40 years of robust control: 1978-2018

2014 American Control Conference

Schedule

8:30-9:10, *Universal laws and architectures* (John Doyle)

9:10-9:35, *Robustness analysis, μ* (Andy Packard)

9:35-10:00, *Multivariable system theory* (Keith Glover)

10:00-10:30, break

10:30-11:00, *H_∞ and H_∞ loopshaping* (Keith Glover)

11:00-11:30, *Signal-weighted design and DK iteration* (Gary Balas)

11:30-12:00, *Design examples* (Roy Smith, Pete Seiler, Gary Balas)

12:00-12:20, *Model Reduction* (Keith Glover)

12:30-1:30, lunch break

1:30-2:00, *Advanced design formulations* (Roy Smith)

2:00-2:30, *Automated tuning of fixed architecture controllers* (Pascal Gahinet)

2:30-3:00, *Integral Quadratic Constraints* (Pete Seiler)

3:00-3:30, break

3:30-4:00, *Robust MPC* (Francesco Borrelli)

4:00-4:30, *Decentralized optimal control* (Laurent Lessard)

4:30-5:30, *2014-2018: what's needed* (John Doyle)

Universal laws and architectures: origins of robust control

John Doyle 道陽
Jean-Lou Chameau Professor
Control and Dynamical Systems, EE, & BioE

Ca#1tech

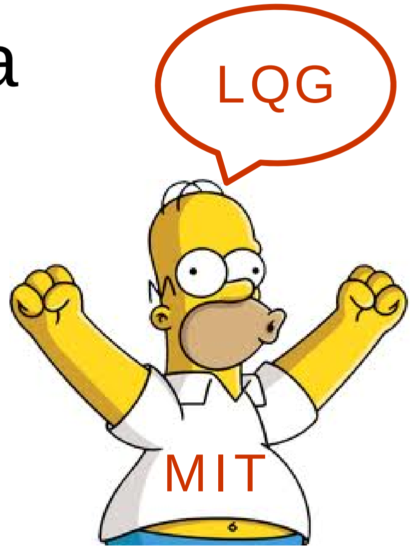
Better, faster, cheaper

Circa late 70s

Trends in the 1970s

- Improved efficiency and performance
- Instability
- Modern control said “no problem”
- Solvable iff stabilizable+detectable+LQG
- Talk math to engineers, and vice versa

- What could go wrong?



Requirements on systems and architectures

accessible
accountable
accurate
adaptable
administrable
affordable
auditable
autonomy
available
credible
process capable
compatible
composable
configurable
correctness
customizable
debugable
degradable
determinable
demonstrable
dependable
deployable
discoverable
distributable
durable
effective
efficient
evolvable
extensible
fail transparent
fast
fault-tolerant
fidelity
flexible
inspectable
installable
Integrity
interchangeable
interoperable
learnable
maintainable
manageable
mobile
modifiable
modular
nomadic
operable
orthogonality
portable
precision
predictable
producible
provable
recoverable
relevant
reliable
repeatable
reproducible
resilient
responsive
reusable
robust
safety
scalable
seamless
self-sustainable
serviceable
supportable
securable
simplicity
stable
standards compliant
survivable
sustainable
tailorable
testable
timely
traceable
ubiquitous
understandable
upgradable
usable

Sustainable ≈ robust + efficient

accessible
accountable
accurate
adaptable
administrable
affordable
auditable
autonomy
available
compatible
composable
configurable
correctness
customizable
debugable
degradable
determinable
demonstrable

dependable
deployable
discoverable
distributable
durable
effective
efficient
evolvable
extensible
fail transparent
fast
fault-tolerant
fidelity
flexible
inspectable
installable
Integrity
interchangeable
interoperable
learnable
maintainable

manageable
mobile
modifiable
modular
nomadic
operable
orthogonality
portable
precision
predictable
producible
provable
recoverable
relevant
reliable
repeatable
reproducible
resilient
responsive
reusable
robust

safety
scalable
seamless
self-sustainable
serviceable
supportable
securable
simple
stable
standards
survivable
sustainable
tailorable
testable
timely
traceable
ubiquitous
understandable
upgradable
usable

PCA ≈ Principal ***Concept*** Analysis ☺

accessible
accountable
accurate
adaptable
adminis
afforda
auditab
autono
availab
compat
compos
configu
correct
custom
debuga
degrad
determ
demons

dependable
deployable
discoverable
distributable
interoperable
learnable
maintainable

manageable
mobile
modifiable
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reusable
robust

safety
scalable
seamless
self-sustainable
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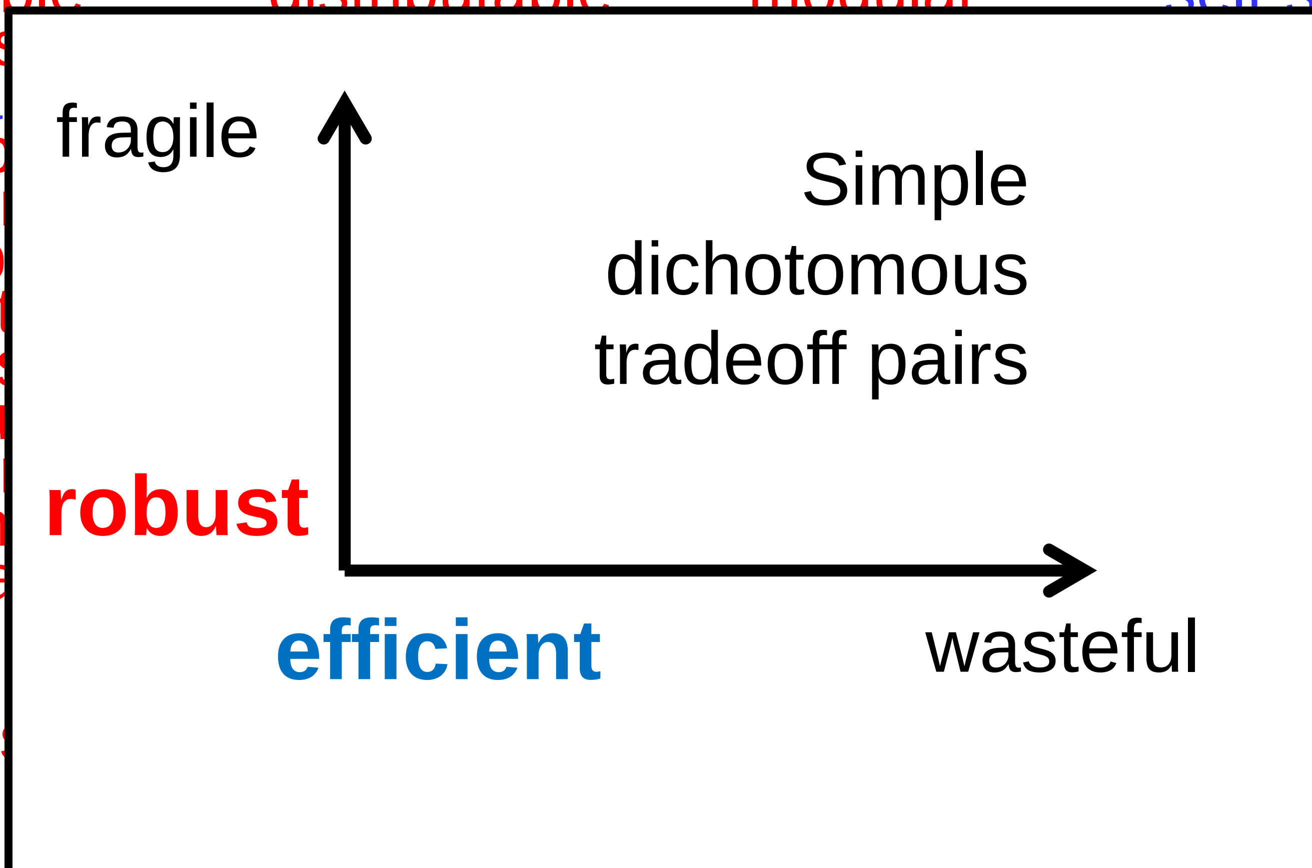
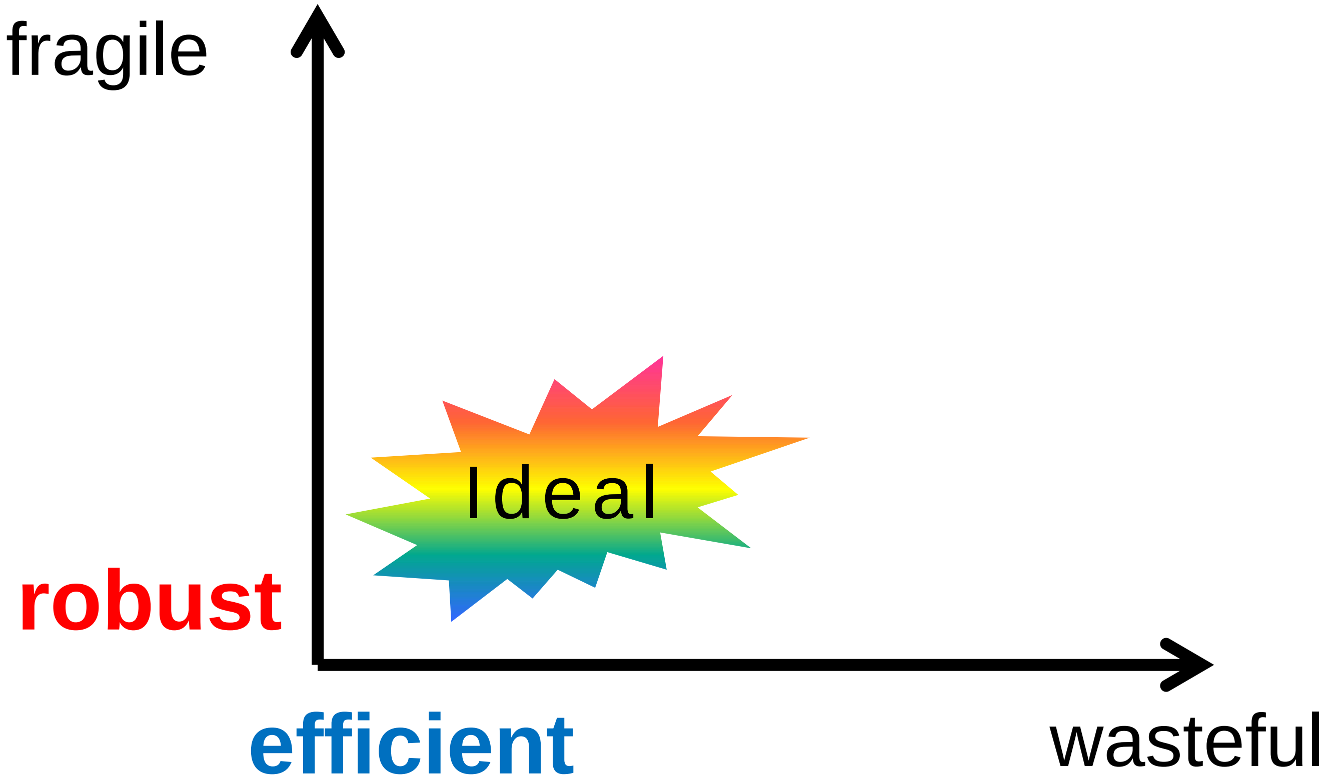



fragile
Ideal
robust
efficient
wasteful



Better, faster, cheaper

Circa late 70s

robust

efficient

wasteful

Robustness?

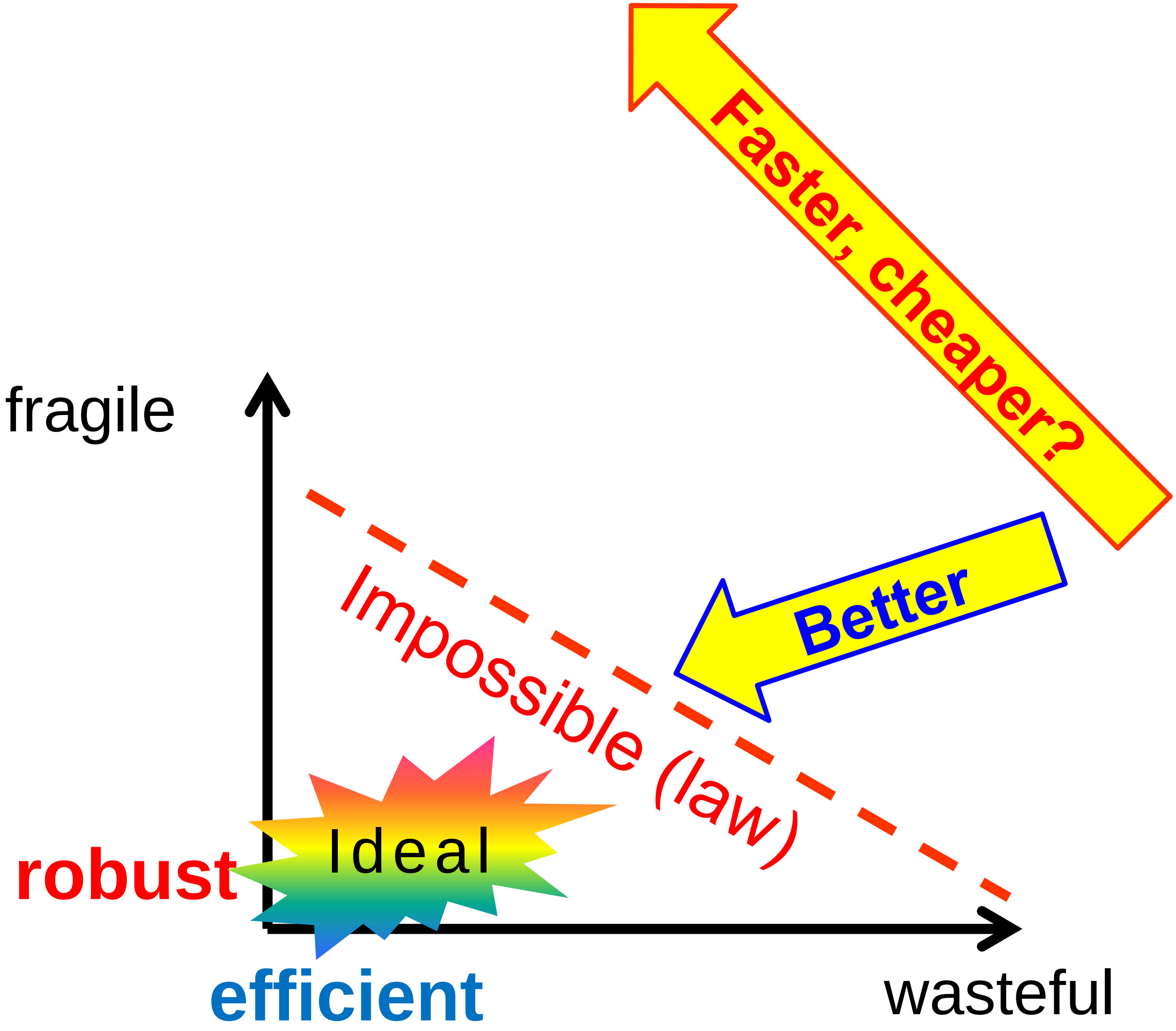
Fragile?

Robust?

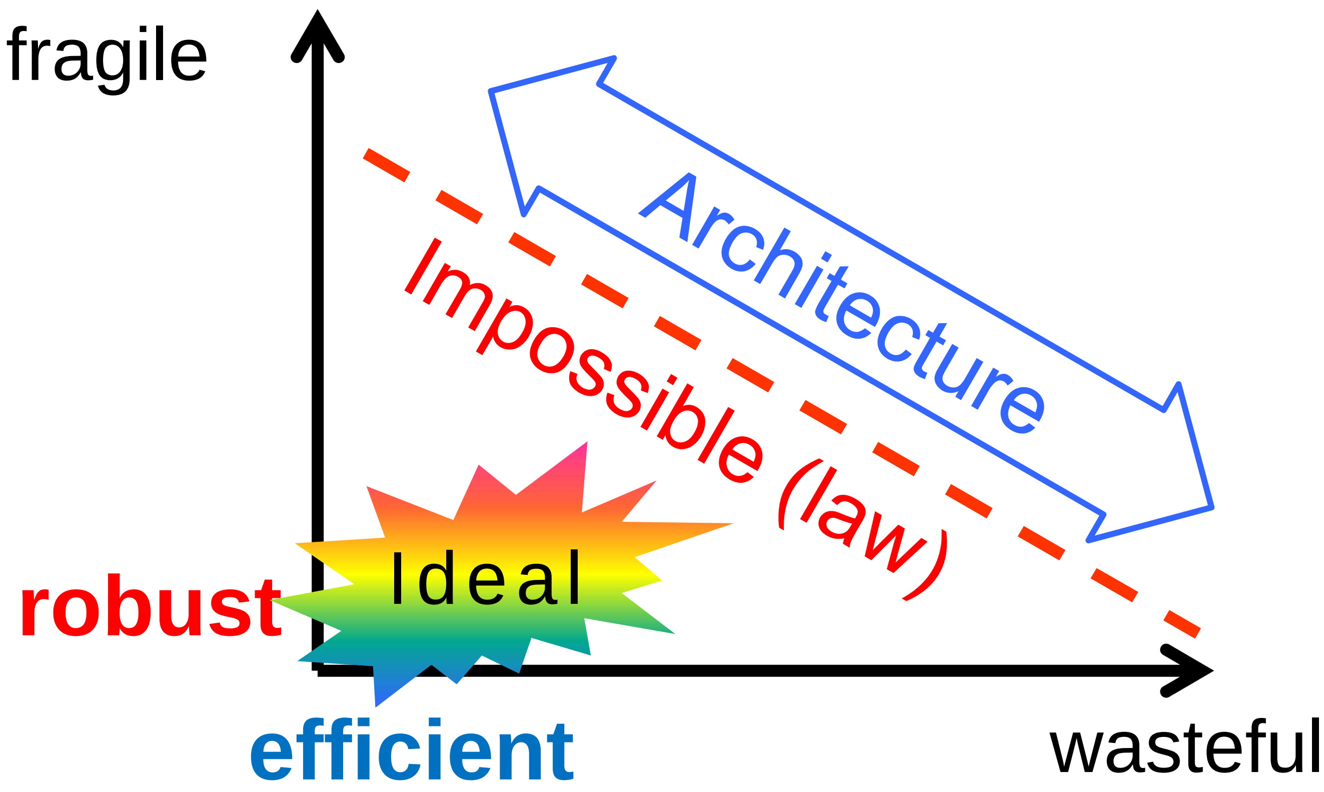
- Modern control said “no problem”
- Solvable iff

  stabilizable+detectable+LQG
- “Guaranteed margins”
- Talk math to engineers & vice versa

- Dissent at fringe (Zames, Horowitz)
- What could go wrong?


Faster, cheaper?
fragile
Better
Impossible (law)
Ideal
robust
efficient
wasteful

Universal laws and architectures



Early Influences (Thanks)

- MIT: Mitter, Sandell, Gould, Safonov, …
- Honeywell: Stein, Wall, Enns, Freudenberg, …
- Zames, Horowitz, Astrom, …
- Glover
- Khargonekar, Francis, Kimura,…
- Berkeley: Sarason, Boyd, Packard, Gohberg, …
- 1981 NATO tour : w/ Stein, Zames, Willems, Wonham, MacFarlane
- 1984 ONR/Honeywell Workshop

Counterexamples and issues?

- Can anything have “guaranteed margins”?
  - No, not in general
  - Depends on plant (d’oh)

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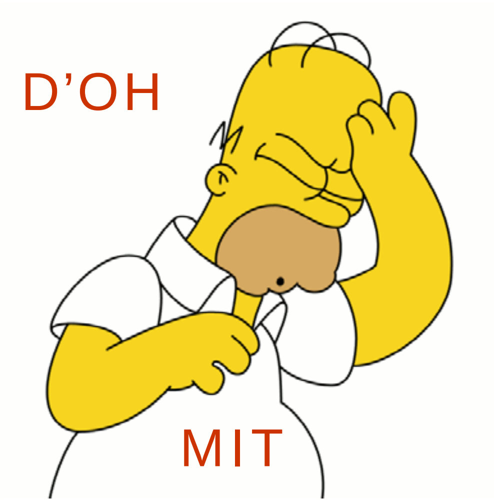
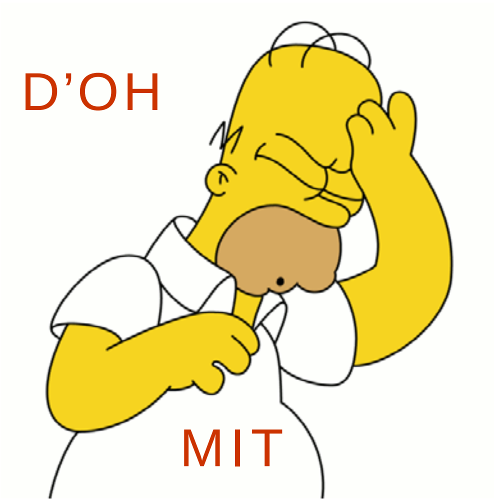
- Is LQG (H_2) special?
  - Yes, it can be gratuitously fragile
  - OK, it isn't completely useless
  - There are tweaks (LTR) that help

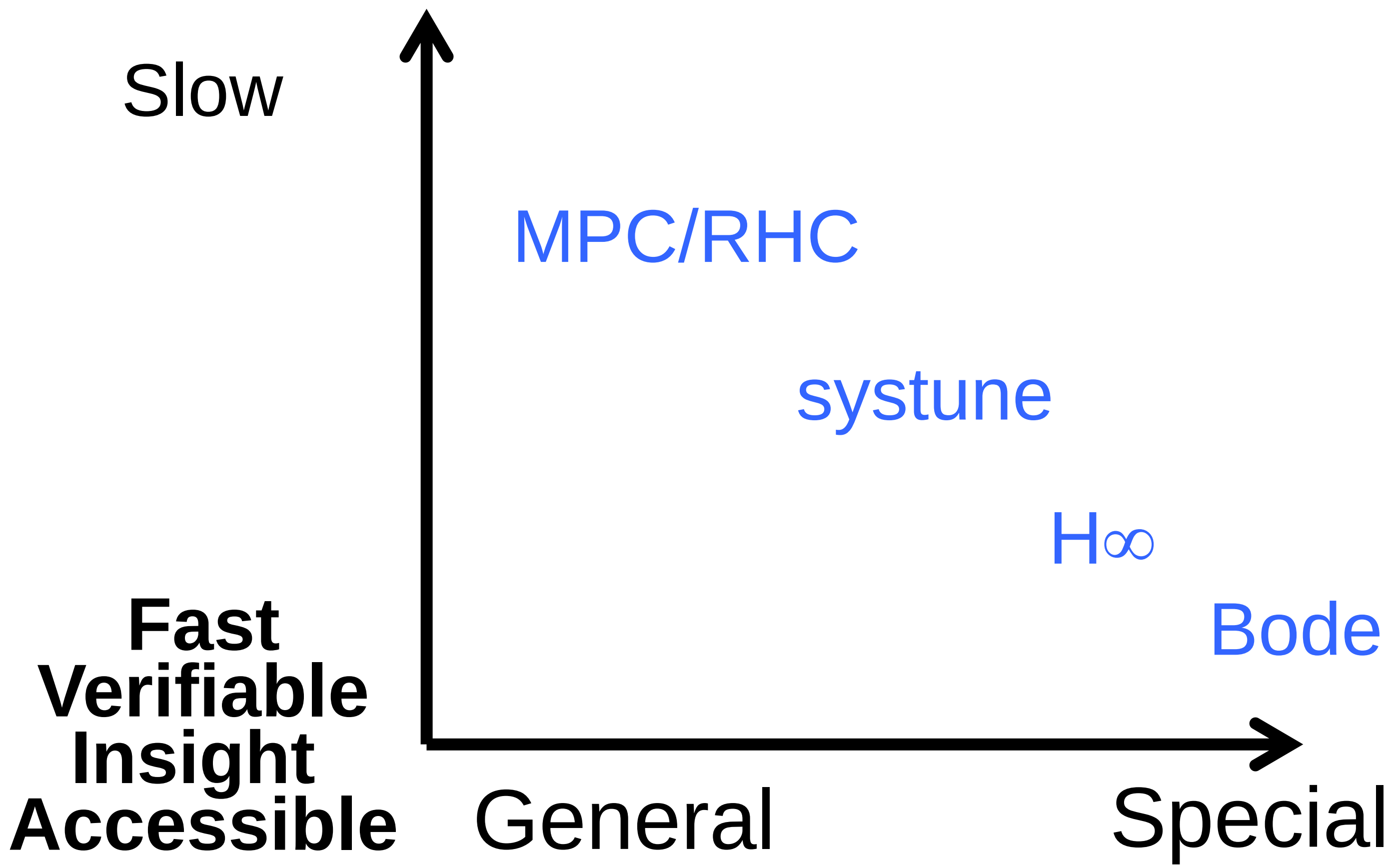


Counterexamples and issues?

