begin{array}{cccc}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{array}\right);\ \Lambda\left(J_2\right)=i\left(\begin{array}{cccc}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{array}\right);\ \Lambda\left(J_3\right)=i\left(\begin{array}{cccc}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right) \tag{1.14}$$

and

$$\Lambda\left(K_1\right)=i\left(\begin{array}{cccc}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right);\ \Lambda\left(K_2\right)=i\left(\begin{array}{cccc}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{array}\right);\ \Lambda\left(K_3\right)=i\left(\begin{array}{cccc}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{array}\right) \tag{1.15}$$

We then have the Lorentz algebra

$$\begin{aligned}\left[\,J_i\,,\,J_j\,\right] &= i\,\varepsilon_{ijk}\,J_k\\ \left[\,J_i\,,\,K_j\,\right] &= i\,\varepsilon_{ijk}\,K_k\\ \left[\,K_i\,,\,K_j\,\right] &= -i\,\varepsilon_{ijk}\,J_k\end{aligned} \tag{1.16}$$

We now introduce

$$N_i = \frac{1}{2}\left(J_i + i\,K_i\right) \;; \qquad \bar{N}_i = \frac{1}{2}\left(J_i - i\,K_i\right) \tag{1.17}$$

and so

$$\begin{aligned} \left[\,N_i\,,\,N_j\,\right] &= i\,\varepsilon_{ijk}\,N_k \\ \left[\,N_i\,,\,\bar{N}_j\,\right] &= 0 \\ \left[\,\bar{N}_i\,,\,\bar{N}_j\,\right] &= i\,\varepsilon_{ijk}\,\bar{N}_k \end{aligned} \tag{1.18}$$

We also have that

$$\Lambda\left(N_1\right) = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{pmatrix} \tag{1.19}$$

$$\Lambda\left(N_2\right) = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{pmatrix} \tag{1.20}$$

$$\Lambda\left(N_3\right) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \tag{1.21}$$

$$\Lambda\left(\bar{N}_1\right) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{pmatrix} \tag{1.22}$$

$$\Lambda\left(\bar{N}_2\right) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 \end{pmatrix} \tag{1.23}$$

$$\Lambda\left(\bar{N}_3\right) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \tag{1.24}$$

Note that

$$\Lambda\left(N_i\right)^2 = \frac{1}{4}\,\mathbb{1} \qquad\qquad \Lambda\left(\bar{N}_i\right)^2 = \frac{1}{4}\,\mathbb{1} \tag{1.25}$$

and so

$$\mathbb{1} \pm 2\,\Lambda\left(N_i\right) \qquad\qquad \text{and} \qquad\qquad \mathbb{1} \pm 2\,\Lambda\left(\bar{N}_i\right) \tag{1.26}$$

are projectors. In fact, if $\mid v\rangle$ is a given state of the representation, then the states

$$\mid v_\pm\rangle = \left(\mathbb{1} \pm 2\,\Lambda\left(N_i\right)\right)\mid v\rangle \qquad\qquad \mid \bar{v}_\pm\rangle = \left(\mathbb{1} \pm 2\,\Lambda\left(\bar{N}_i\right)\right)\mid v\rangle \tag{1.27}$$

are eigenstates

$$\Lambda\left(N_i\right)\mid v_\pm\rangle = \pm\frac{1}{2}\,\mid v_\pm\rangle \qquad\qquad \Lambda\left(\bar{N}_i\right)\mid \bar{v}_\pm\rangle = \pm\frac{1}{2}\,\mid \bar{v}_\pm\rangle \tag{1.28}$$

Of course, we can not diagonalize (have eigenstates) of all N_i's and $\bar{N}_i$'s at the same time. Usualy we diagonilize N_3 and $\bar{N}_3$.