- Can anything have "guaranteed margins"?
  - No, not in general
  - Depends on plant (d'oh)
- Is LQG (H2) special?
  - Yes, it can be gratuitously fragile
  - OK, it isn't completely useless
  - There are tweaks (LTR) that help
- Are μ and H_∞ a panacea for everything?
  - Yes! Or so it seemed at the time?
  - No! See everything else today, including me



Today: μ to H_∞ to systune to MPC



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- ebates over riable")
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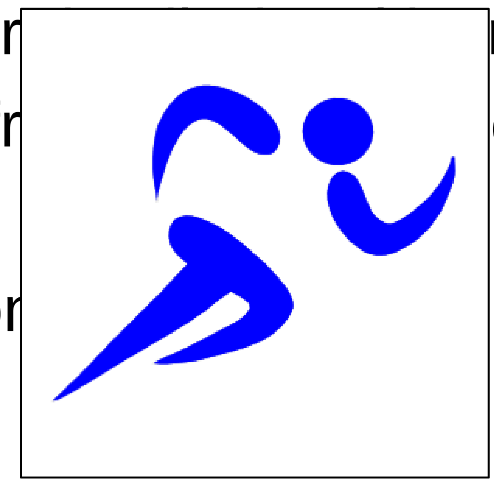
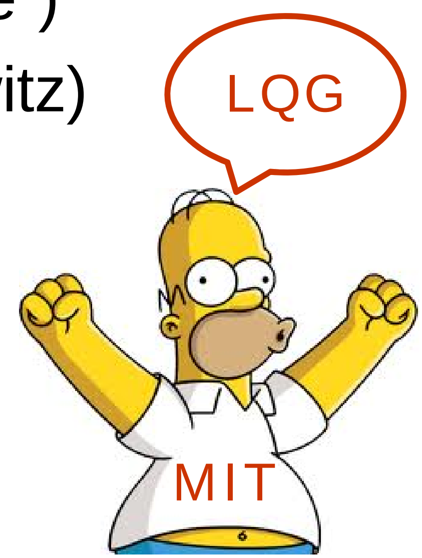
- What could go wror

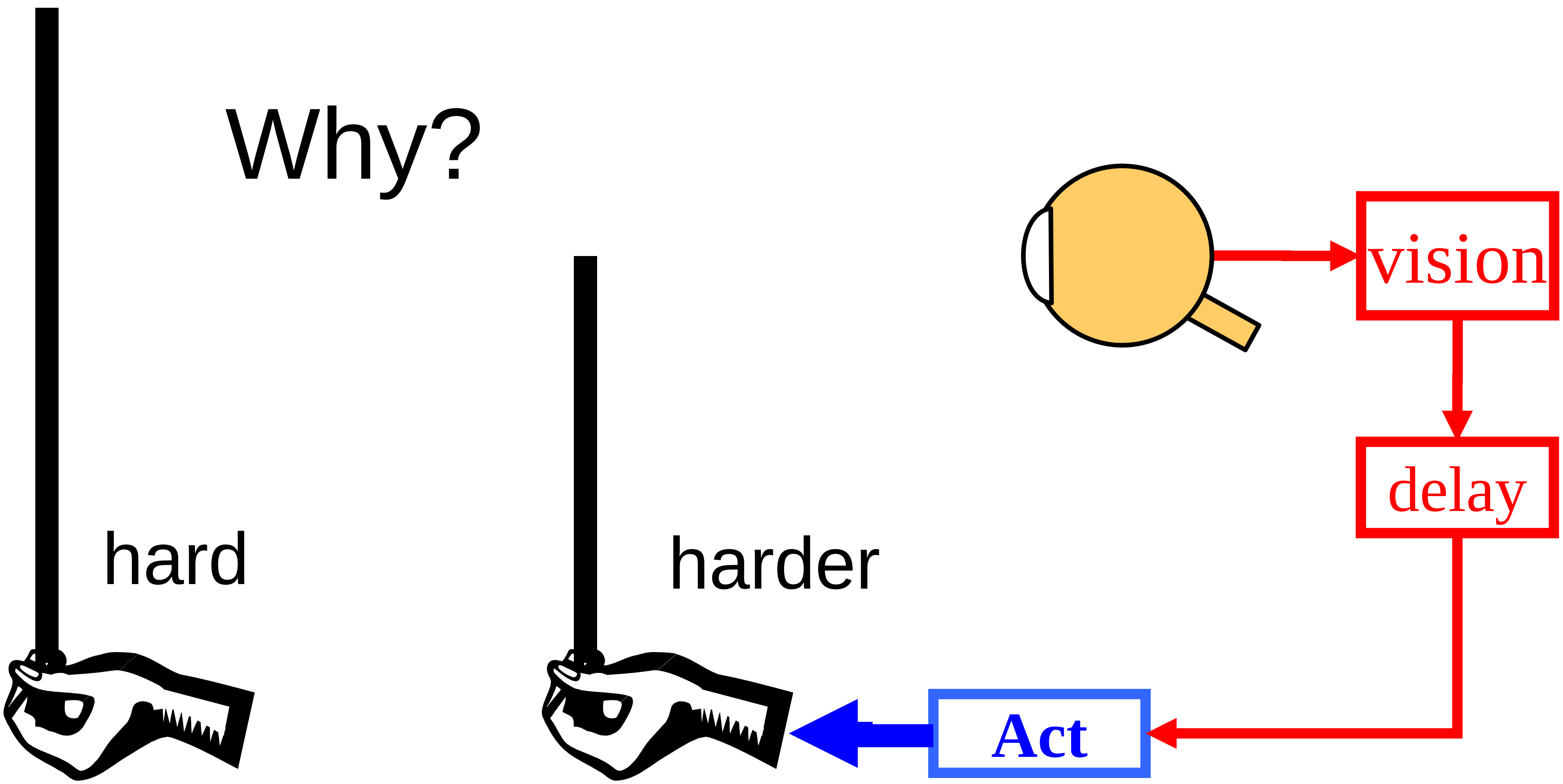
What I would have done differently



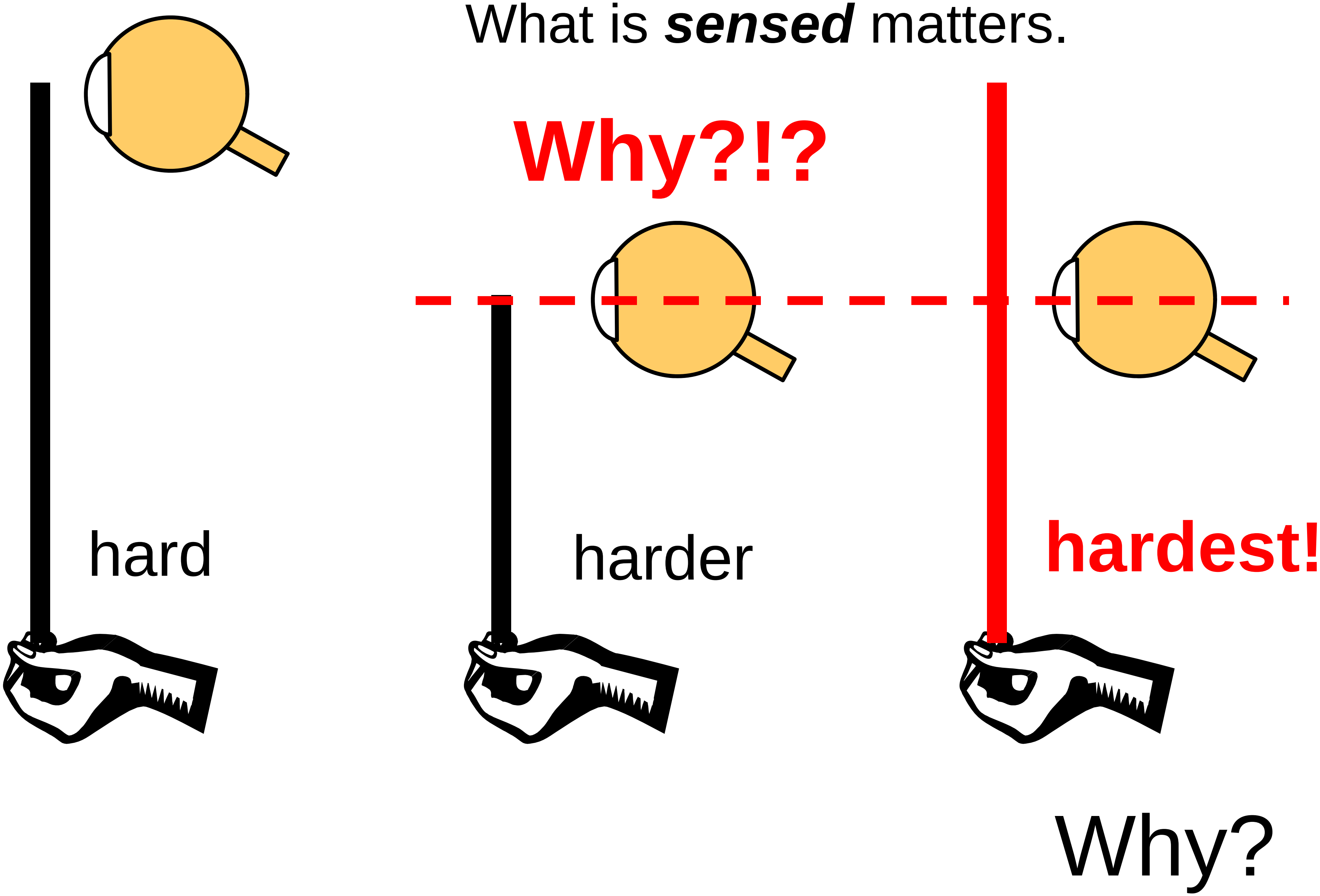


Respect the unstable.
Gunter Stein

Respect Gunter Stein.
The Unstable


Why?
hard
harder
vision
delay
Act

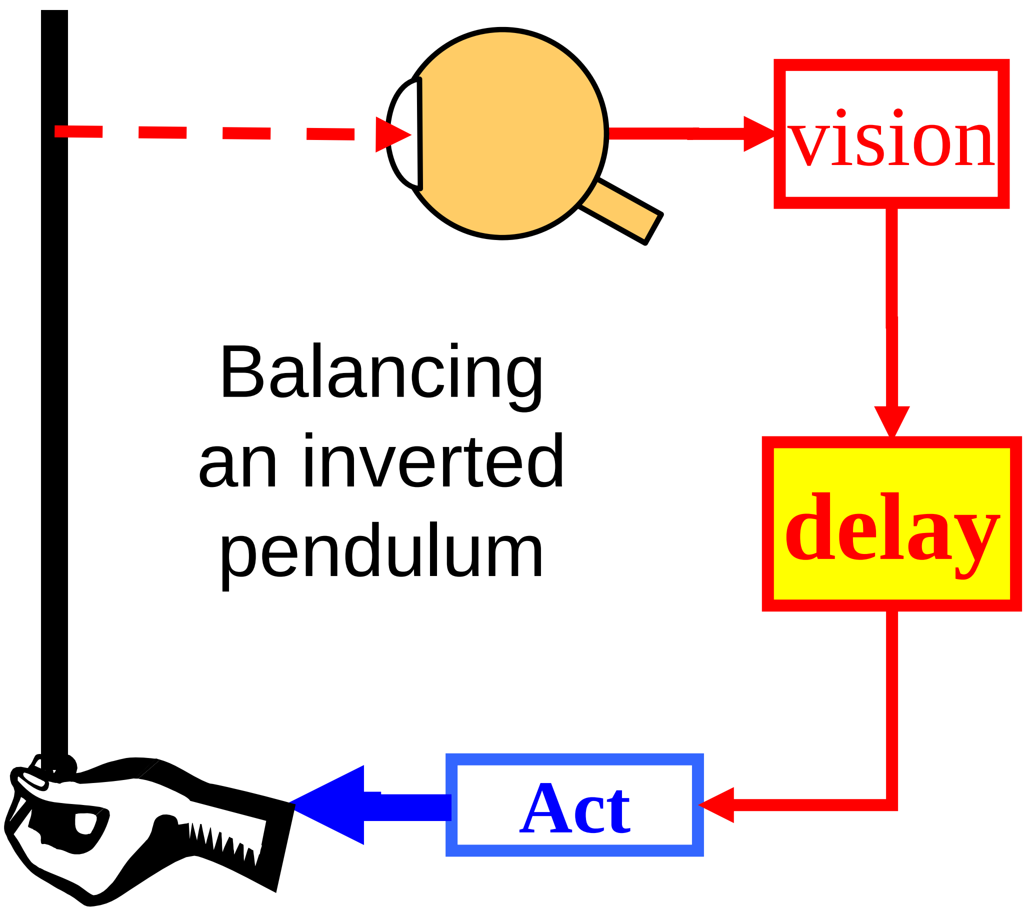
What is ***sensed*** matters.



Easy to ***prove*** using simple models.

Universal laws

Mechanics+
Gravity +
Light +



Balancing
an inverted
pendulum

Control theory
+ Neuroscience

Crashes ***can*** be made rare with active control.

Law #1 : Mechanics
Law #2 : Gravity

Gravity is stabilizing

Gravity is destabilizing

More unstable

Efficiency/instability/layers/feedback

- All create new efficiencies but also instabilities
- Requires new active/layered/complex/active control

- Money/finance/lobbyists/etc
- Society/agriculture/weapons/etc
- Bipedalism
- Maternal care
- Warm blood
- Flight
- Mitochondria
- Translation (ribosomes)
- Glycolysis (2011 *Science*)

Major transitions

Efficiency/instability/layers/feedback

- All create new efficiencies but also instabilities
- Requires new active/layered/complex/active control

- Money/finance/lobbyists/etc
- Society/agriculture/weapons/etc
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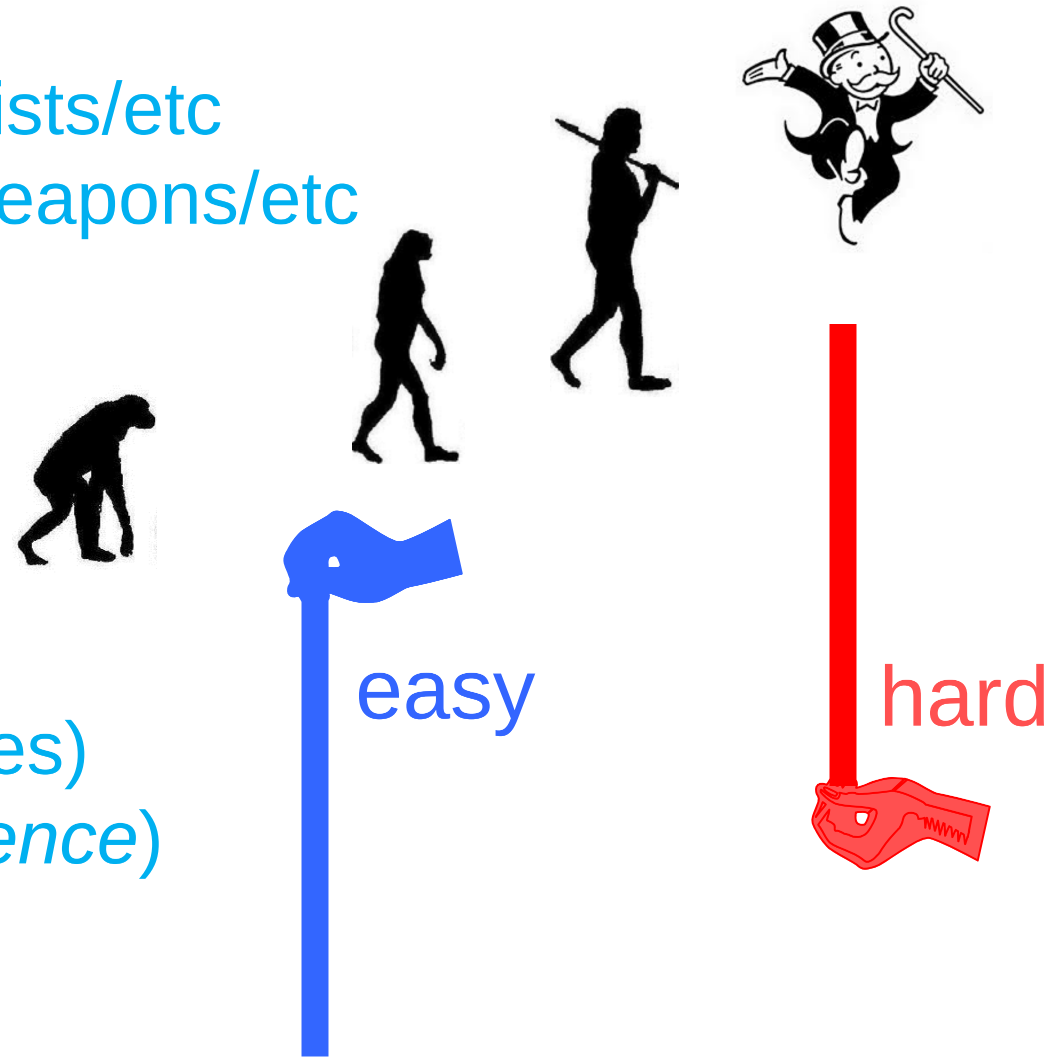
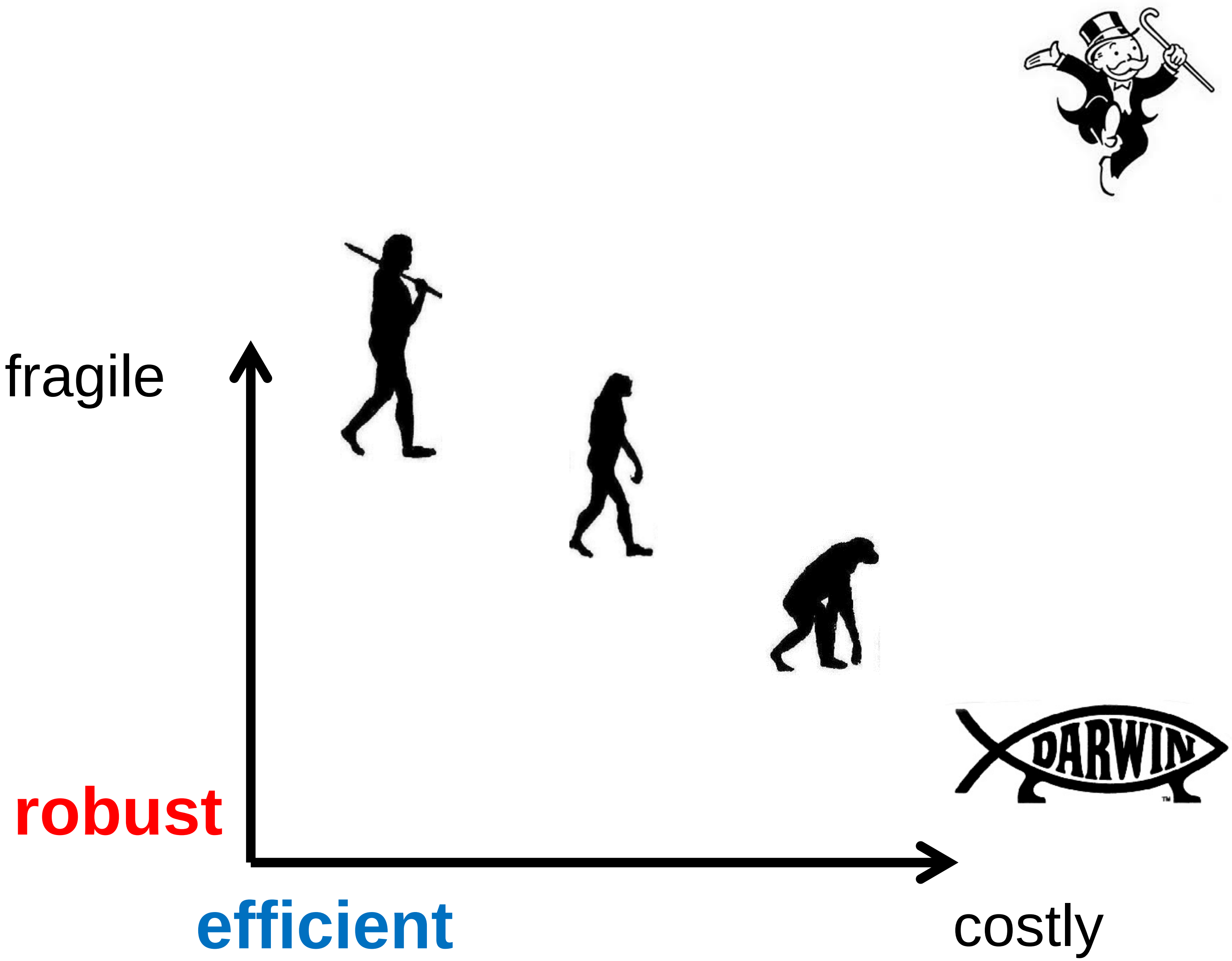
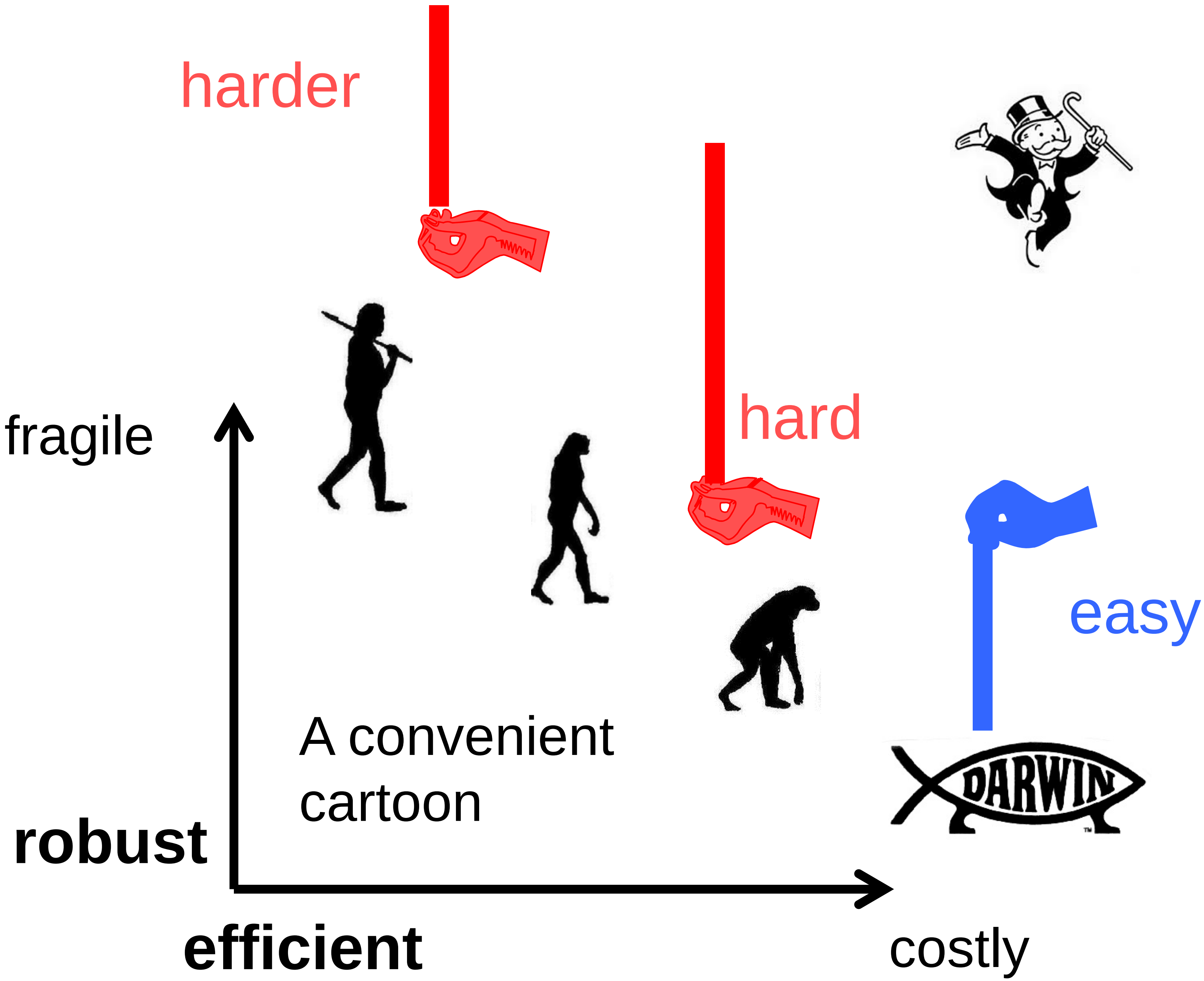
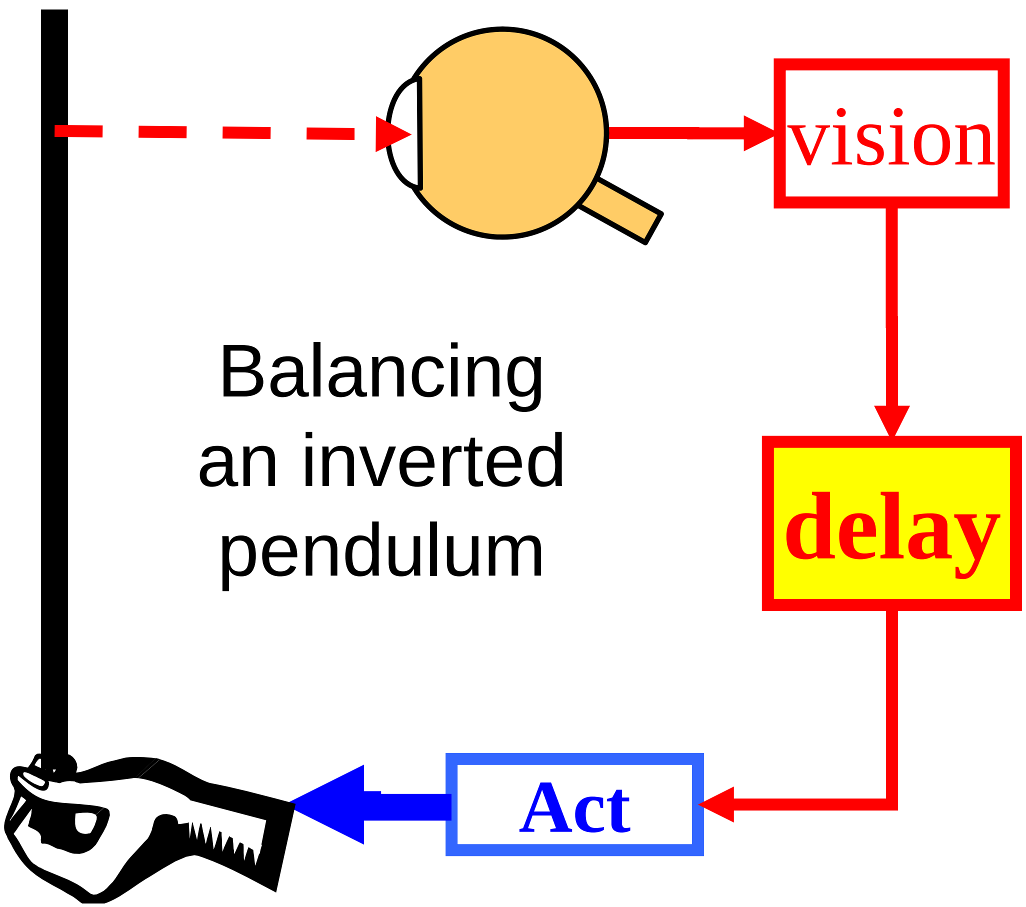