1.1 The vector field A_μ

As an example take a vector field A_μ transforming as the derivative (see (1.4))

$$A'_\mu = \Lambda^{-1\,\nu}_{\quad\mu}\,A_\nu \qquad\qquad \text{or} \qquad\qquad A' = \Lambda^{-1}\cdot A \tag{1.29}$$

Let us denote

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{1.30}$$

We can construct the eigenstates of N_3 and $\bar{N}_3$ by using the projectors. However, note that $\Lambda\left(N_3\right)$ and $\Lambda\left(\bar{N}_3\right)$ mix the components $A_0 \leftrightarrow A_3$, and $A_1 \leftrightarrow A_2$, but not otherwise. Then we can split each eigenvector into two. So, we write

$$\begin{aligned} A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} &= \frac{\left(A_1 - i A_2\right)}{2}\,\mid 1/2\,,\,1/2\rangle + \frac{\left(A_1 + i A_2\right)}{2}\,\mid -1/2\,,\,-1/2\rangle \\ &+ \frac{\left(A_3 - A_0\right)}{2}\,\mid 1/2\,,\,-1/2\rangle + \frac{\left(A_3 + A_0\right)}{2}\,\mid -1/2\,,\,1/2\rangle \end{aligned} \tag{1.31}$$

with

$$\begin{aligned} \mid 1/2\,,\,1/2\rangle &= \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \qquad\qquad \mid 1/2\,,\,-1/2\rangle = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ \mid -1/2\,,\,1/2\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad\qquad \mid -1/2\,,\,-1/2\rangle = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \end{aligned} \tag{1.32}$$

One can check that

$$\Lambda\left(N_3\right) \mid s_1\,,\, s_2\rangle = s_1 \mid s_1\,,\, s_2\rangle \qquad \Lambda\left(\bar{N}_3\right) \mid s_1\,,\, s_2\rangle = s_2 \mid s_1\,,\, s_2\rangle \qquad s_a = \pm 1/2 \tag{1.33}$$

Therefore, A_μ transforms under the doublet of the $SU(2)$ generated by N_i, and also by the doublet of the $SU(2)$ generated by $\bar{N}_i$.

Note that under the $SO(3)$ (or $SU(2)$) group of spatial rotations, generated by $J_i = N_i + \bar{N}_i$, we have a triplet representation given by

$$\begin{aligned} \mid 1\rangle &= \frac{1}{2} \mid 1/2\,,\, 1/2\rangle \\ \mid 0\rangle &= \frac{1}{2} \left(\mid -1/2\,,\, 1/2\rangle + \mid 1/2\,,\, -1/2\rangle\right) && (1.34) \\ \mid -1\rangle &= \frac{1}{2} \mid -1/2\,,\, -1/2\rangle && (1.35) \end{aligned}$$

and a singlet given by

$$\mid 0\rangle_s = \frac{1}{2} \left(\mid -1/2\,,\, 1/2\rangle - \mid 1/2\,,\, -1/2\rangle\right) \tag{1.36}$$

Therefore, 3 degrees of freedom of A_μ correspond to a spin-one particle, and one degree of freedom to a spin-zero particle. Indeed, we can write

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \left(A_1 - iA_2\right) \mid 1\rangle + A_3 \mid 0\rangle + \left(A_1 + iA_2\right) \mid -1\rangle + A_0 \mid 0\rangle_s \tag{1.37}$$

1.2 Other fields

We give here the representations of the Lorentz group under which some other fields transform.

1. The first one is the real scalar field ϕ that transform under the scalar representations of the two $SU(2)$'s generated by N_i and $\bar{N}_i$, which we denote $(0\,,\,0)$:

   $$\mid \phi\rangle \equiv \mid 0\rangle \otimes \mid 0\rangle \tag{1.38}$$

   In the case of a complex scalar field $\phi = \phi_1 + i\phi_2$, the real and imaginary parts of the field transform under the scalar representation.

2. The spinors fields transform under the spinor representations of the Lorentz group. In fact, the Weyl left and right spinors ψ_L and ψ_R transform under the $(1/2\,,\,0)$ and $(0\,,\,1/2)$ representations respectively, i.e.

   $$\mid \psi_L\rangle = \mid \pm 1/2\rangle \otimes \mid 0\rangle \qquad \mid \psi_R\rangle = \mid 0\rangle \otimes \mid \pm 1/2\rangle \tag{1.39}$$

So, each one has two independent components. The Dirac spinor transforms under the $(1/2\,,\,0)+(0\,,\,1/2)$ representation, i.e.

$$\mid \psi\rangle = \mid \pm 1/2\rangle \otimes \mid 0\rangle + \mid 0\rangle \otimes \mid \pm 1/2\rangle \tag{1.40}$$

and so it has four independent components.

3. As we have seem above the vector field A_μ transforms under the $(1/2\,,\,1/2)$ representation, i.e.

$$\mid A_\mu\rangle \equiv \mid \pm 1/2\rangle \otimes \mid \pm 1/2\rangle \tag{1.41}$$

and so it has four independent components, and as representations of the rotation subgroup $SO(3)$ it has a spin 1 and a spin 0 components, i.e.

$$\mid A_\mu\rangle = \mid 0\rangle + \mid 0, \pm 1\rangle \tag{1.42}$$

4. The field tensor $F_{\mu\nu}$ or any other antisymmetric rank 2 tensor $B_{\mu\nu}$ transform under the $(1\,,\,0)+(0\,,\,1)$ representation, i.e.

$$\mid B_{\mu\nu}\rangle \equiv \mid 0, \pm 1\rangle \otimes \mid 0\rangle + \mid 0\rangle \otimes \mid 0, \pm 1\rangle \tag{1.43}$$

and so it has 6 independent components. Its self-dual and anti-self-dual components transform under the $(1\,,\,0)$ and $(0\,,\,1)$ respectively, i.e.

$$\mid B^{(+)}_{\mu\nu}\rangle \equiv \mid 0, \pm 1\rangle \otimes \mid 0\rangle \qquad\qquad \mid B^{(-)}_{\mu\nu}\rangle \equiv \mid 0\rangle \otimes \mid 0, \pm 1\rangle \tag{1.44}$$

with

$$B^{(\pm)}_{\mu\nu} = \frac{1}{2}\left(\eta_{\mu\rho}\,\eta_{\nu\sigma} \pm \frac{i}{2}\,\varepsilon_{\mu\nu\rho\sigma}\right) B^{\rho\sigma} \tag{1.45}$$

where $\eta_{\mu\nu} = \text{diag.}\,(1\,,\,-1\,,\,-1\,,\,-1)$ is the Minkowski metric and $\varepsilon_{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor with $\varepsilon_{0123} = 1$.

5. The metric field described by a symmetric matrix $g_{\mu\nu}$, transforms under the representation $(1\,,\,1)+(0\,,\,0)$ of the Lorentz group, i.e.

$$\mid g_{\mu\nu}\rangle \equiv \mid 0, \pm 1\rangle \otimes \mid 0, \pm 1\rangle + \mid 0\rangle \otimes \mid 0\rangle \tag{1.46}$$

and so it has 10 independent components. In terms of representations of the rotation subgroup $SO(3)$ it decomposes into a spin 2, a spin 1 and 2 spin 0 representations, i.e.

$$\mid g_{\mu\nu}\rangle \equiv = \mid 0, \pm 1, \pm 2\rangle + \mid 0, \pm 1\rangle + \mid 0\rangle + \mid 0\rangle \tag{1.47}$$

The particle associated to the graviton itself has spin 2 and so only 5 independent components. The extra 5 degrees of freedom can be eliminated by the 5 Lorentz covariant conditions

$$g^{\mu\nu}{}_{,\nu} = 0 \qquad\qquad g^{\mu}_{\mu} = 0 \tag{1.48}$$

2 The Poincaré group

The Poincaré group is an extension of the Lorentz group by the addition of the space-time translations

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu} + a^{\mu} \qquad \text{or} \qquad x' = \Lambda \cdot x + a \tag{2.49}$$