fragile
robust
efficient
costly
DARWIN


harder
hard
easy
fragile
robust
A convenient cartoon
DARWIN
efficient
costly

Some minimal math details

Universal laws



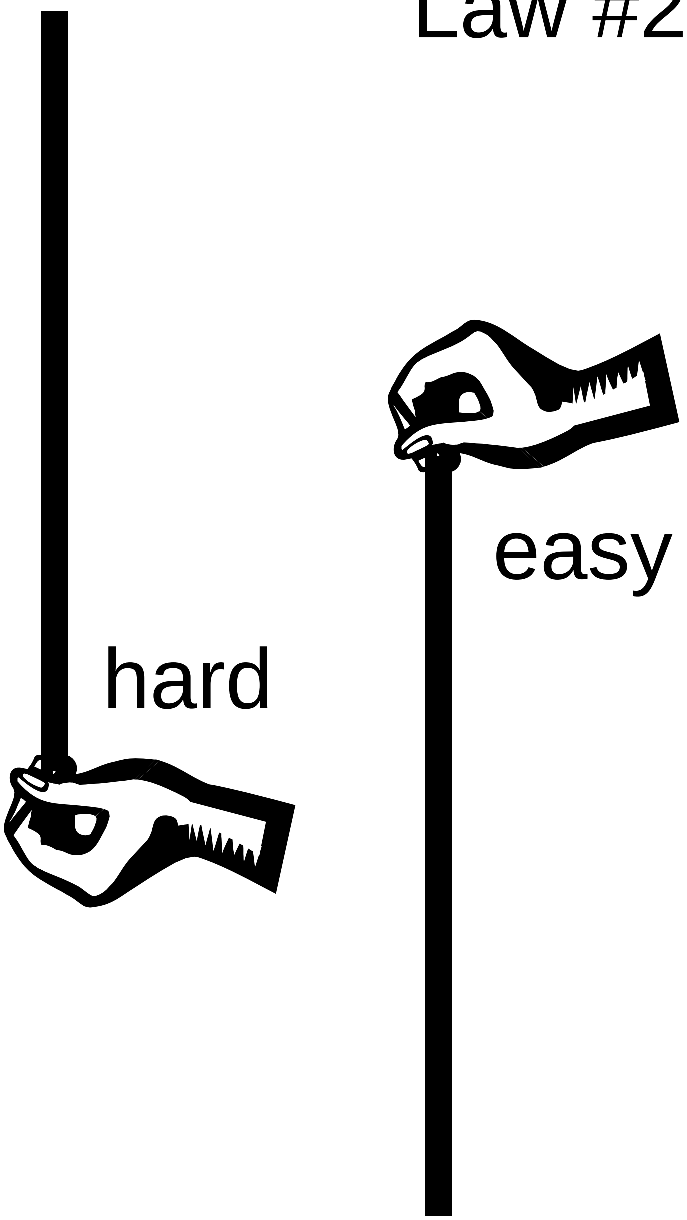
Mechanics+
Gravity +
Light +

$$\|T\|_{\infty} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$

+ Neuroscience

Law #1 : Mechanics

Law #2 : Gravity



$$(M+m)\ddot{x} + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u$$

$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0$$

$$y = x + \alpha l \sin\theta$$

linearize

$$(M+m)\ddot{x} + ml\ddot{\theta} = u$$

$$\ddot{x} + l\ddot{\theta} \pm g\theta = 0$$

$$y = x + \alpha l\theta$$

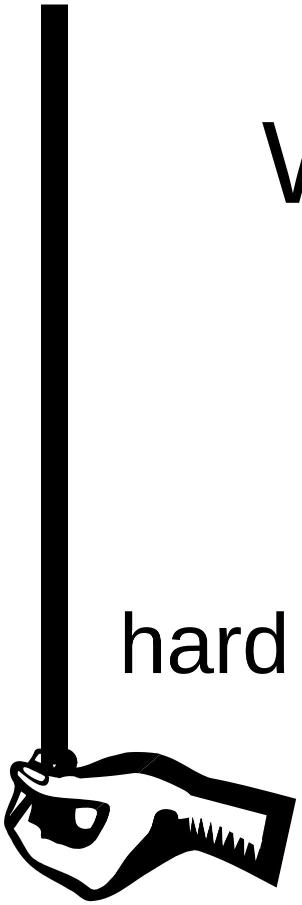
Law #3 : Light

$$\left(M+m\right)\ddot{x}+ml\ddot{\theta}=u$$

$$\ddot{x}+l\ddot{\theta}\pm g\theta=0$$

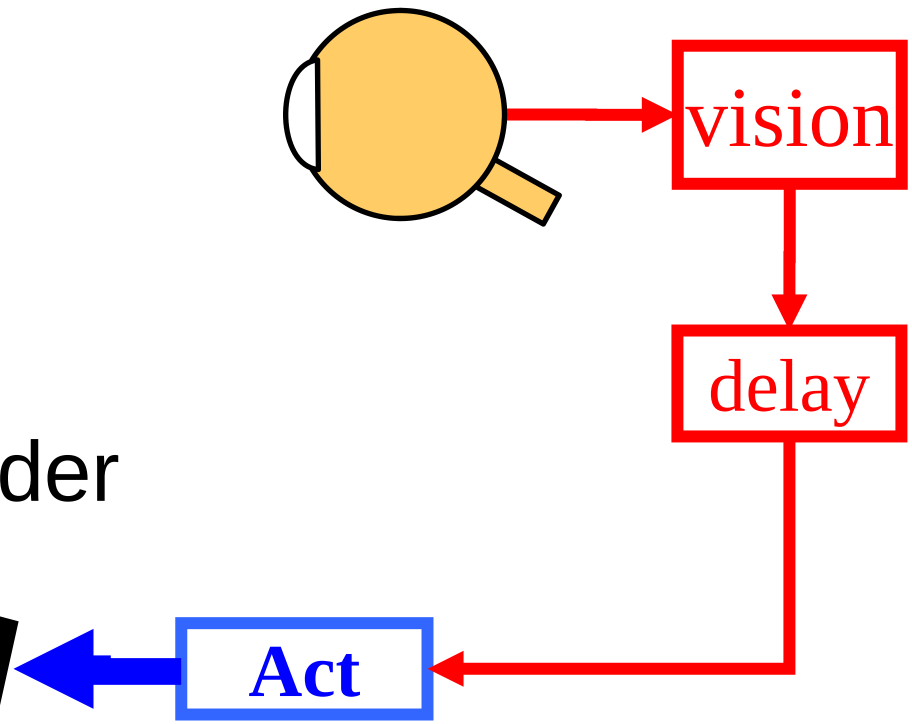
$$y=x+\alpha l\theta$$

Why?

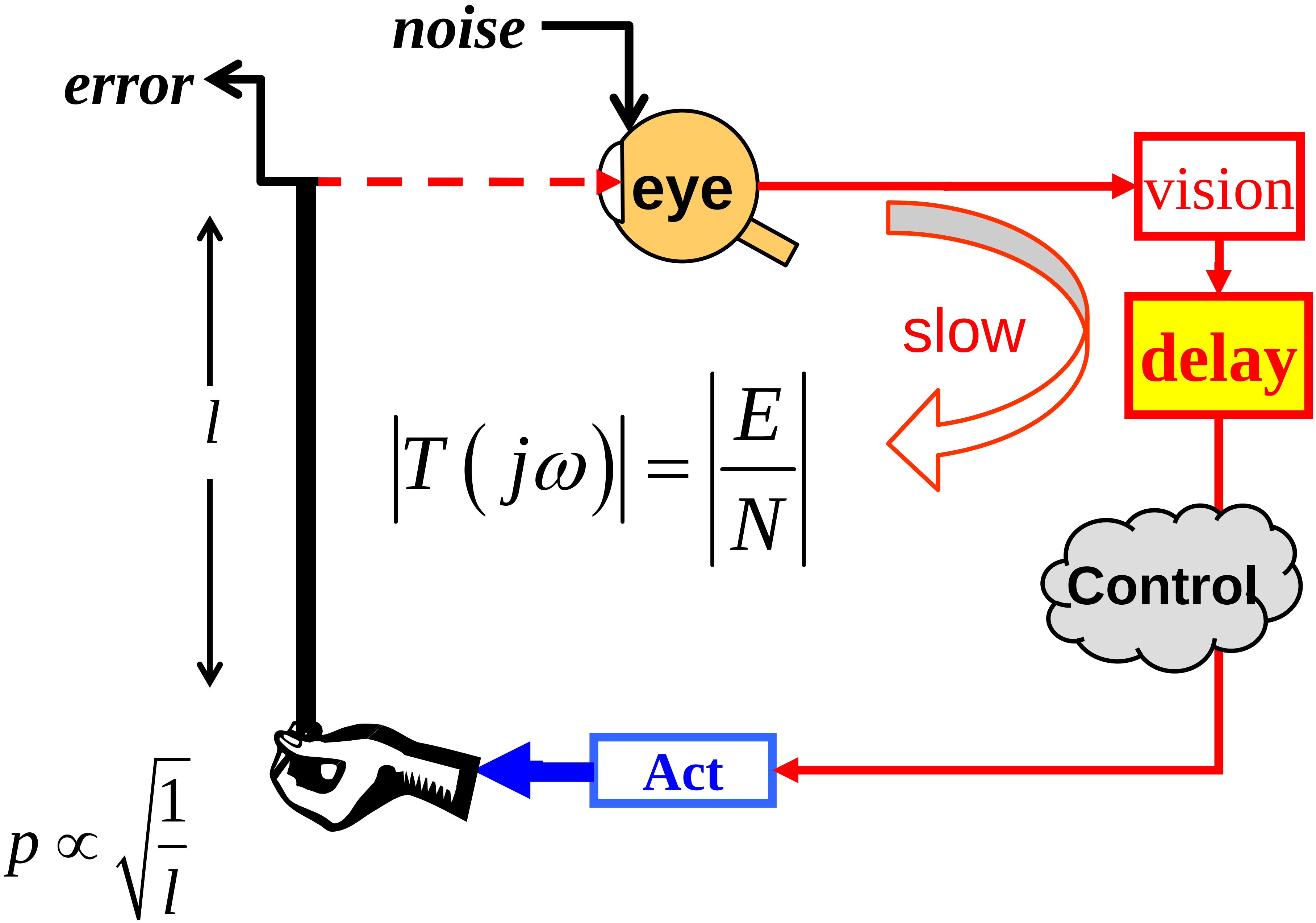


hard

harder

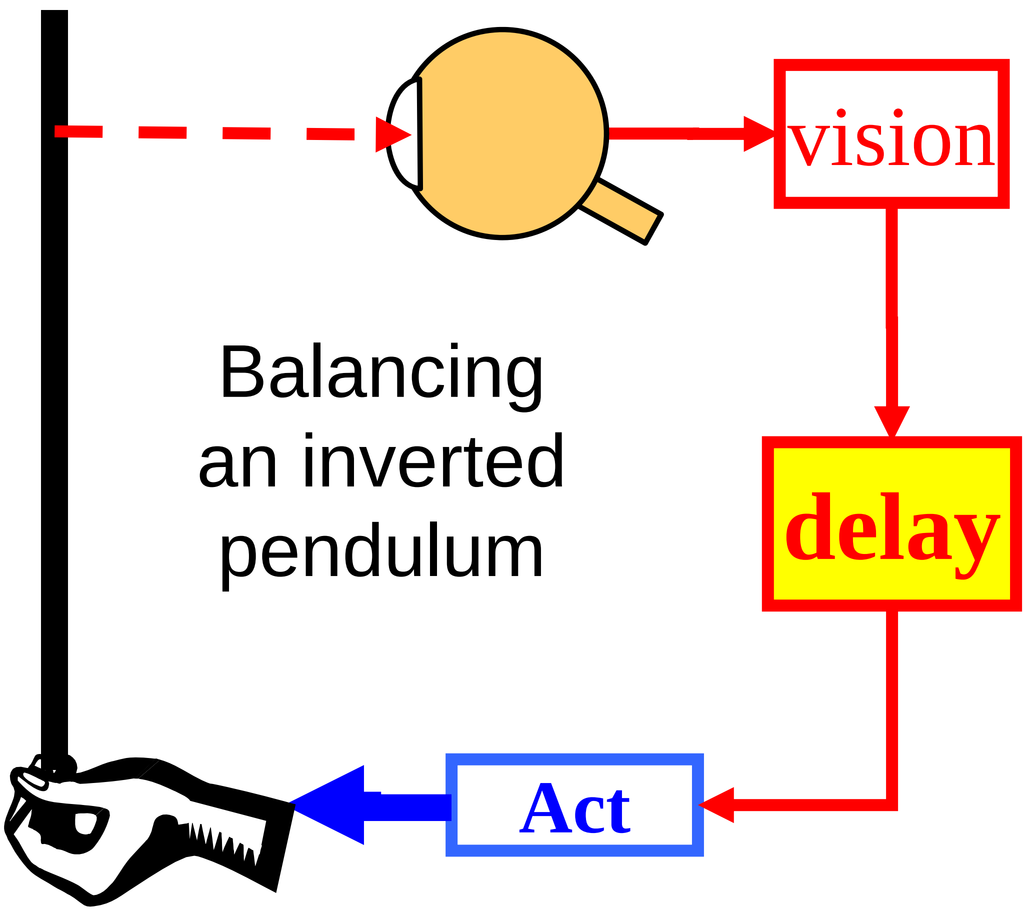


Easy to ***prove*** using simple models.


noise
error
eye
vision
delay
slow
Control
Act
l
$|T(j\omega)| = \left|\frac{E}{N}\right|$
$p \propto \sqrt{\frac{1}{l}}$

Universal laws

Mechanics+
Gravity +
Light +

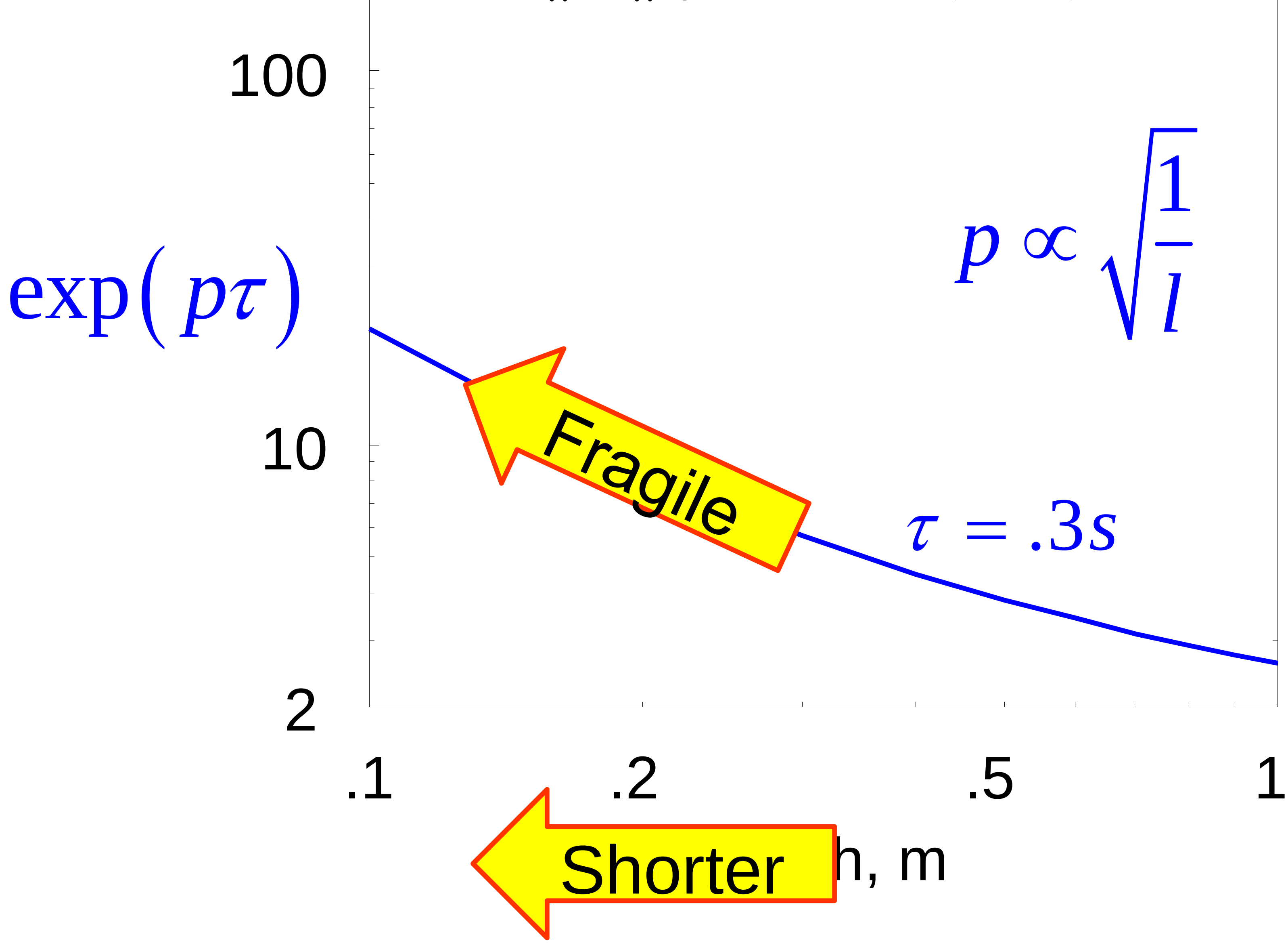


$$\|T\|_{\infty} \geq \exp(p\tau)$$

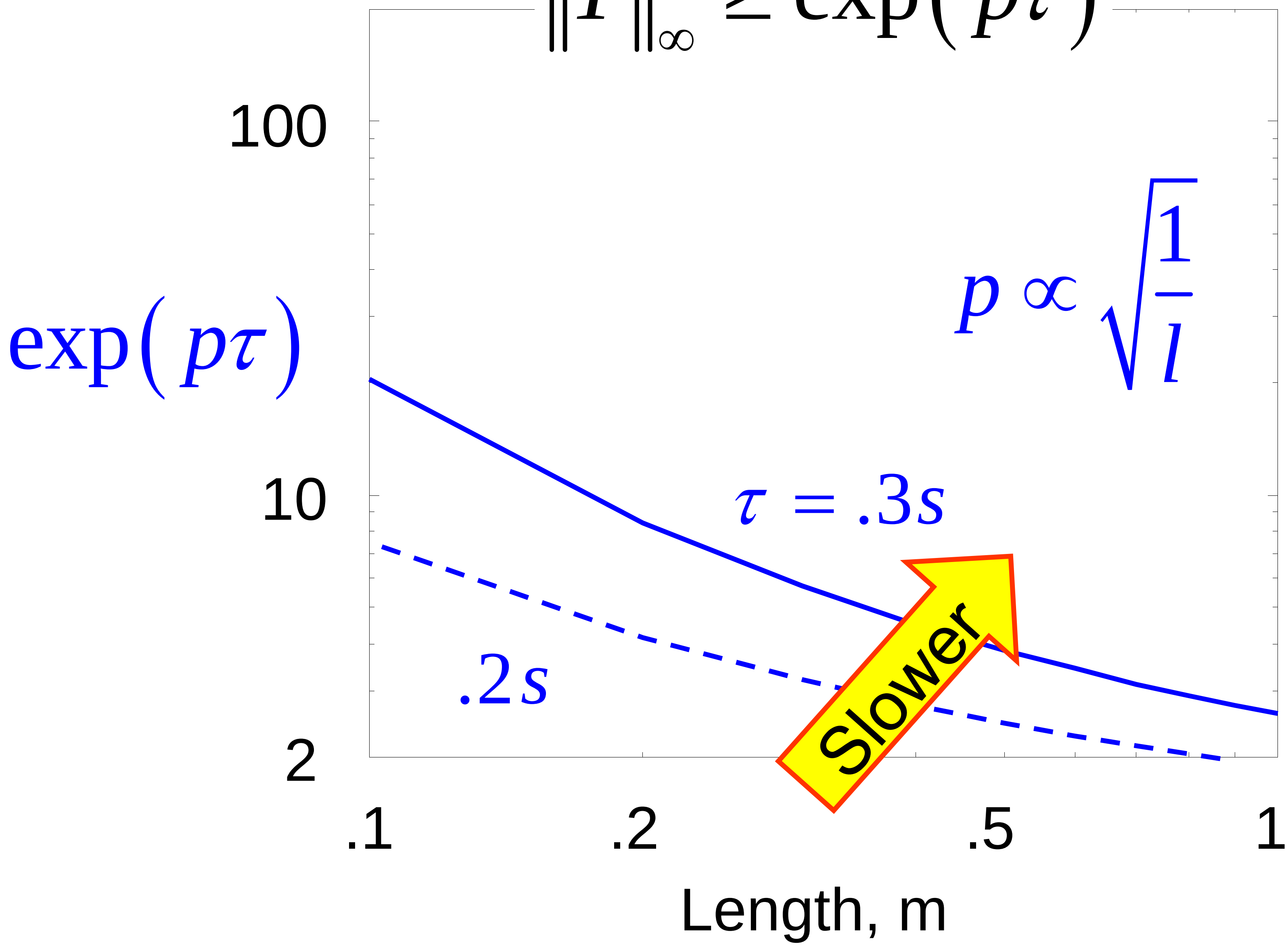
$$p \propto \sqrt{\frac{1}{l}}$$

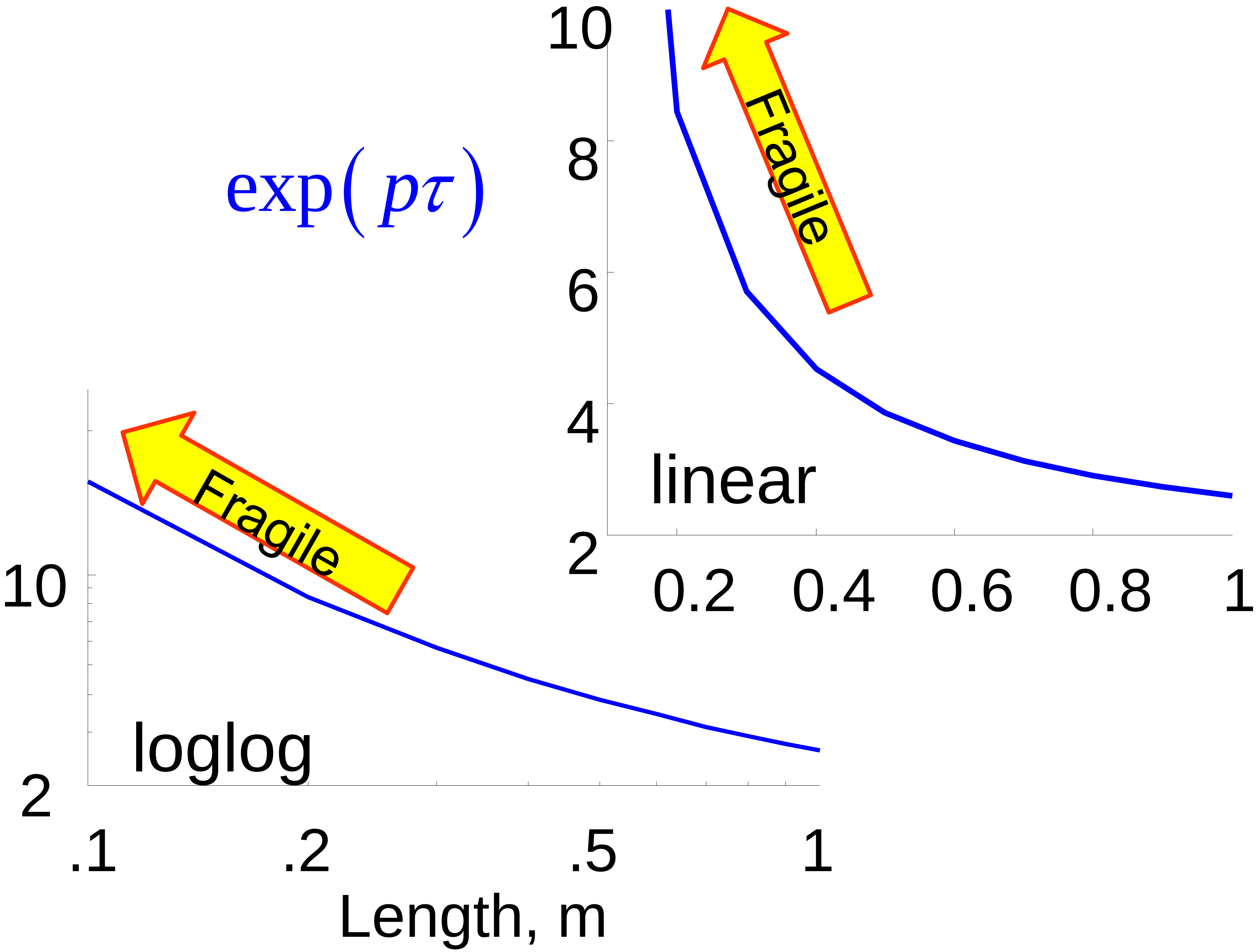
$$\tau \approx .3s$$

$$\|T\|_\infty \geq \exp(p\tau)$$

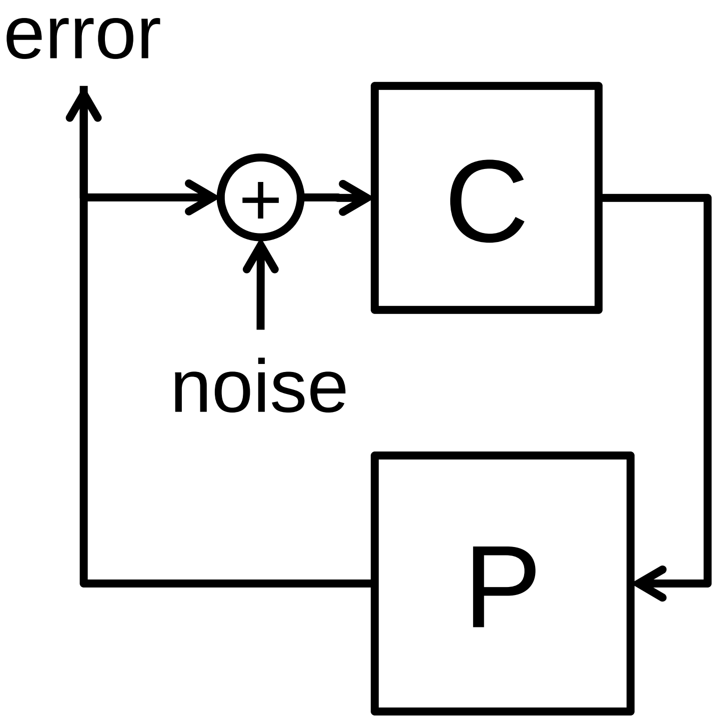


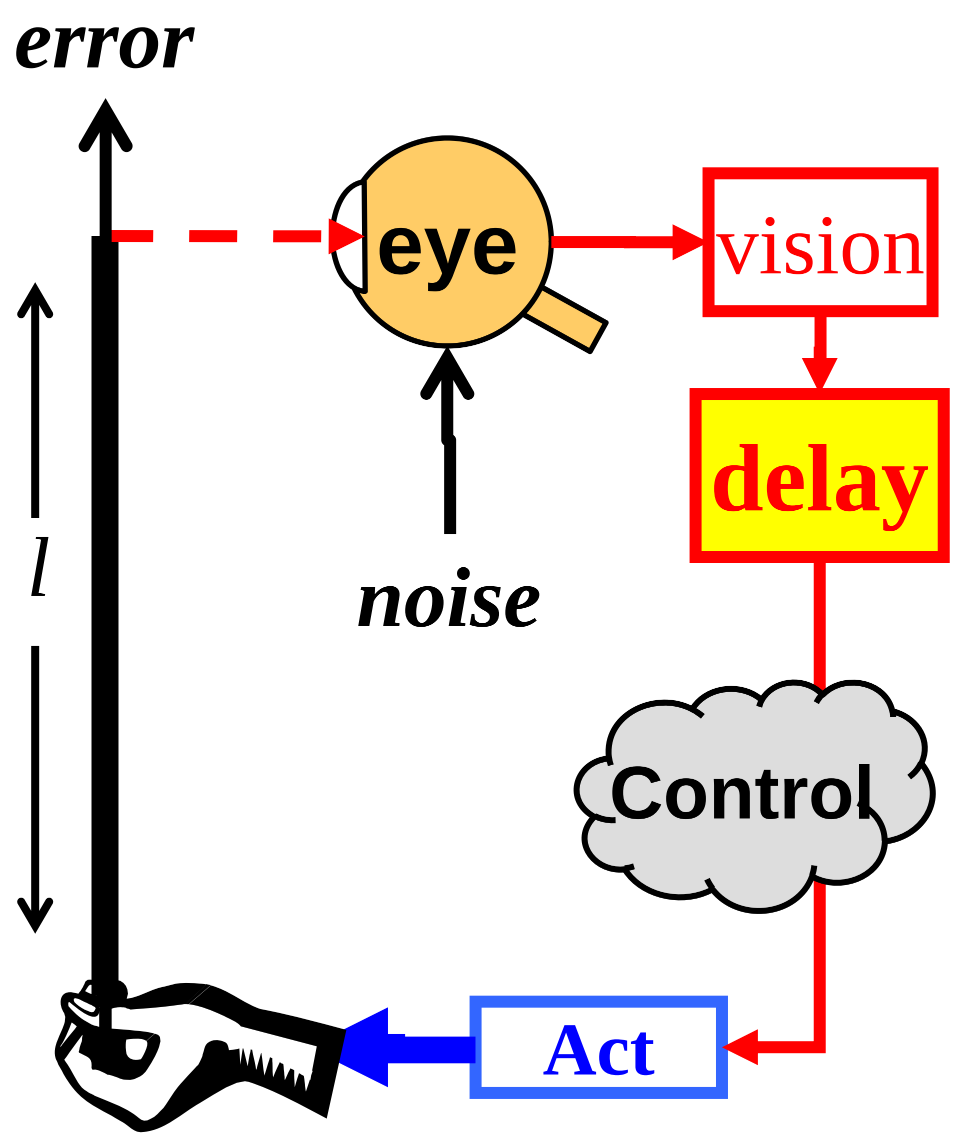
$$\|T\|_\infty \geq \exp(p\tau)$$




exp(pτ)
10
8
6
4
2
linear
0.2
0.4
0.6
0.8
1
Fragile
Fragile
10
2
loglog
.1
.2
.5
1
Length, m

$$\left|T(j\omega)\right| = \left|\frac{E}{N}\right|$$

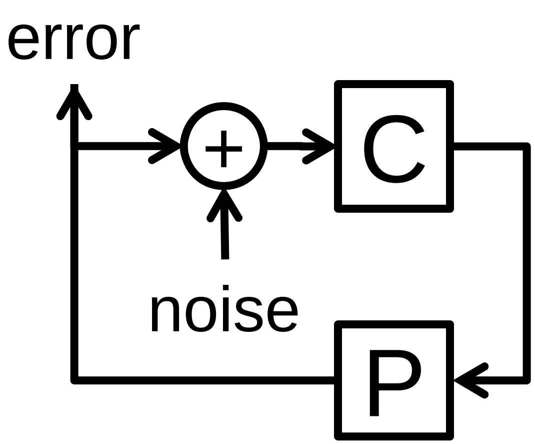




Proof?

$$\|T\|_\infty = \sup_{\omega} |T(j\omega)| = \sup\left\{ |T(s)| \,\middle|\, \mathrm{Re}(s) \geq 0 \right\}$$

Max modulus



$$T(s) = M(s)\Theta(s) \quad |\Theta(j\omega)| = 1$$

$$|T(j\omega)| = \left|\frac{E}{N}\right|$$

$$P(p) = \infty \Rightarrow T(p) = 1$$

$$\Rightarrow M(p) = \Theta(p)^{-1}$$

$$P(s) = P_M(s)\exp(-\tau s) \Rightarrow$$

$$\|T\|_\infty = \|M\|_\infty \geq |M(p)| \geq \left|\Theta(p)^{-1}\right| \geq \exp(\tau s)$$

$$\Rightarrow \|T\|_\infty \geq \exp(\tau s)$$

Proof?

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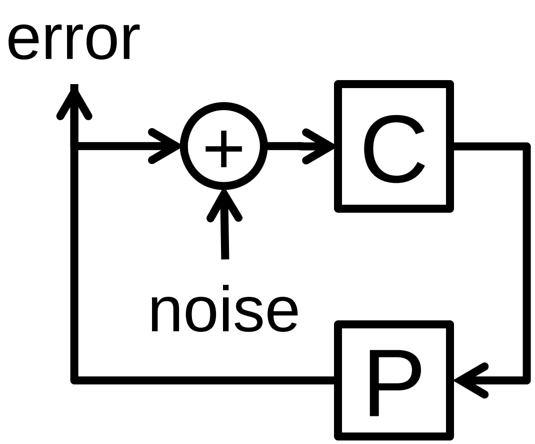
Max modulus



$$T(s) = M(s)\Theta(s) \quad |\Theta(j\omega)| = 1$$

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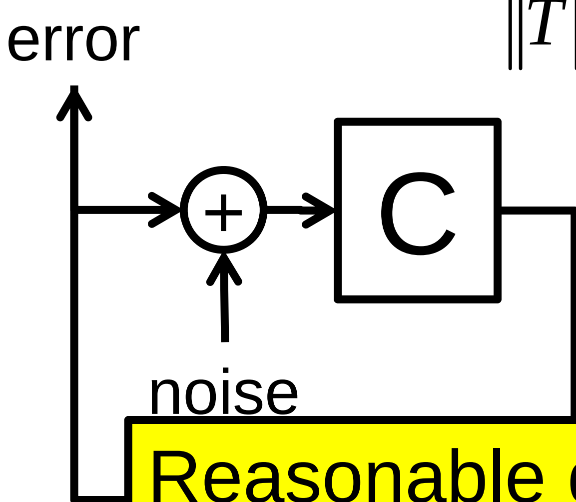
$$\Rightarrow \|T\|_\infty \geq \exp(\tau s)$$



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Max modulus

$$T(s) = M(s)\Theta(s) \qquad |\Theta(j\omega)| = 1$$

Reasonable questions from biologists/doctors:

- Why complex variables for robust control?
- Do we really need to learn *this* math too?
- Why can't we do this with optimization/duality?
- (which we need to learn anyway)

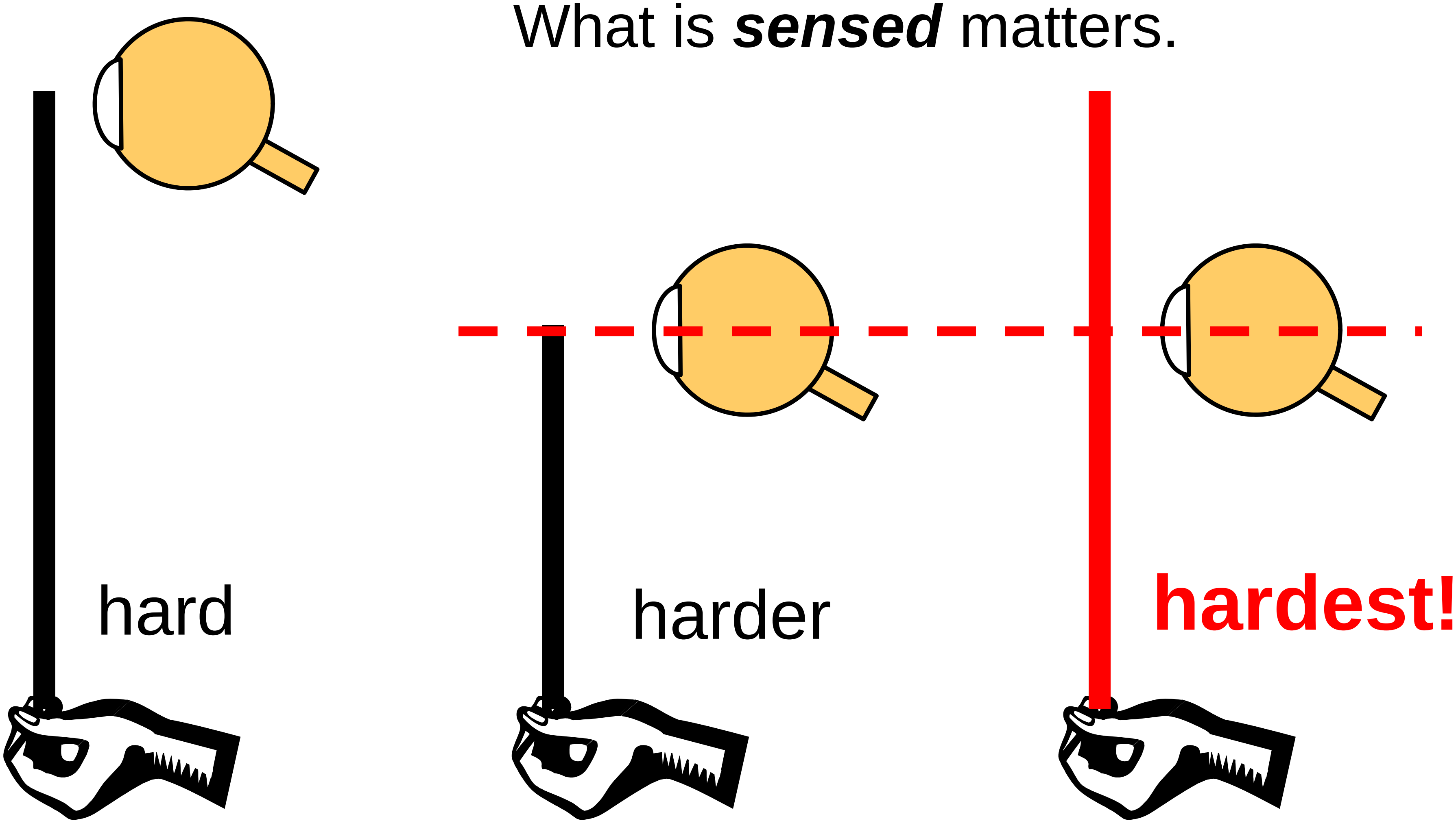
$|T$

$$P(s) = P_M(s)\exp(-\tau s) \Rightarrow$$

$$\|T\|_\infty = \|M\|_\infty \ge |M(p)| \ge \left|\Theta(p)^{-1}\right| \ge \exp(\tau s)$$

$$\Rightarrow \|T\|_\infty \ge \exp(\tau s)$$

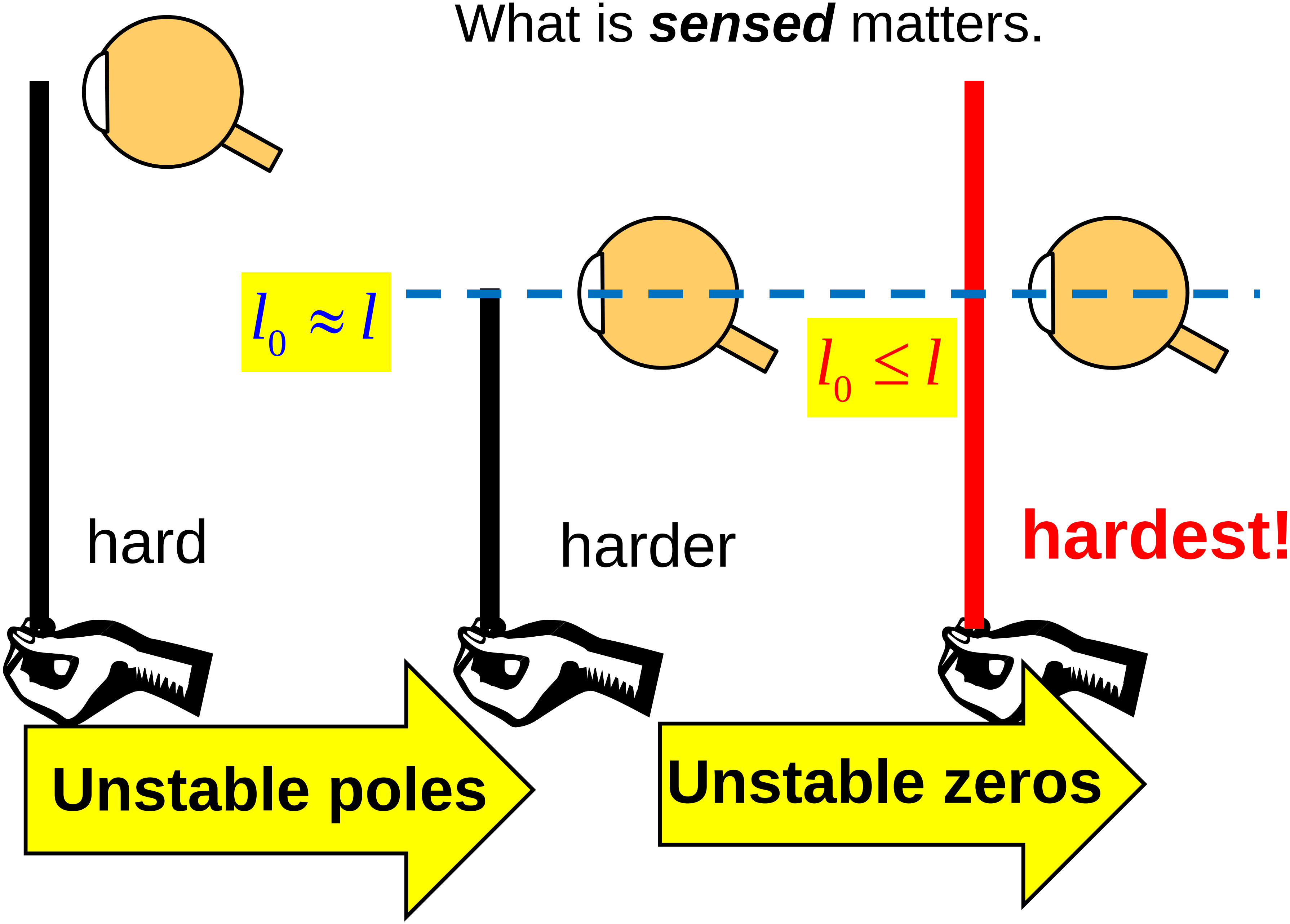
What is ***sensed*** matters.



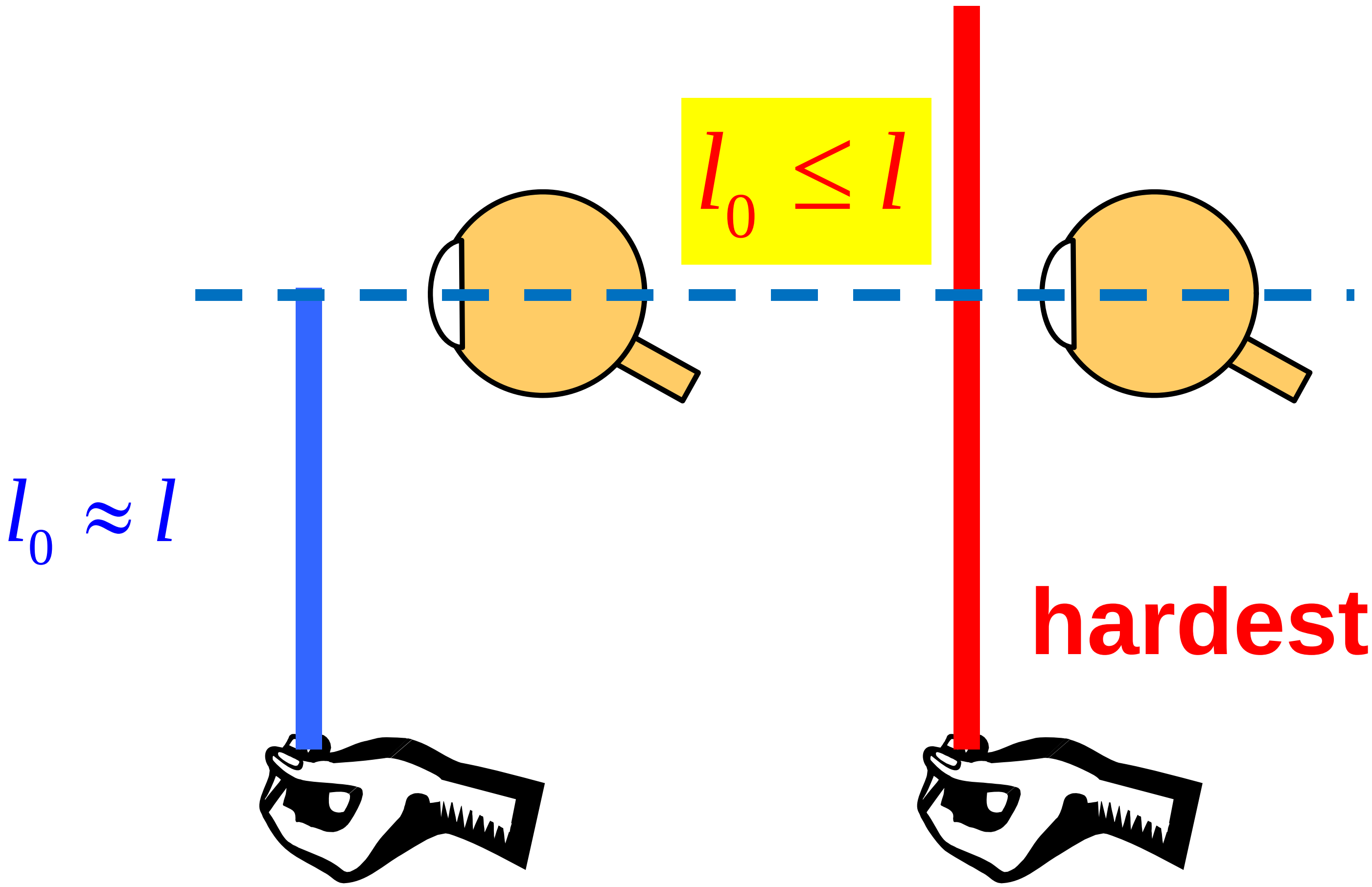
Why?

Easy to ***prove*** using simple models.

What is ***sensed*** matters.