It can be realised in matrix notation as

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \\ 1 \end{pmatrix} = \left(\begin{array}{c|c} & a^0 \\ \Lambda & a^1 \\ & a^2 \\ & a^3 \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} \tag{2.50}$$

For infinitesimal translations we have

$$T_{\mu} = 1 + \varepsilon\, P_{\mu} + O\left(\varepsilon^2\right) \tag{2.51}$$

The generators of infinitesimal translations being

$$P_0 = \left(\begin{array}{c|c} & 1 \\ 0 & 0 \\ & 0 \\ & 0 \\ \hline 0 & 0 \end{array}\right) \quad P_1 = \left(\begin{array}{c|c} & 0 \\ 0 & 1 \\ & 0 \\ & 0 \\ \hline 0 & 0 \end{array}\right) \quad P_2 = \left(\begin{array}{c|c} & 0 \\ 0 & 0 \\ & 1 \\ & 0 \\ \hline 0 & 0 \end{array}\right) \quad P_3 = \left(\begin{array}{c|c} & 0 \\ 0 & 0 \\ & 0 \\ & 1 \\ \hline 0 & 0 \end{array}\right)$$

Adding a fifth row and column of zeros to the matrices (1.14) and (1.15) we get that

$$[\, J_i\,,\, P_0\,] = 0 \qquad\qquad [\, J_i\,,\, P_j\,] = i\,\varepsilon_{ijk}\, P_k \tag{2.52}$$

and

$$[\, K_i\,,\, P_0\,] = -i\, P_i \qquad\qquad [\, K_i\,,\, P_j\,] = -i\,\delta_{ij}\, P_0 \tag{2.53}$$

In addition,

$$[\, P_{\mu}\,,\, P_{\nu}\,] = 0 \tag{2.54}$$

So, the translations constitute an abelian invariant sub-algebra of the Poincaré Lie algebra, and so the Poincaré algebra and the Poincaré group are not semisimple.

One can use a four dimensional notation and introduce the antisymmetric generators $M_{\mu\nu}$ as

$$K_i \equiv M_{0i} \qquad\qquad J_i \equiv \frac{1}{2}\,\varepsilon_{ijk}\, M_{jk} \tag{2.55}$$

The commutation relations for the Lorentz group become

$$[\, M_{\mu\nu}\,,\, M_{\rho\sigma}\,] = i\,\eta_{\nu\rho}\, M_{\mu\sigma} - i\,\eta_{\mu\rho}\, M_{\nu\sigma} - i\,\eta_{\nu\sigma}\, M_{\mu\rho} + i\,\eta_{\mu\sigma}\, M_{\nu\rho} \tag{2.56}$$

The commutation relations (2.52) and (2.53) for the Poincaré group become

$$[\, M_{\mu\nu}\,,\, P_{\rho}\,] = -i\,\eta_{\mu\rho}\, P_{\nu} + i\,\eta_{\nu\rho}\, P_{\mu} \tag{2.57}$$

The Poicaré group has two Casimir operators. The first one in the square of momenta

$$P^2 = p_[\mu \, P^\mu \tag{2.58}$$

One can check that

$$\Big[P^2 \, , \, M_{\mu\nu} \Big] = \Big[P^2 \, , \, P_\mu \Big] = 0 \tag{2.59}$$

The other Casimir is less trivial and it is constructed form the so-called Pauli-Lubansky vector

$$W_\mu \equiv \frac{1}{2} \, \varepsilon_{\mu\nu\rho\sigma} \, P^\nu \, M^{\rho\sigma} \tag{2.60}$$

which satisfies

$$[\, M_{\mu\nu} \, , \, W_\rho \,] = -i \, \eta_{\mu\rho} \, W_\nu + i \, \eta_{\nu\rho} \, W_\mu \qquad\qquad [\, W_\mu \, , \, P_\nu \,] = 0 \tag{2.61}$$

The second Casimir operator is

$$W^2 = W_\mu \, W^\mu \tag{2.62}$$

which satisfies

$$\Big[W^2 \, , \, M_{\mu\nu} \Big] = \Big[W^2 \, , \, P_\mu \Big] = 0 \tag{2.63}$$

The representations of the Poincaré group have studied by Eugene P. Wigner and fall into three classes[1]:

1. $P^2 = m^2 \geq 0$, (m^2 real and positive), and $W^2 = -m^2 \, s \, (s+1)$, where s is the spin and $s = 0 \, , \, \frac{1}{2} \, , \, 1 \, , \, \frac{3}{2} \, , \, 2 \ldots$. The representation is labelled by the mass m and the spin s, and the states are labelled the third eigenvalues s_3 of S_3 component of the spin operator, and $s_3 = -s \, , \, -s+1 \ldots s-1 \, , \, s$, and the continuous eigenvalues of P_i. Massive particle have therefore $2\,s+1$ degrees of freedom.

2. $P^2 = 0$ and $W^2 = 0$. Since $P_\mu \, W^\mu = 0$, it follows that P_μ and W_μ are proportional, with the constant of proportionality called helicity, and has values $\pm s$, where s is the spin of the representation, and $s = 0 \, , \, \frac{1}{2} \, , \, 1 \, , \, \frac{3}{2} \, , \, 2 \ldots$. Therefore massless particles with $s \neq 0$ have only 2 degrees of freedom. Note that

$$\begin{aligned} P_0^2 - \vec{P}^2 &= 0 \quad \rightarrow \quad P_0^2 \vec{P}^2 \quad \rightarrow \quad P_0 = \varepsilon_1 \mid \vec{P} \mid \qquad \varepsilon_1 = \pm 1 \\ W_0^2 - \vec{W}^2 &= 0 \quad \rightarrow \quad W_0^2 \vec{W}^2 \quad \rightarrow \quad W_0 = \varepsilon_2 \mid \vec{W} \mid \qquad \varepsilon_2 = \pm 1 \\ P_0 \, W_0 - \vec{P} \cdot \vec{W} &= 0 \quad \rightarrow \quad \vec{P} \cdot \vec{W} = \varepsilon_1 \, \varepsilon_2 \mid \vec{P} \mid \mid \vec{W} \mid \quad \rightarrow \quad \cos\theta = \varepsilon_1 \, \varepsilon_2 = \pm 1 \end{aligned}$$

So

$$\vec{W} = \lambda \, \vec{P} \qquad \rightarrow \qquad W_0 = \varepsilon_1 \, \varepsilon_2 \mid \lambda \mid \, P_0 \tag{2.64}$$

and

$$\vec{P} \cdot \vec{W} = \lambda \, \vec{P}^2 = \mid \vec{P} \mid \mid \vec{W} \mid \cos\theta = \mid \lambda \mid \, \vec{P}^2 \cos\theta \quad \rightarrow \quad \lambda = \varepsilon_1 \, \varepsilon_2 \mid \lambda \mid \quad \rightarrow \quad W_0 = \lambda P_0$$

[1] Eugene P. Wigner, Reviews of Modern Physics, vol. 29, n. 3, 255 (1957)

So

$$W_\mu = \lambda\, P_\mu \tag{2.65}$$

But

$$W^0 = \frac{1}{2}\varepsilon^{0ijk}\, P_i\, M_{jk} = \vec{P}\cdot\vec{J} = \lambda\, P^0 = \varepsilon_1 \mid \lambda \mid\mid \vec{P} \mid \quad \rightarrow \quad \lambda = \varepsilon_1 \frac{\vec{P}\cdot\vec{J}}{\mid \vec{P} \mid} \tag{2.66}$$

3. $P^2 = 0$ but the spin is continuous. The length of W_μ is minus the square of a positive number. Such a representation describes a massless particle with an infinite number of polarization states labeled by a continuous variable. A s far we know it is not realisable in Nature.