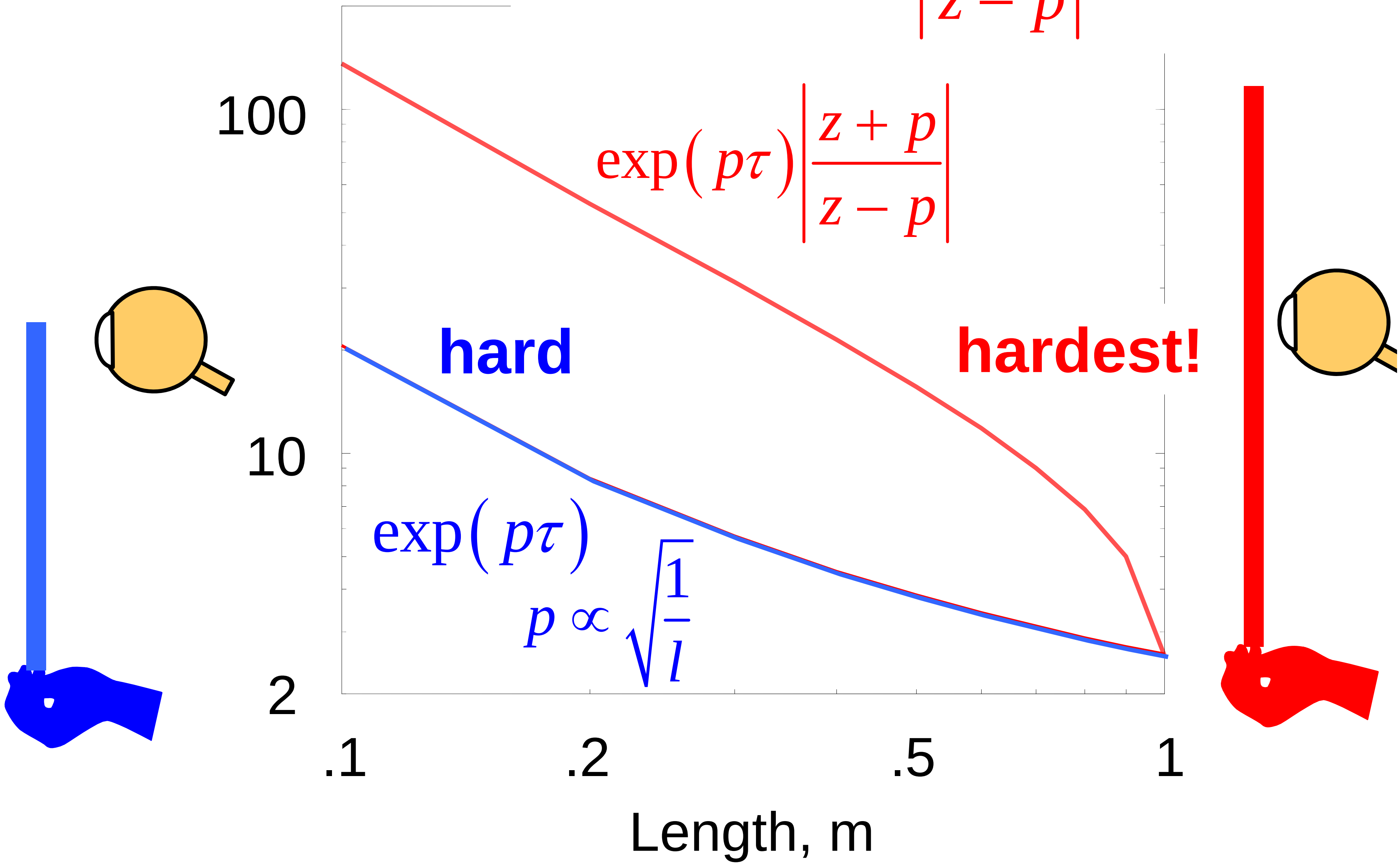


$$\|T\|_\infty \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right| \geq \exp(p\tau)$$



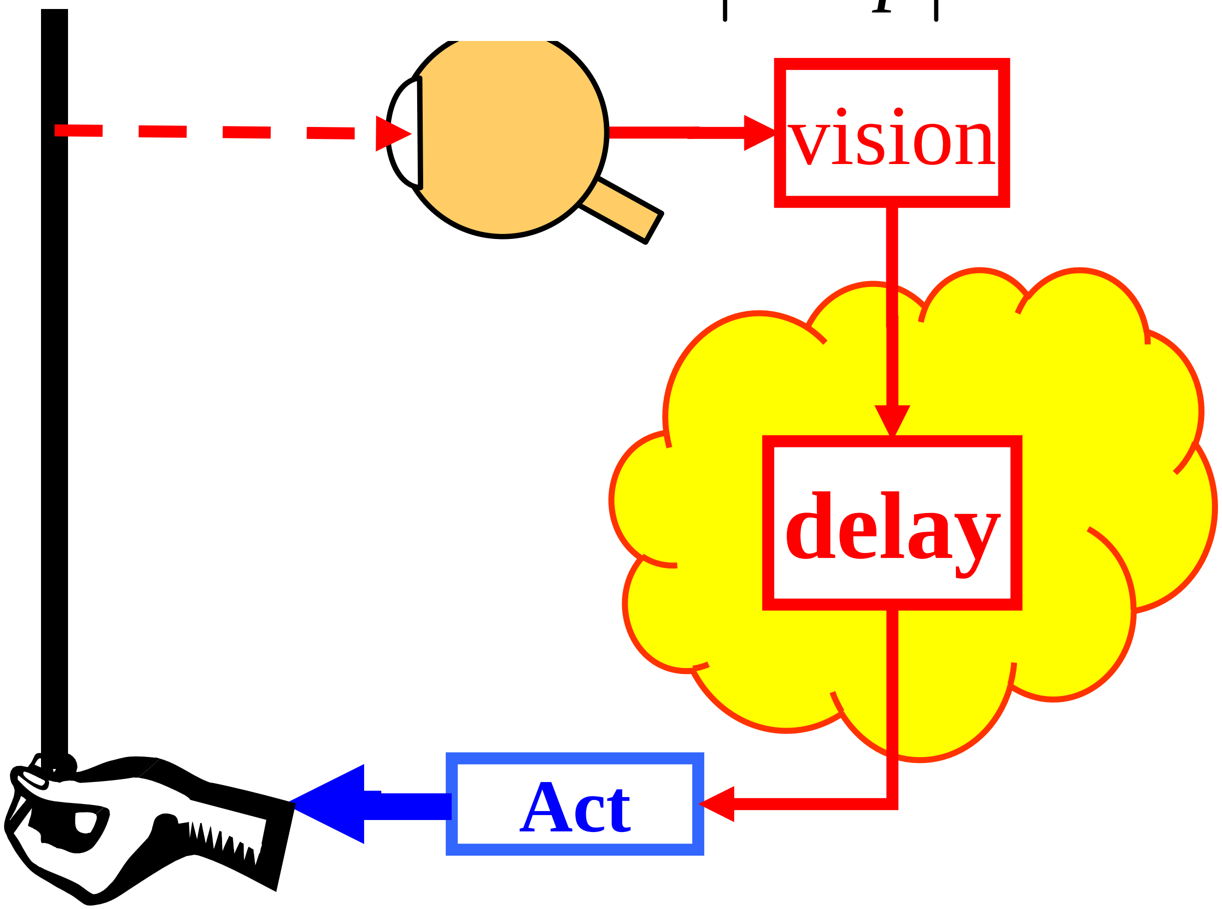
$\tau = .3s$

$$\|T\|_\infty \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right| \geq \exp(p\tau)$$



Holds for *all* controllers.

$$\|T\|_\infty \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$



A "law" about intrinsic problem difficulty (a la Turing).

"Guaranteed margins?" Impossible!

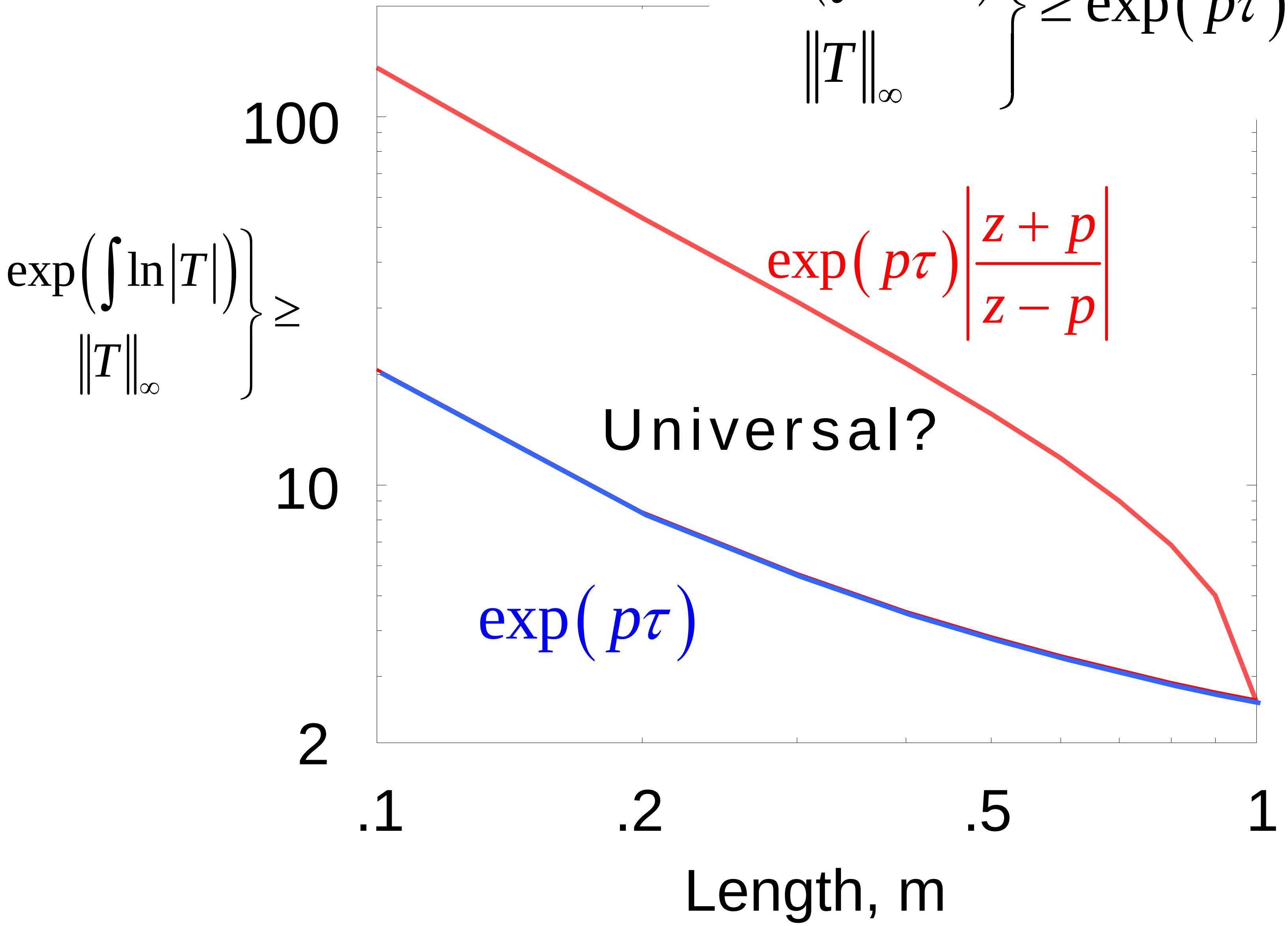
Fragility two ways (Bode* and Zames):

$$\exp\left(\int \ln|T|\right) \square \exp\left(\frac{1}{\pi}\int_0^\infty \ln\left|T(j\omega)\right|\left(\frac{p}{p^2+\omega^2}\right)d\omega\right)$$

$$\left.\begin{matrix}\exp\left(\int \ln|T|\right) \\ \|T\|_\infty\end{matrix}\right\} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$

* With key help from Freudenberg and Seron et al

$$\left.\begin{array}{c}\exp\left(\int \ln|T|\right)\\ \|T\|_\infty\end{array}\right\} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$



RESEARCH ARTICLES

UG biochem, math, control theory

Glycolytic Oscillations and Limits on Robust Efficiency

Fiona A. Chandra,[1]* Gentian Buzi,[2] John C. Doyle[2]

Both engineering and evolution are constrained by trade-offs between efficiency and robustness, but theory that formalizes this fact is limited. For a simple two-state model of glycolysis, we explicitly derive analytic equations for hard trade-offs between robustness and efficiency with oscillations as an inevitable side effect. The model describes how the trade-offs arise from individual parameters, including the interplay of feedback control with autocatalysis of network products necessary to power and catalyze intermediate reactions. We then use control theory to prove that the essential features of these hard trade-off "laws" are universal and fundamental, in that they depend minimally on the details of this system and generalize to the robust efficiency of any autocatalytic network. The theory also suggests worst-case conditions that are consistent with initial experiments.

the cell's use of ATP. In glycolysis, two ATP molecules are consumed upstream and four are produced downstream, which normalizes to $q = 1$ (each y molecule produces two downstream) with kinetic exponent $a = 1$. To highlight essential trade-offs with the simplest possible analysis, we normalize the concentration such that the unperturbed ($\delta = 0$) steady states are $\bar{y} = 1$ and $\bar{x} = 1/k$ [the system can have one additional steady state, which is unstable when $(1, 1/k)$ is stable]. [See the supporting online material (SOM) part I]. The basal rate of the PFK reaction and the consumption rate have been normalized to 1 (the 2 in the numerator and feedback coefficients of the reactions come from these normalizations). Our results hold for more general systems

Chandra, Buzi, and Doyle

Insight
Accessible
Verifiable

Science

AAAS

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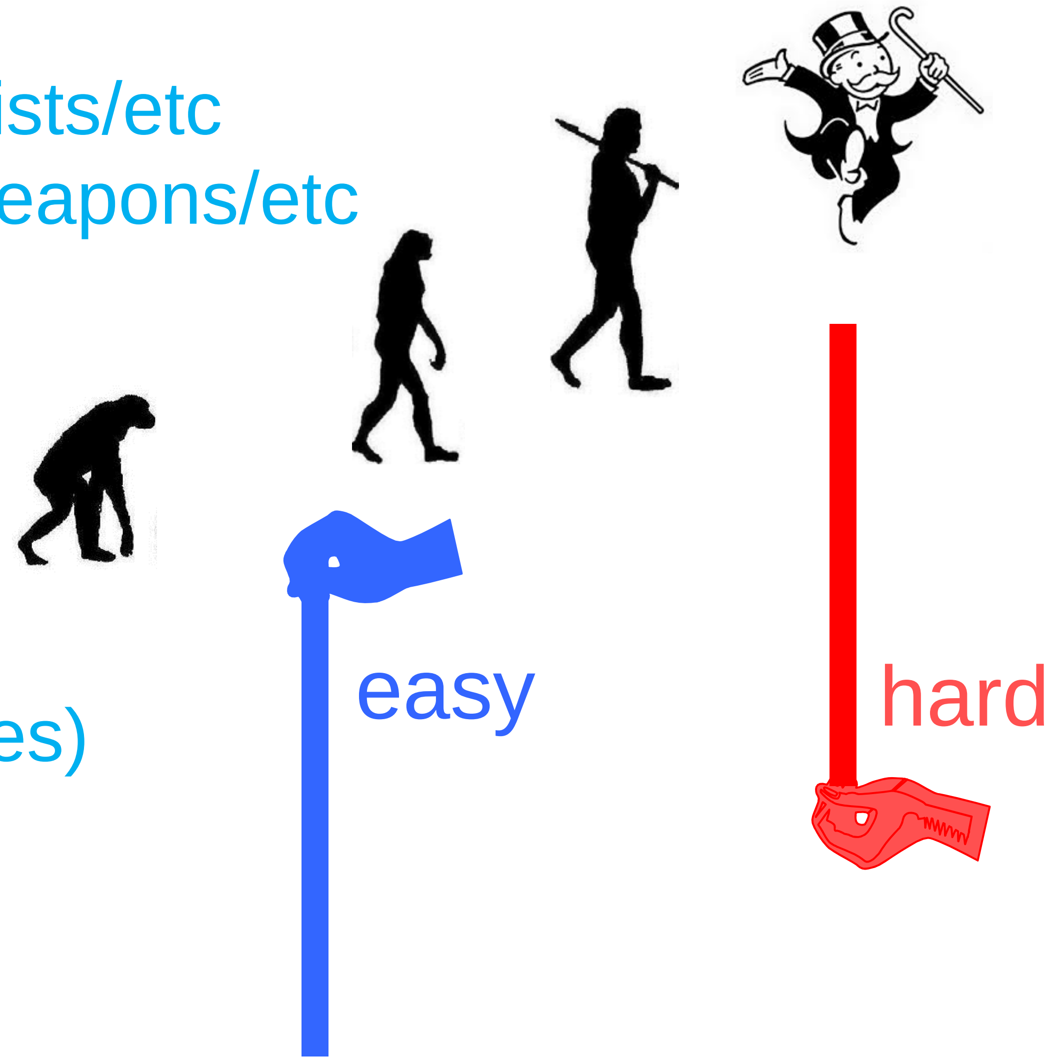
Efficiency/instability/layers/feedback

- All create new efficiencies but also instabilities
- Requires new active/layered/complex/active control

- Money/finance/lobbyists/etc
- Society/agriculture/weapons/etc
- Bipedalism
- Maternal care
- Warm blood
- Flight
- Mitochondria
- Translation (ribosomes)
- Glycolysis



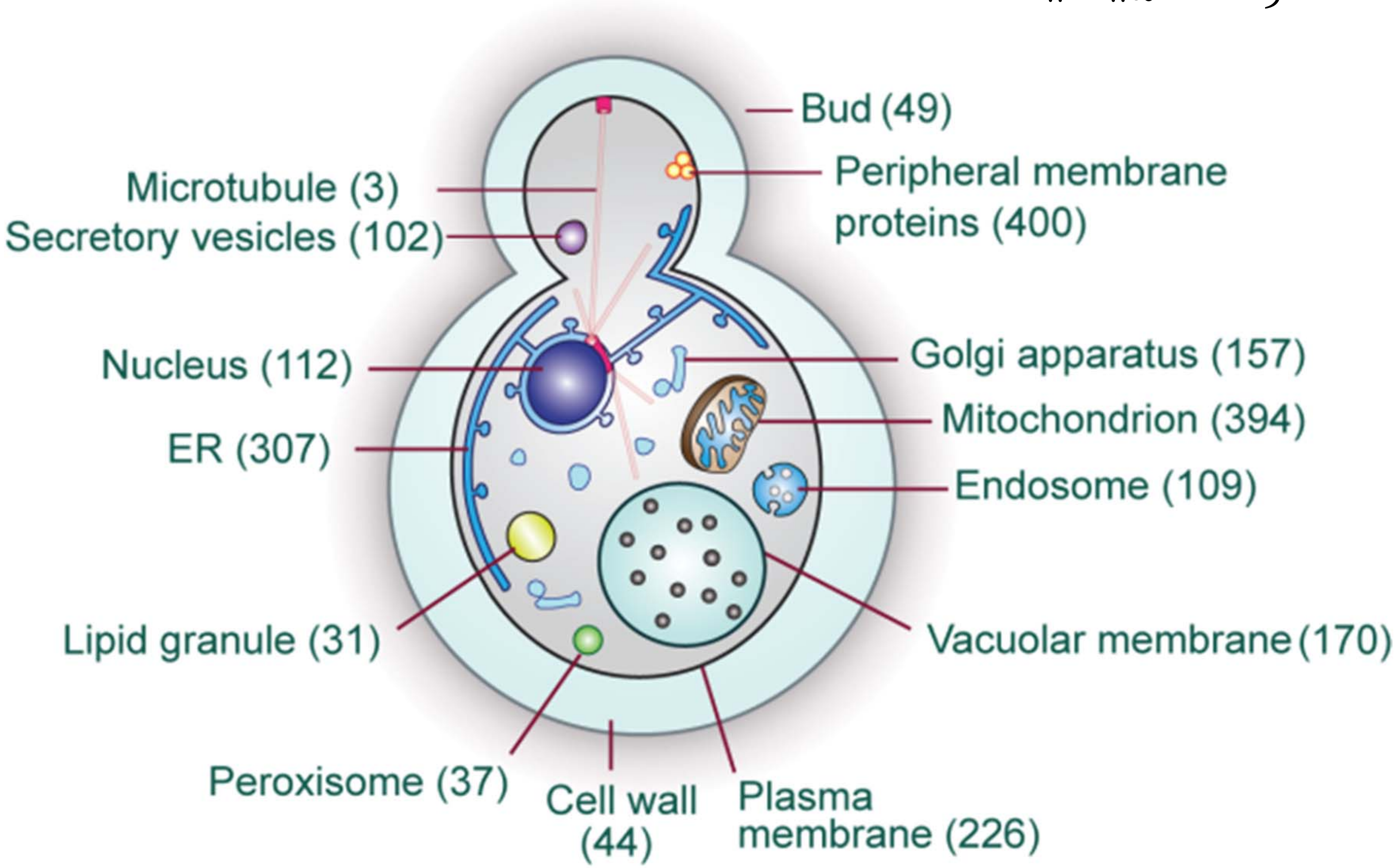
Law #1 : Chemistry

Law #2 : Autocatalysis

(→ RHP p and z)

Law #3:

$$\left.\begin{array}{c}\exp\left(\int \ln|T|\right)\\ \|T\|_\infty\end{array}\right\} \geq \left|\frac{z+p}{z-p}\right|$$



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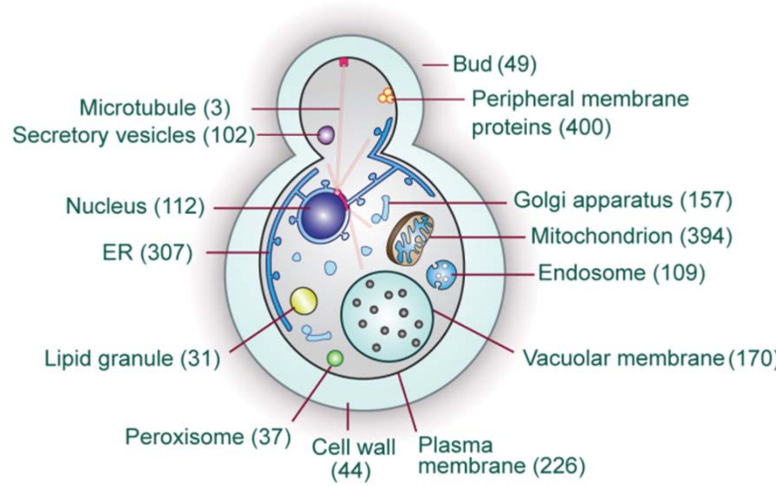
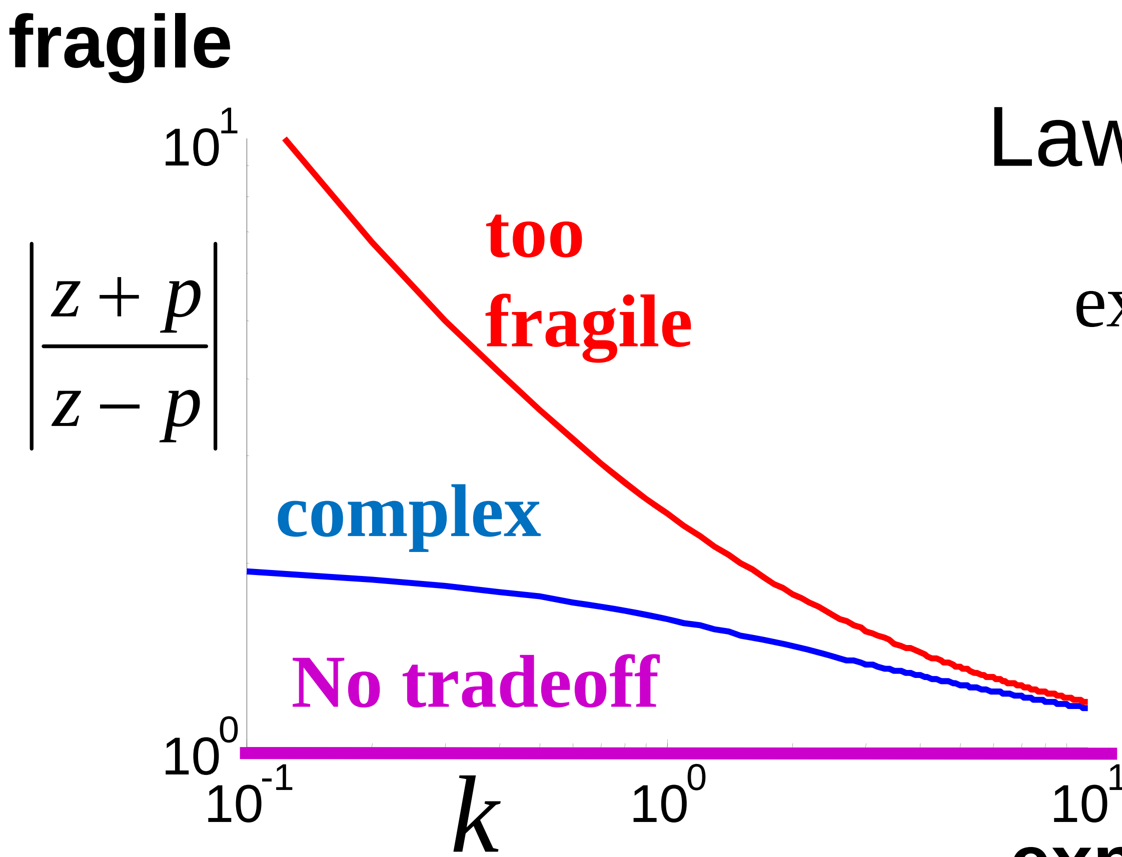




expensive

Law #3:

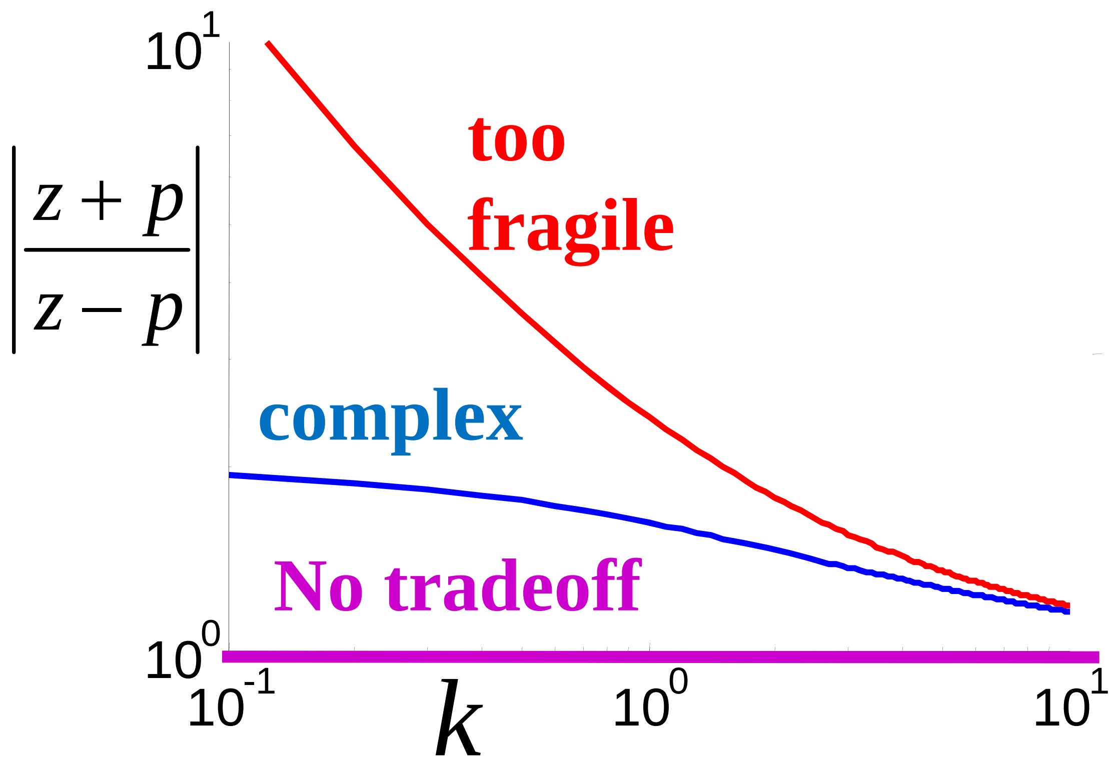
$$\left.\begin{matrix}\exp\left(\int \ln|T|\right)\\ \|T\|_\infty\end{matrix}\right\} \geq \left|\frac{z+p}{z-p}\right|$$

Robust Efficiency in Energy Supply

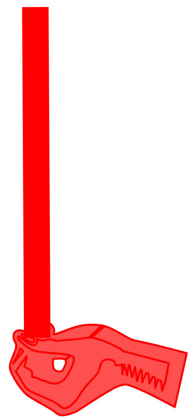
$$\left.\begin{array}{c}\exp\left(\int \ln|T|\right)\\ \|T\|_\infty\end{array}\right\} \geq \left|\frac{z+p}{z-p}\right|$$

fragile

Robust to
Δ in supply
and demand

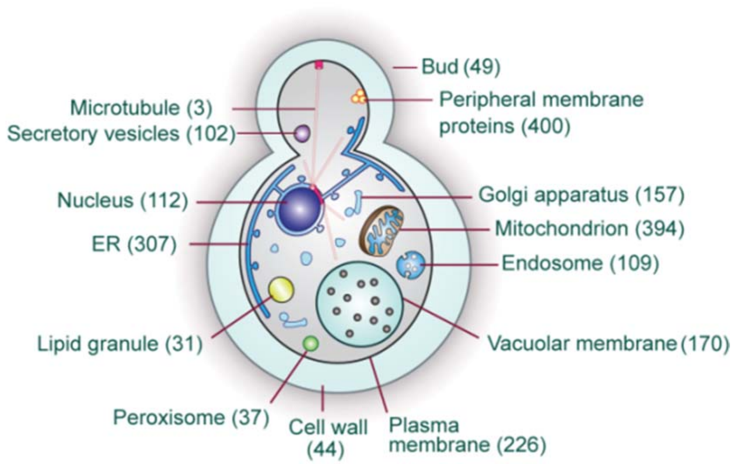


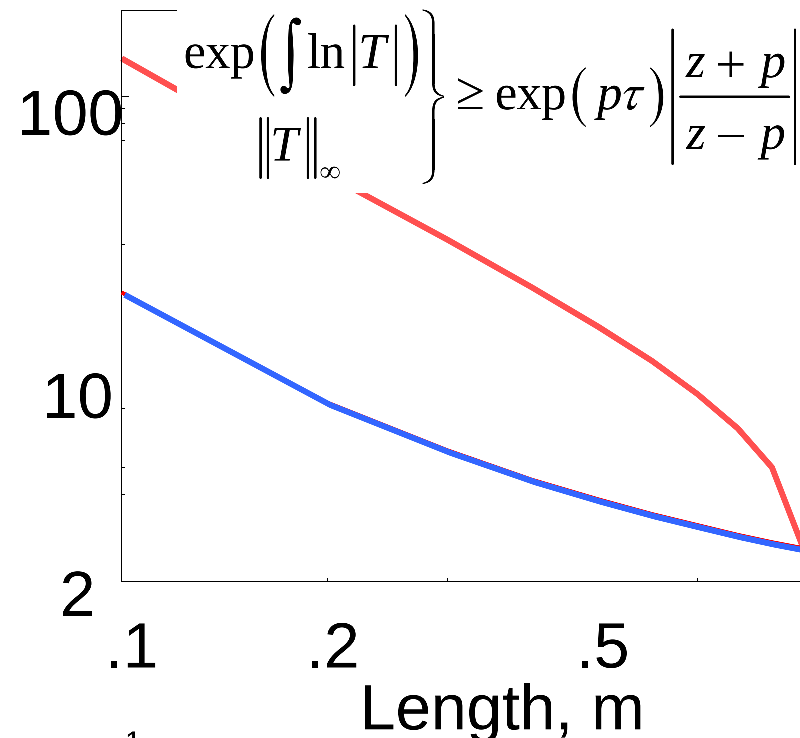
Metabolic overhead
to make enzymes

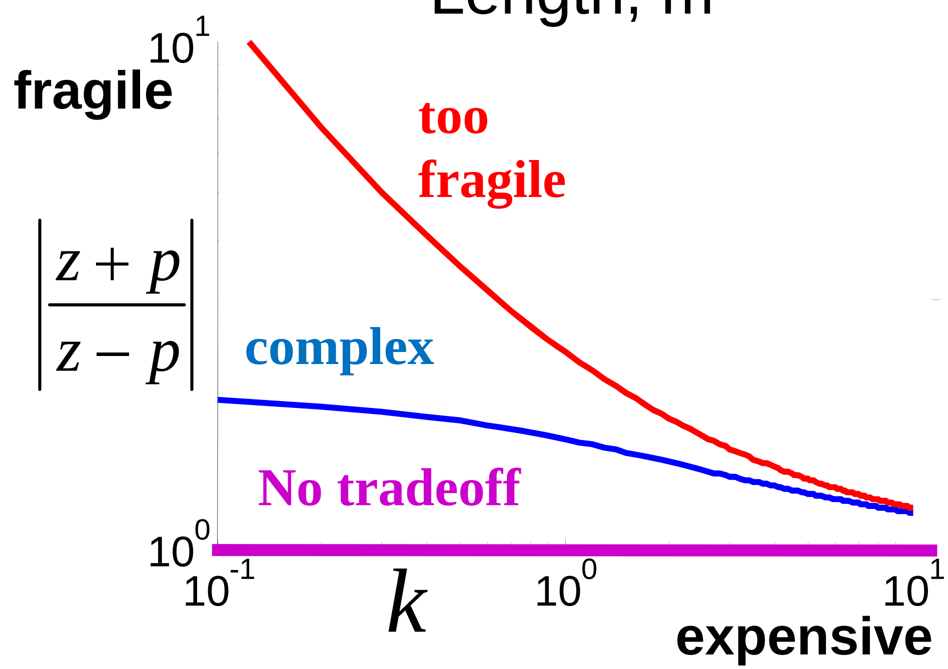


Universal laws?

$$\left.\begin{matrix}\exp\left(\int \ln|T|\right)\\ \|T\|_\infty\end{matrix}\right\} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$







54

What (some) reviewers say

- “…to establish universality … is **simply wrong**. It cannot be done…
- … a mathematical scheme **without any real connections to biological or medical**…
- …universality is well justified in physics… for biological and physiological systems …**a dream …never be realized**, due to the vast diversity in such systems.
- …**does not seem to understand or appreciate** the vast diversity of biological and physiological systems…
- …a high degree of abstraction, which …make[s] the model **useless** …

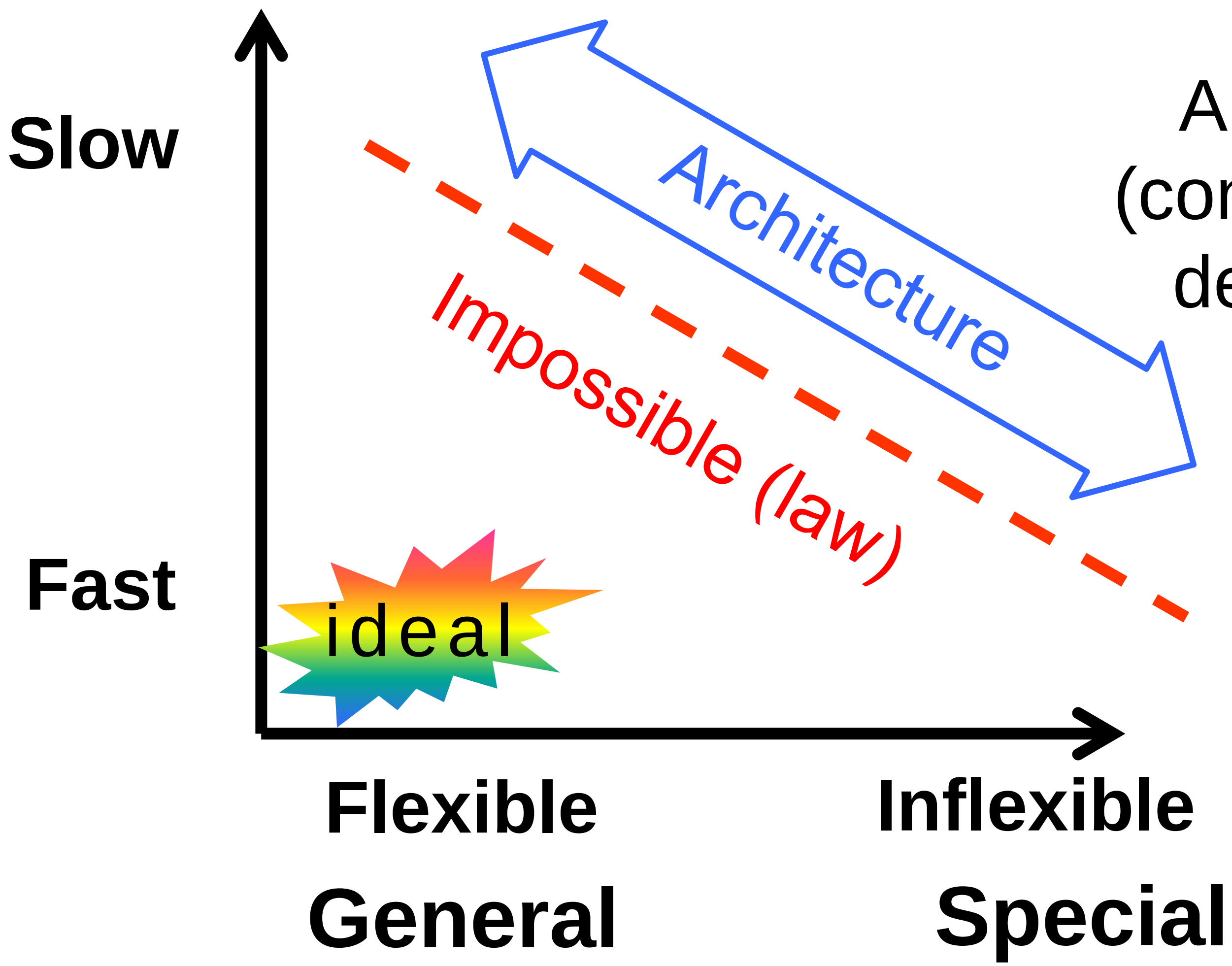
What (some) reviewers say

- "…to establish universality … is **simply wrong**. It cannot be done…
- … a mathematical scheme **without any real connections to biological or medical**

> If you agree
> - You're in good company
> - See Andy at break about refund policy
> - Stay off commercial aircraft

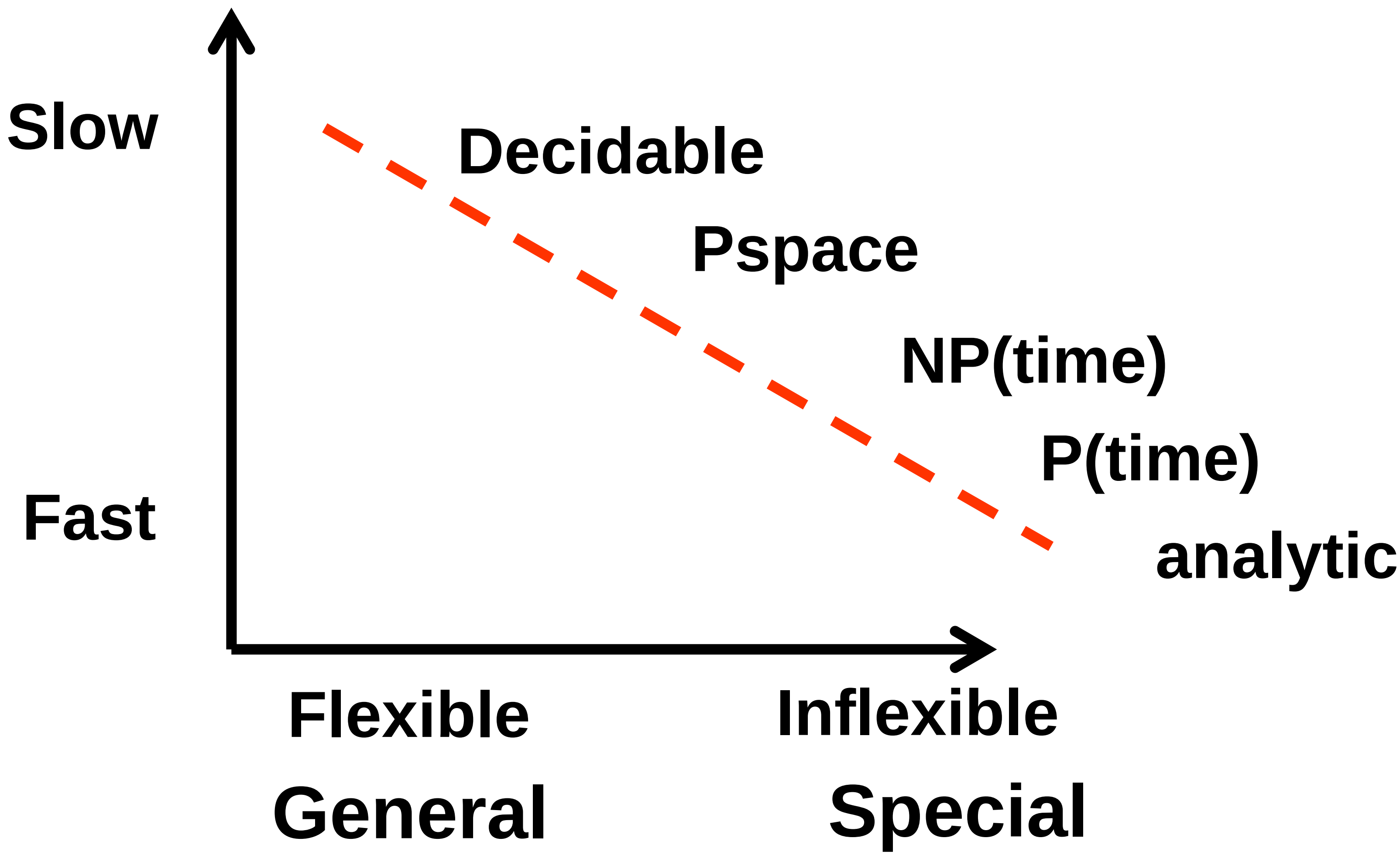
- **m** in
- **te** the vast diversity of biological and physiological systems…
- …a high degree of abstraction, which …make[s] the model **useless** …

Universal laws and architectures (Turing)

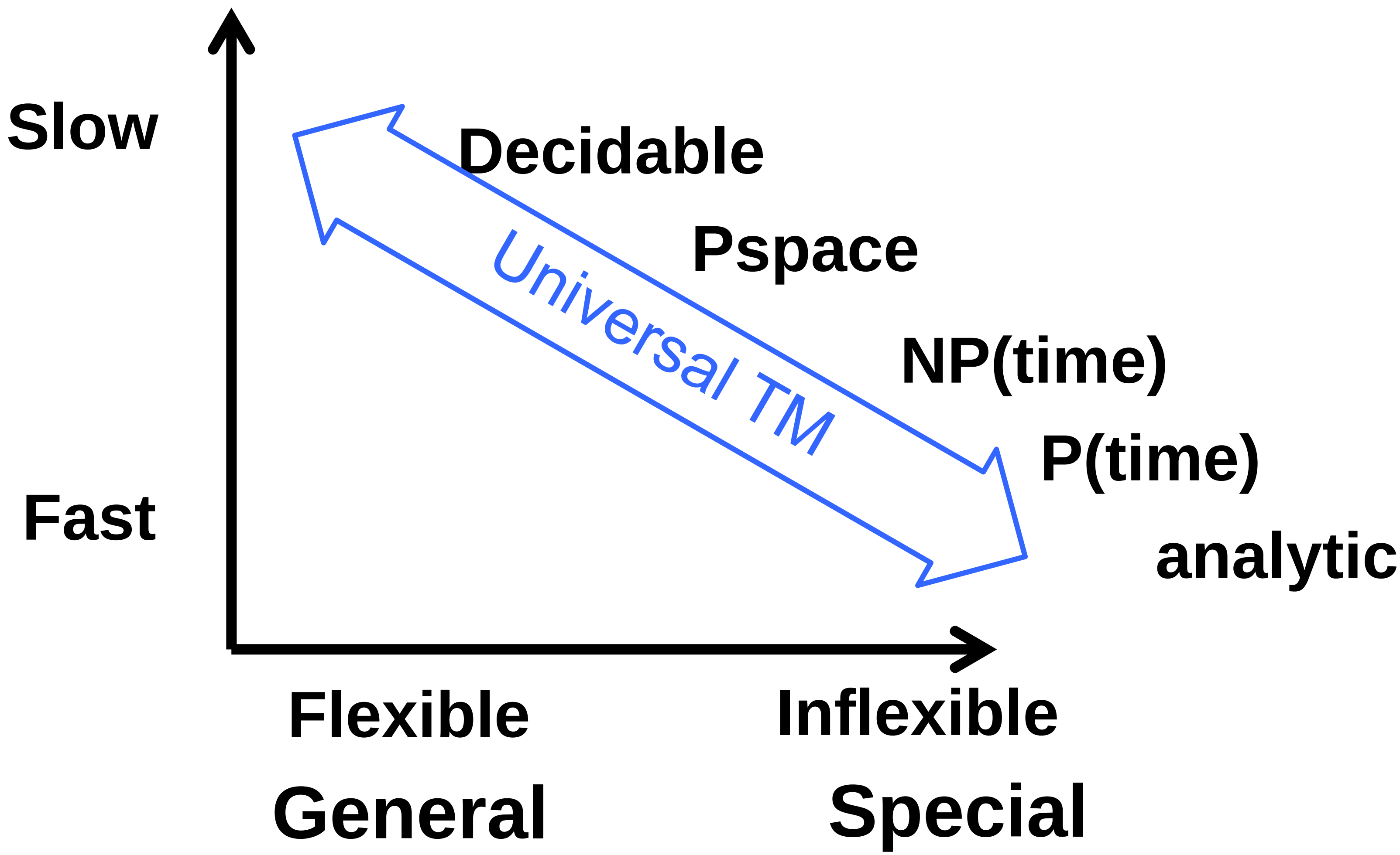


Architecture (constraints that deconstrain)

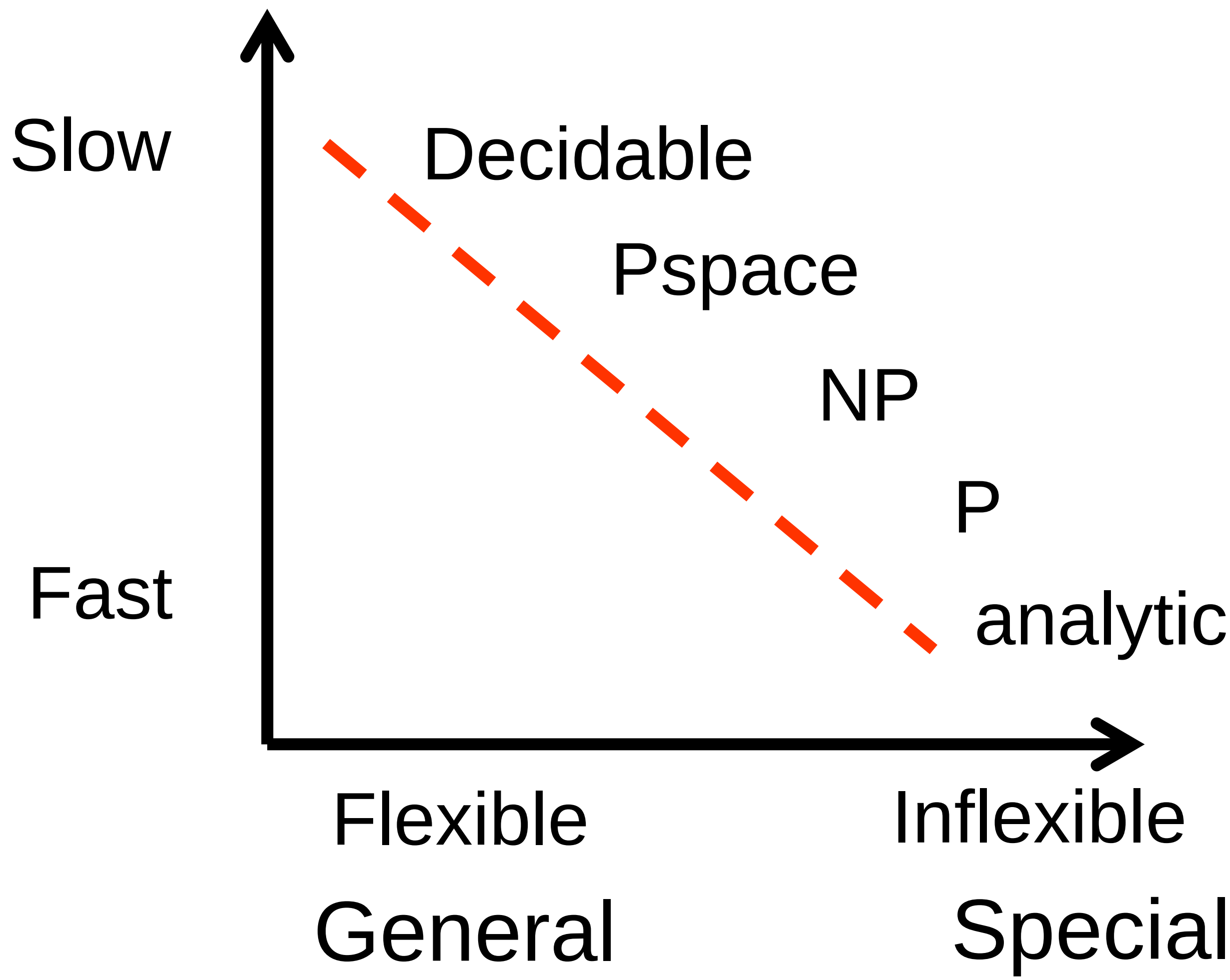
Computation
(on and off-line)



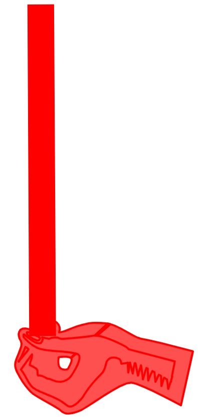
Universal architecture

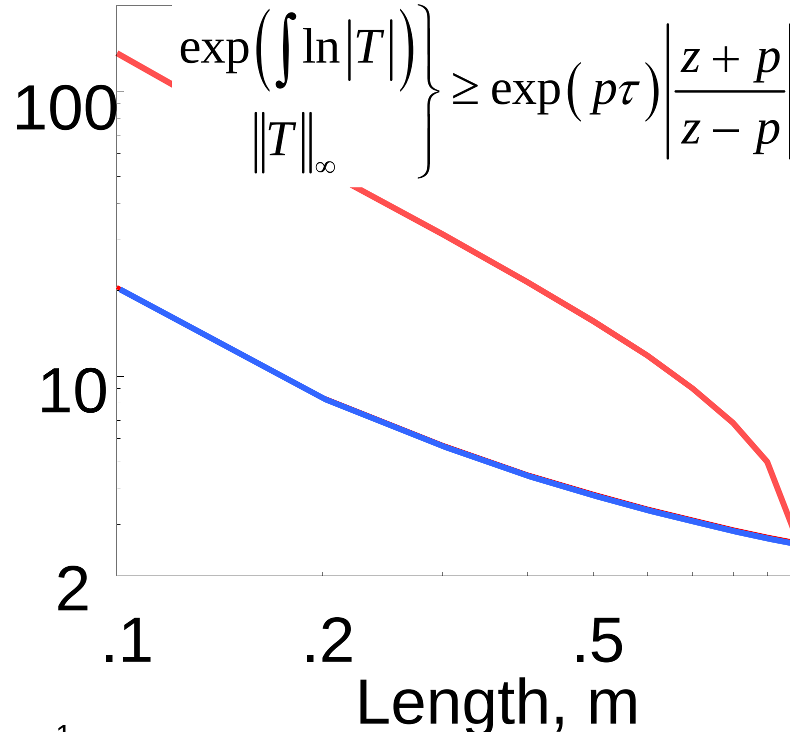


Computation (on and off-line)

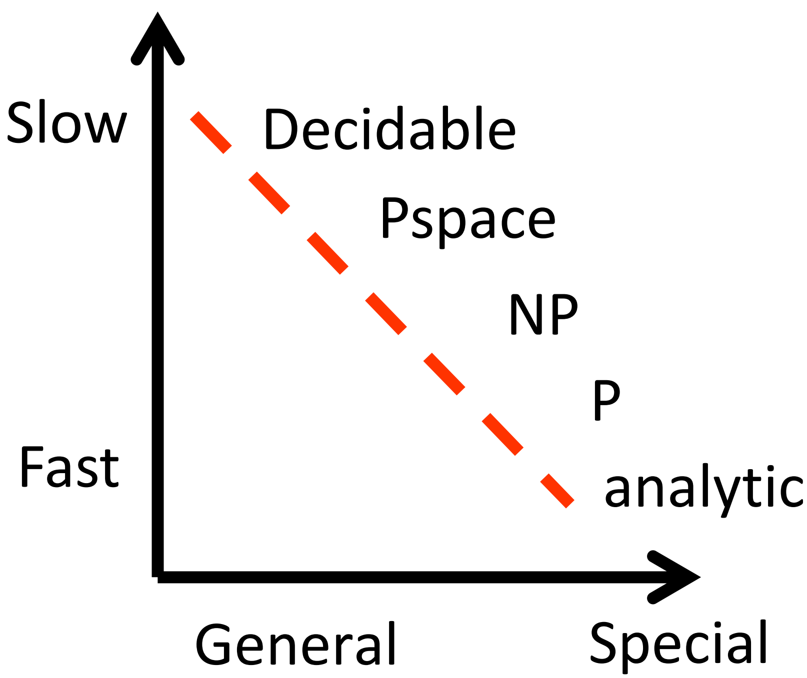


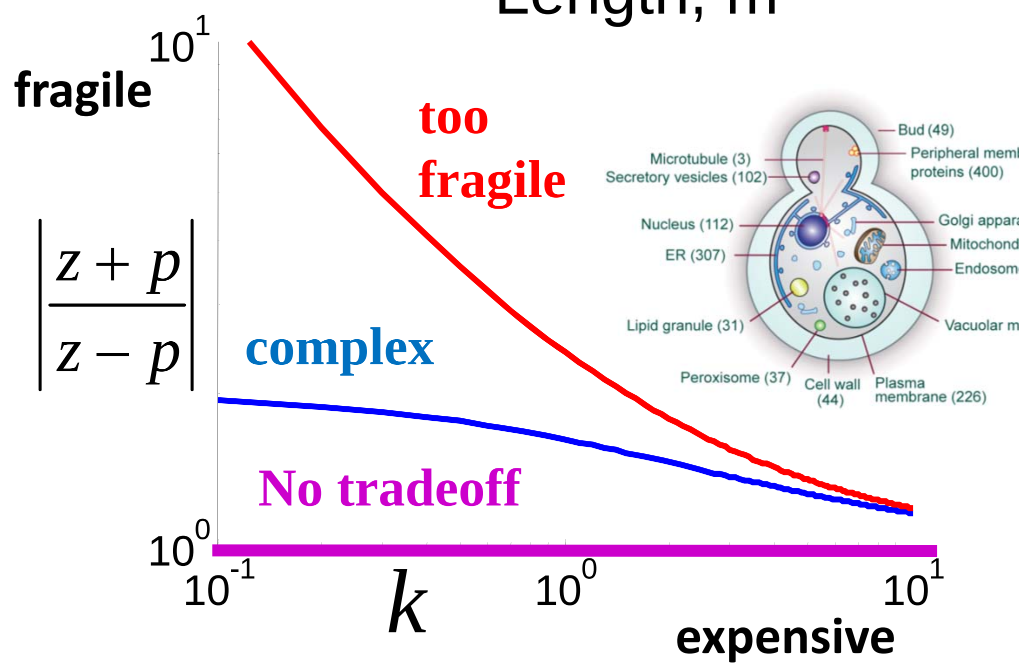
Fast
Insight
Accessible
Verifiable

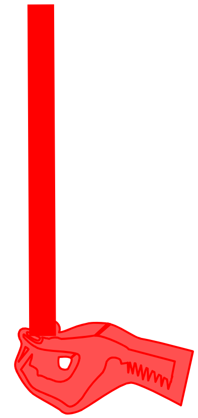


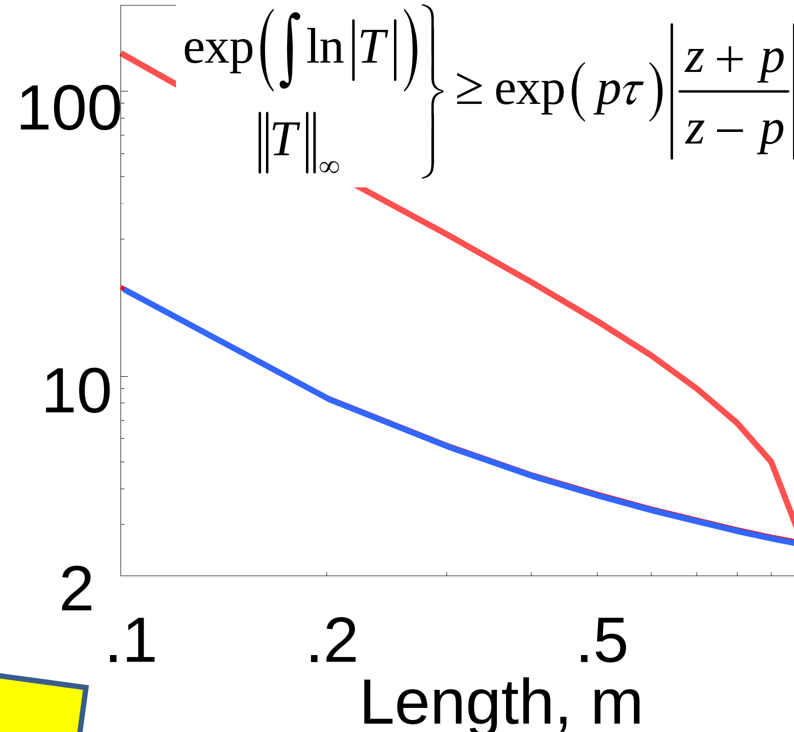


Universal laws?

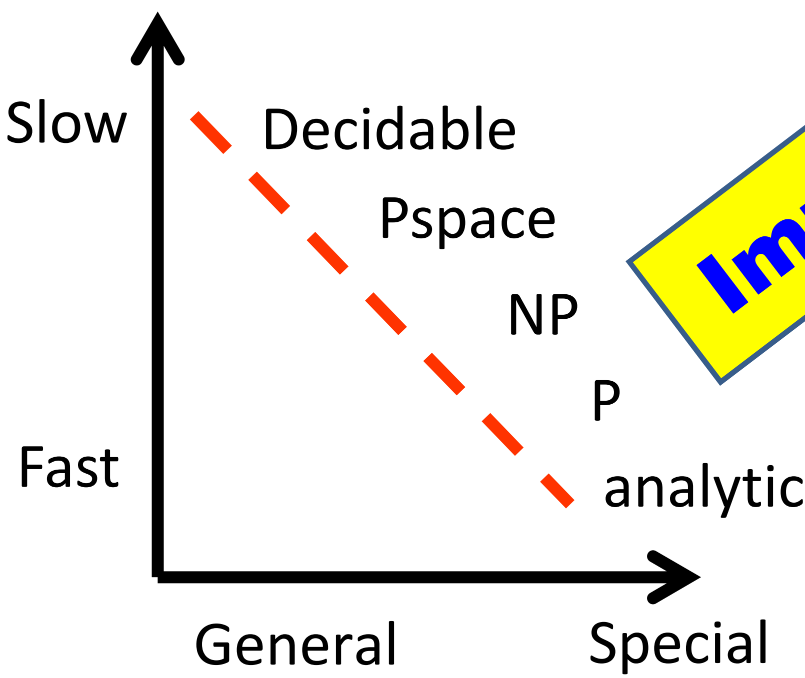


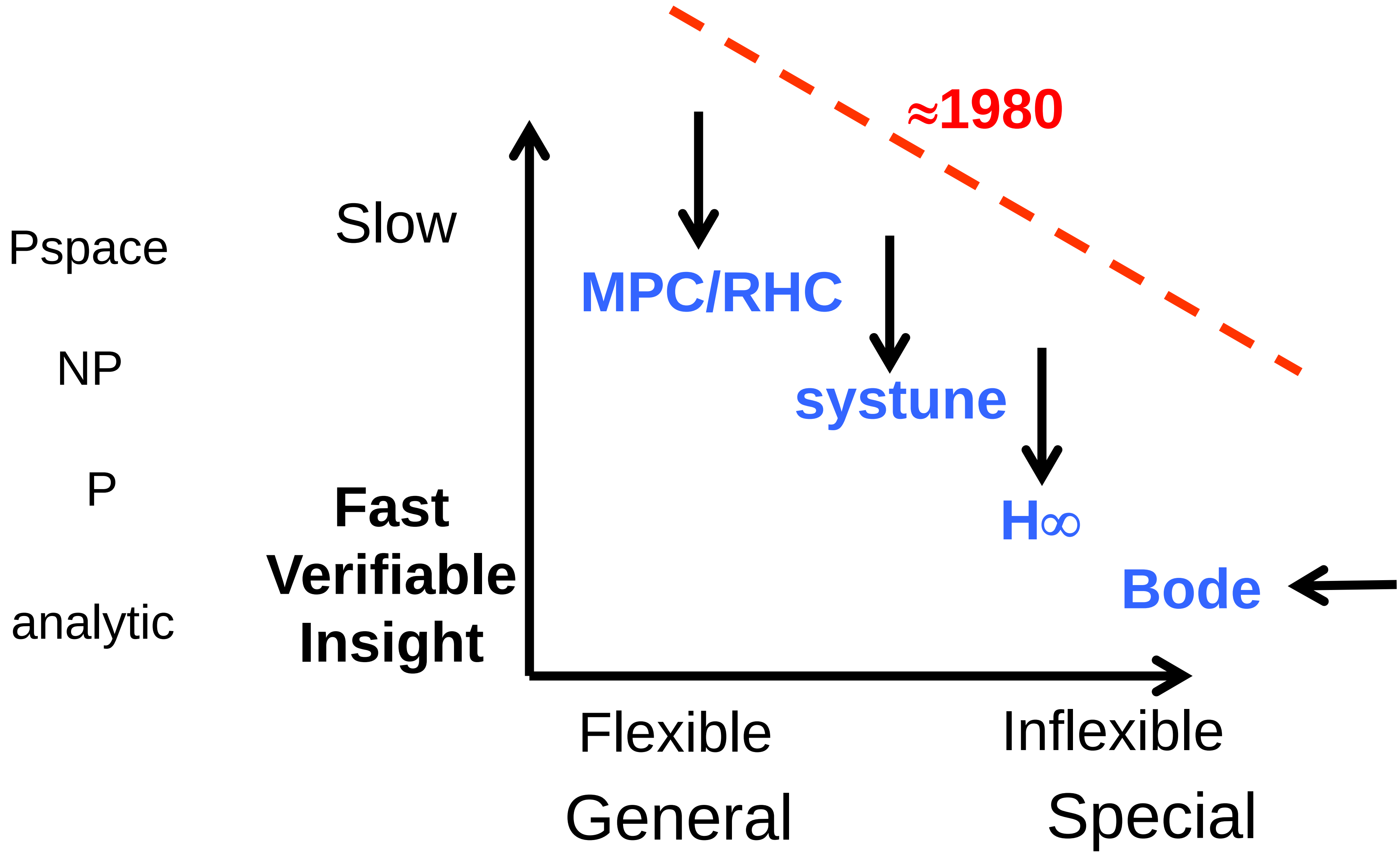




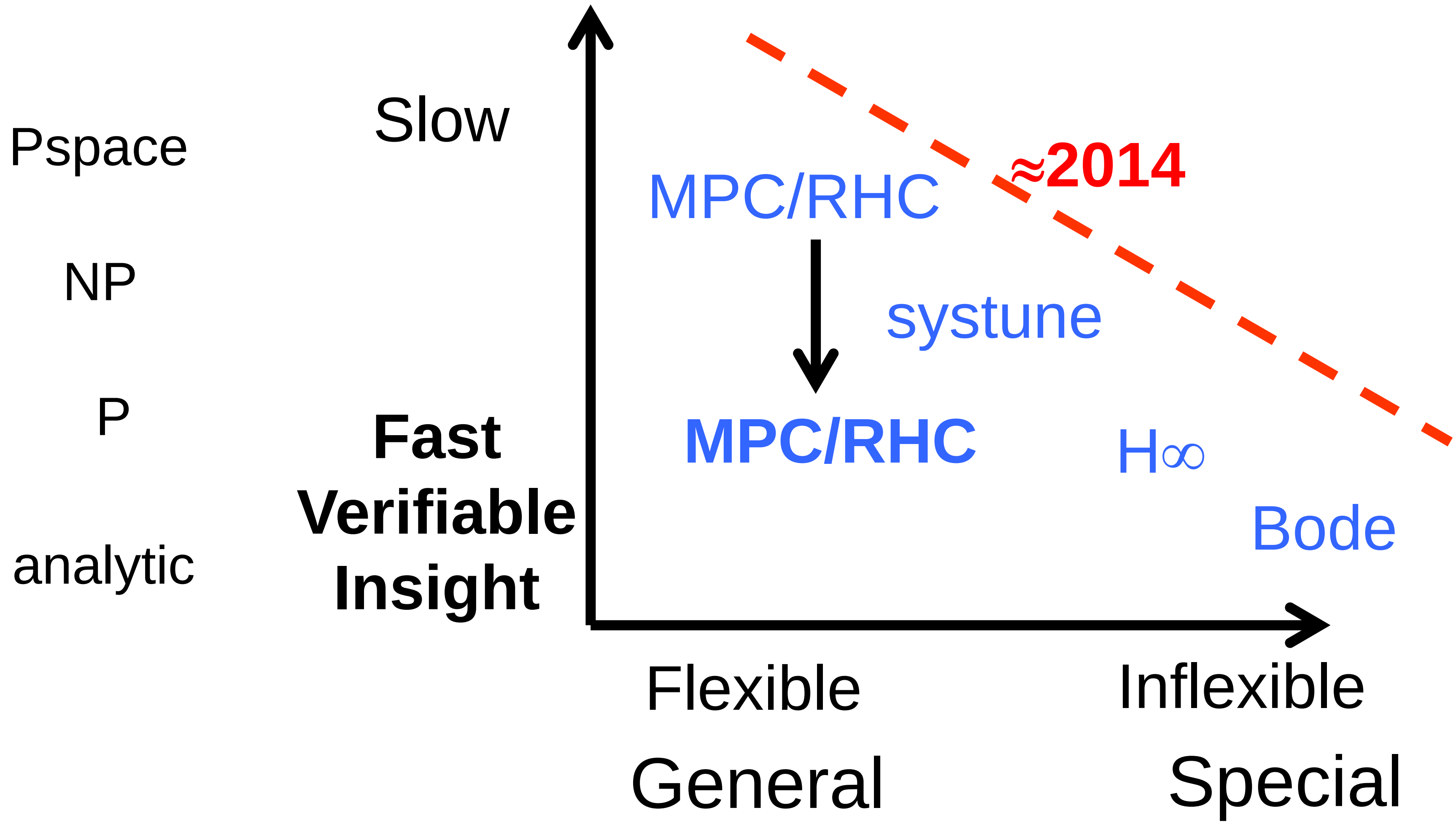


Universal laws?





Convenient cartoon



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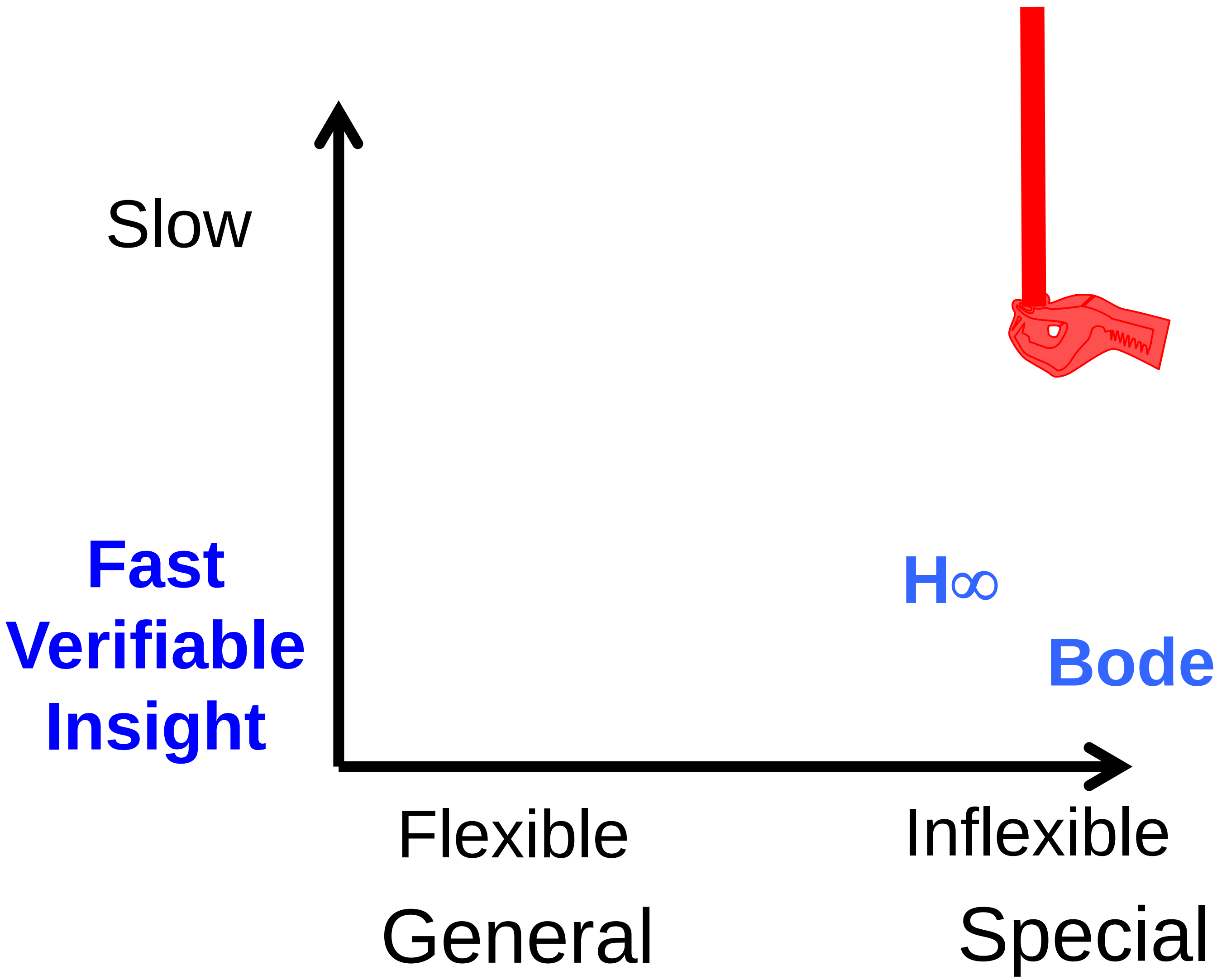

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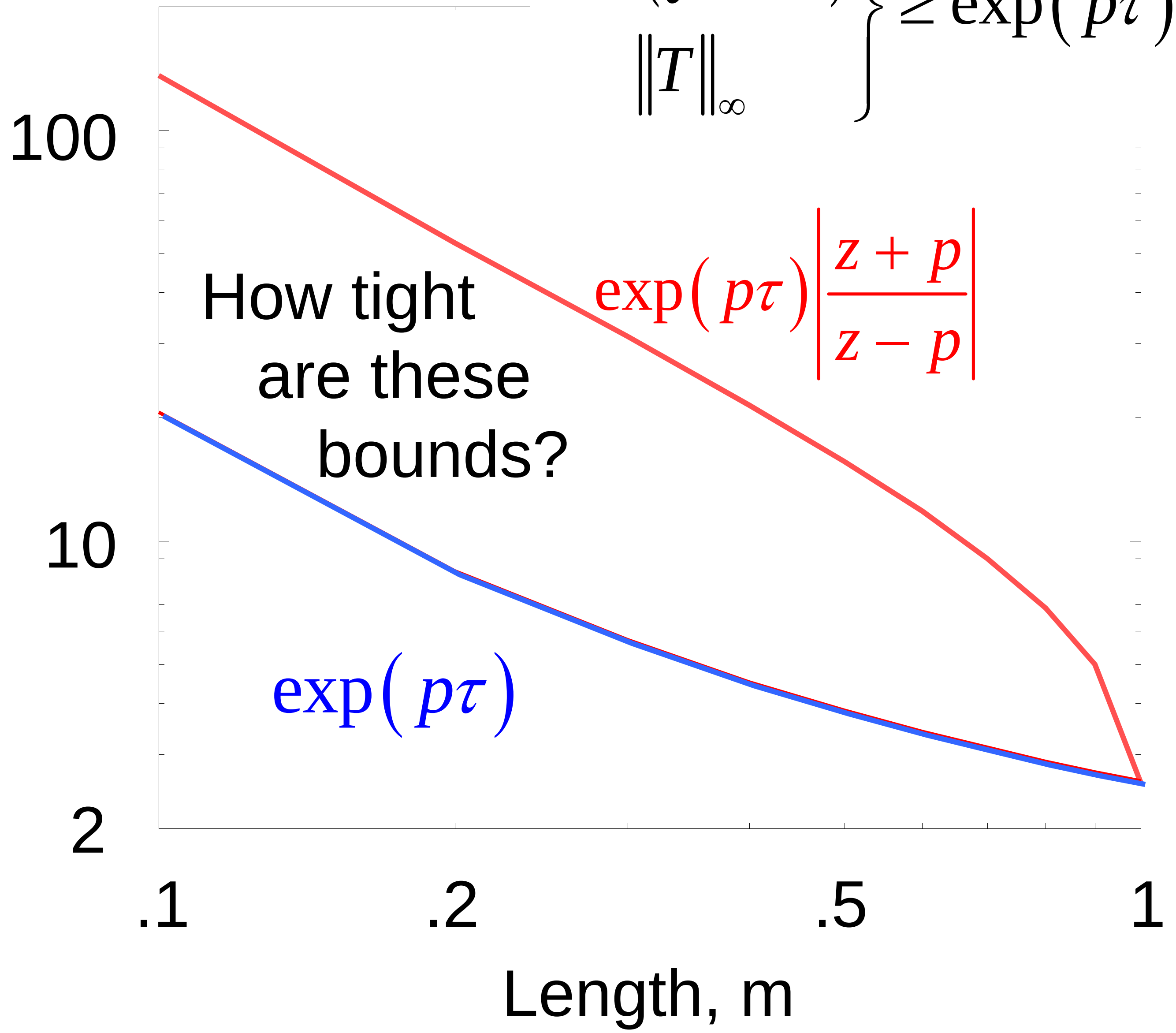
$$\left.\begin{matrix}\exp\left(\int \ln|T|\right) \\ \|T\|_\infty\end{matrix}\right\} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$



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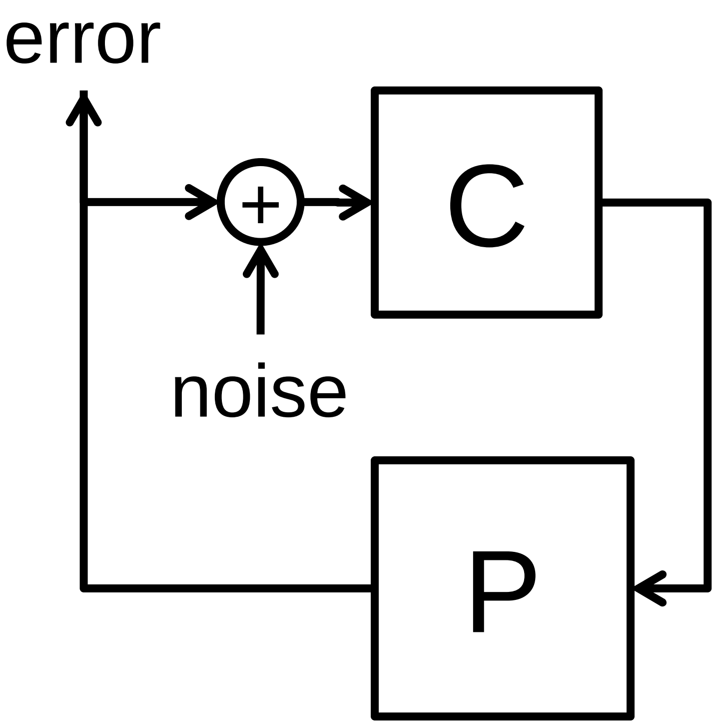
$$\exp\left(\int \ln|T|\right) = \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$

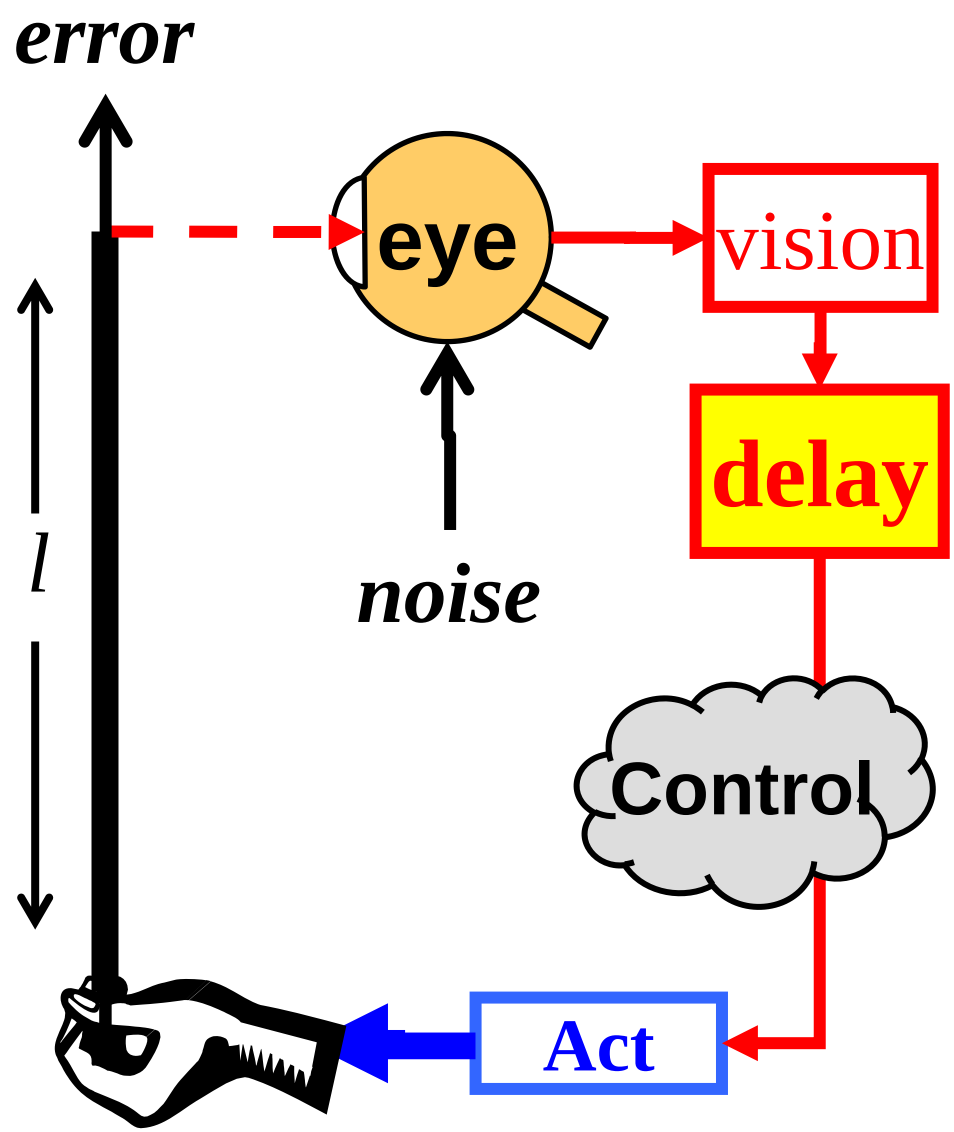
"conservation law"

$$\|WT\|_\infty \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right||W(\mathrm{p})|$$

bound

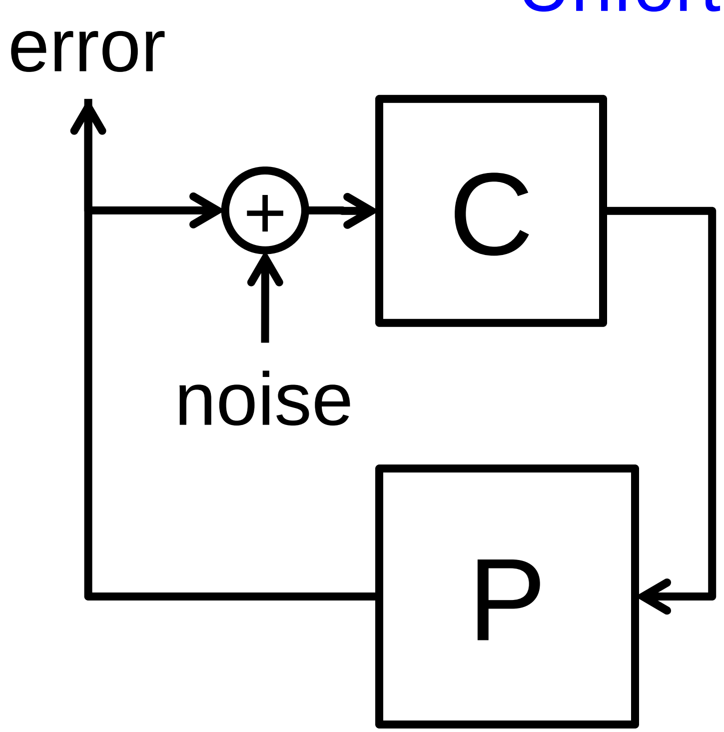
$$\left|T(j\omega)\right| = \left|\frac{E}{N}\right|$$





Solve optimal $\|T\|_2$ and $\|T\|_\infty$?

Unfortunately ill-posed.



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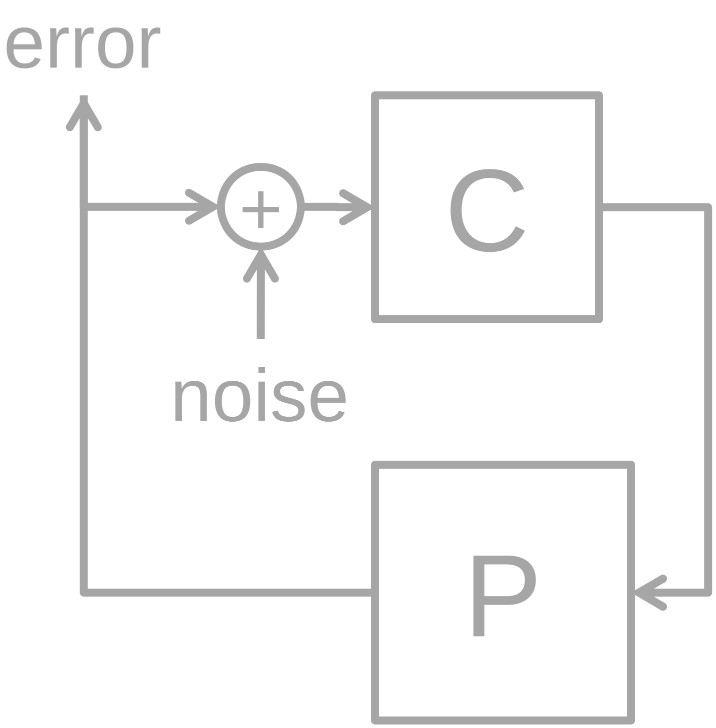
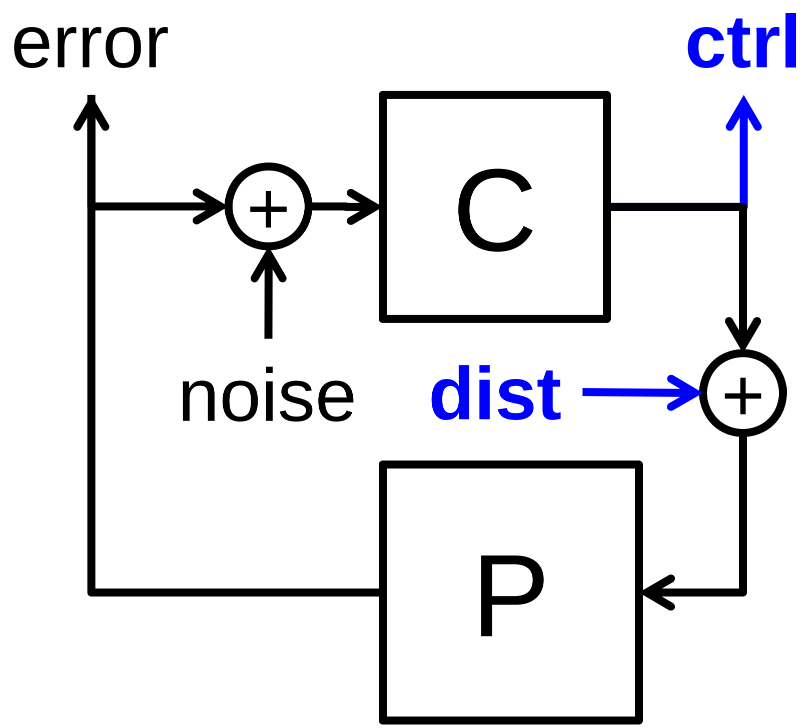
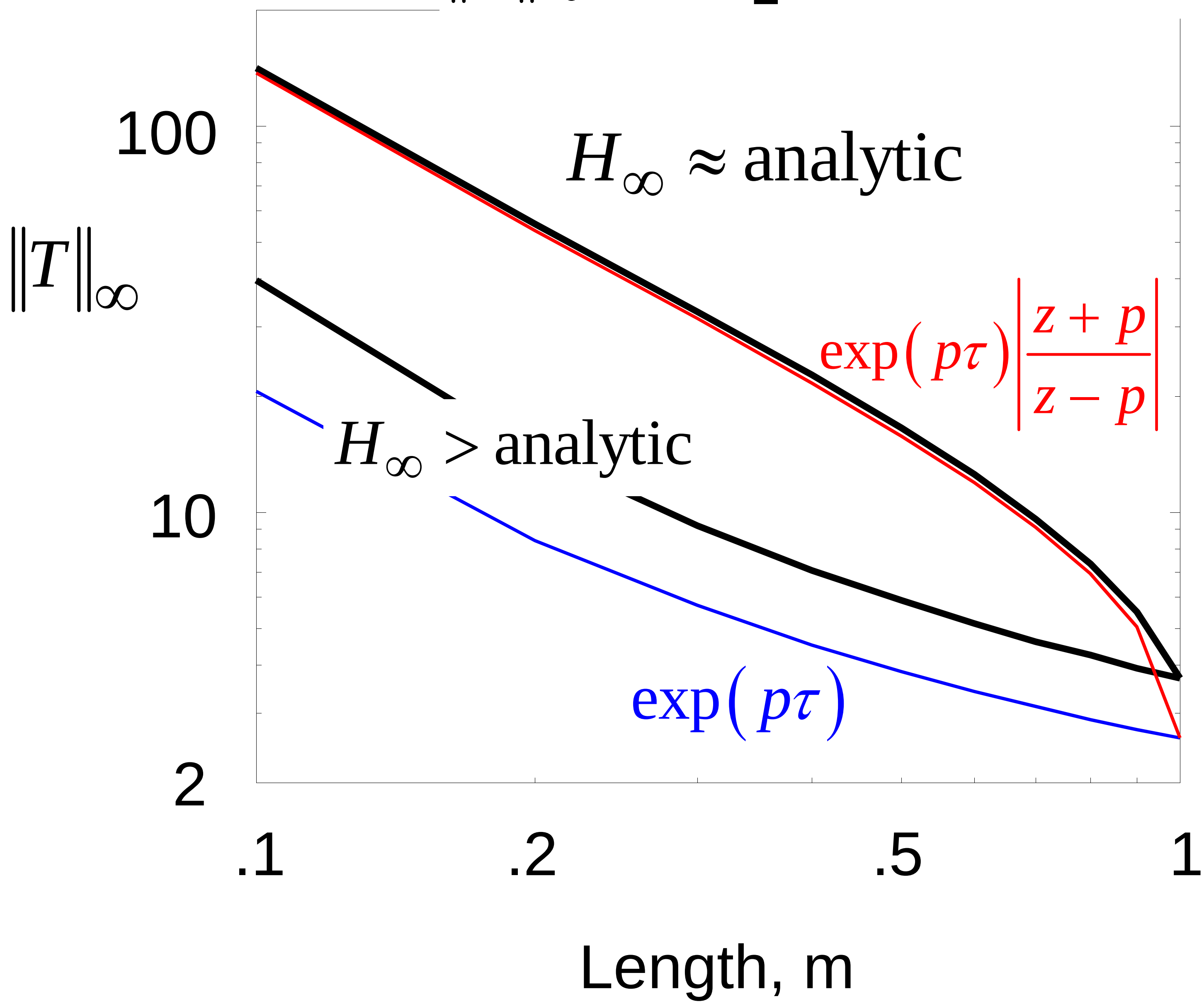
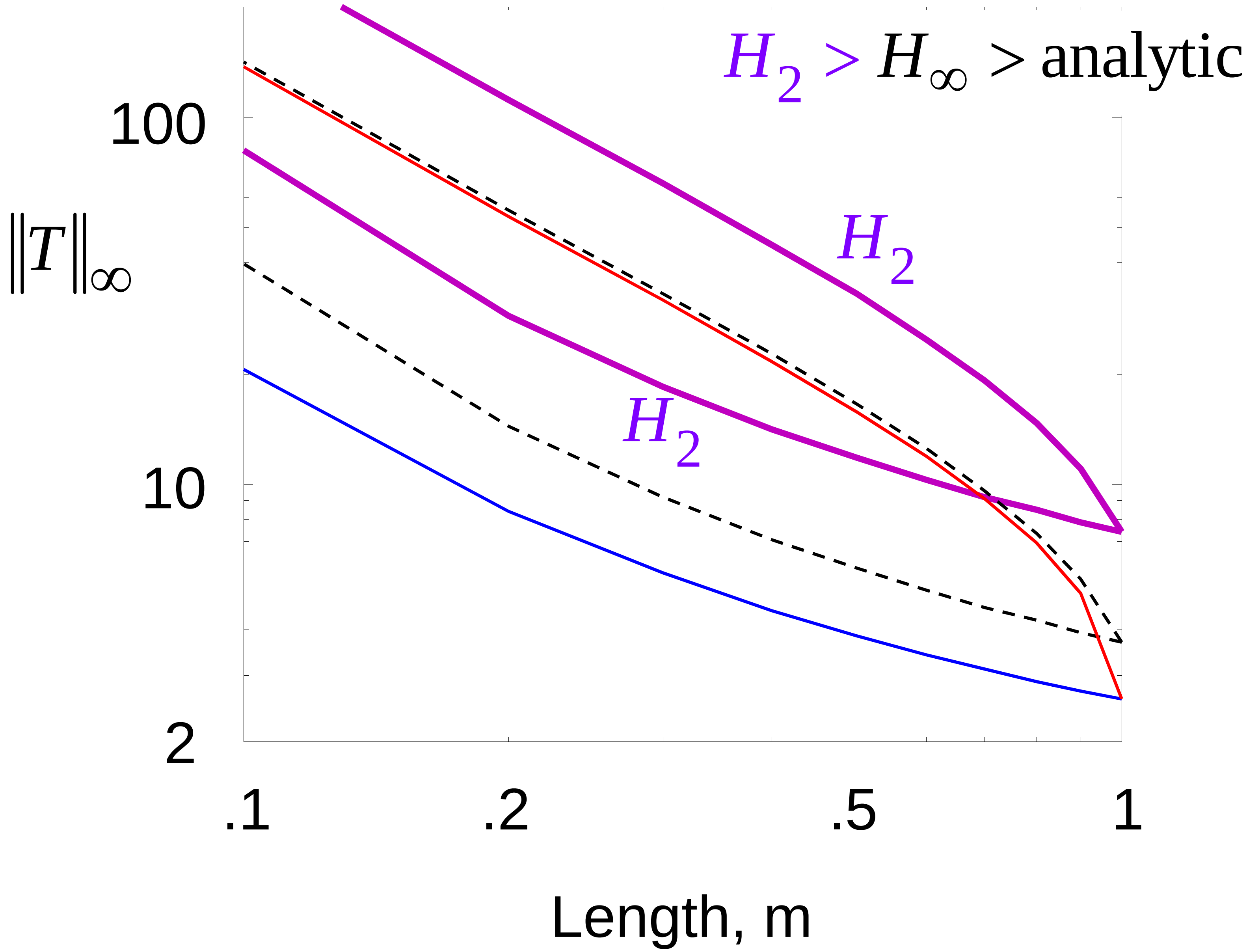




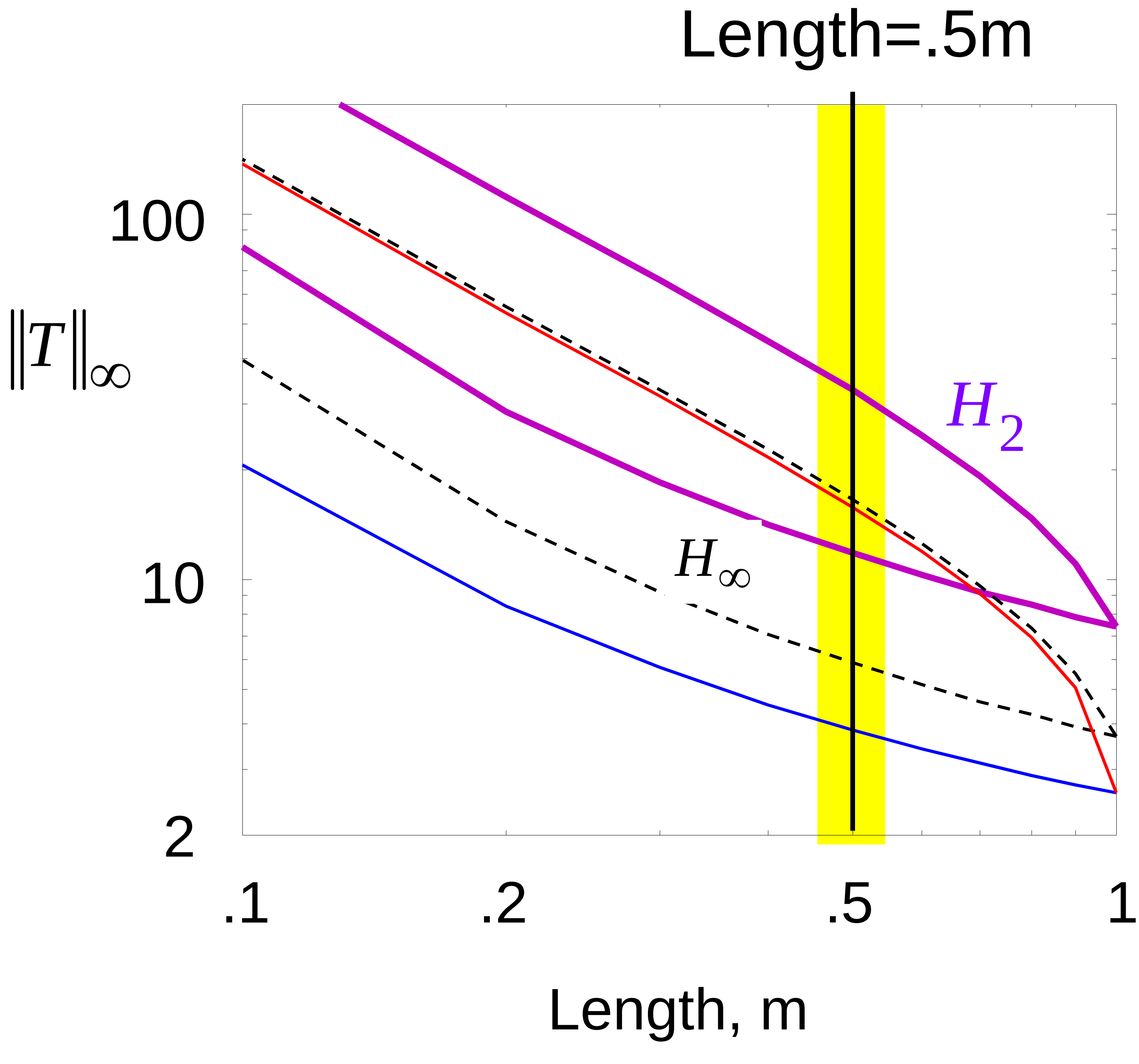
Well posed
(w/ even small weights)

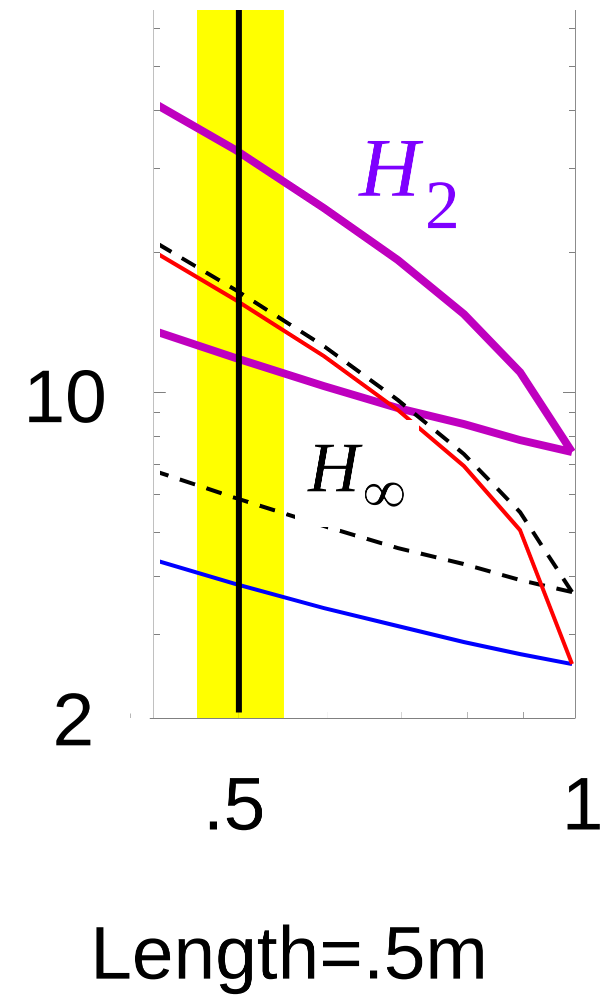
$$\|T\|_\infty \Rightarrow H_2 > H_\infty > \text{analytic}$$

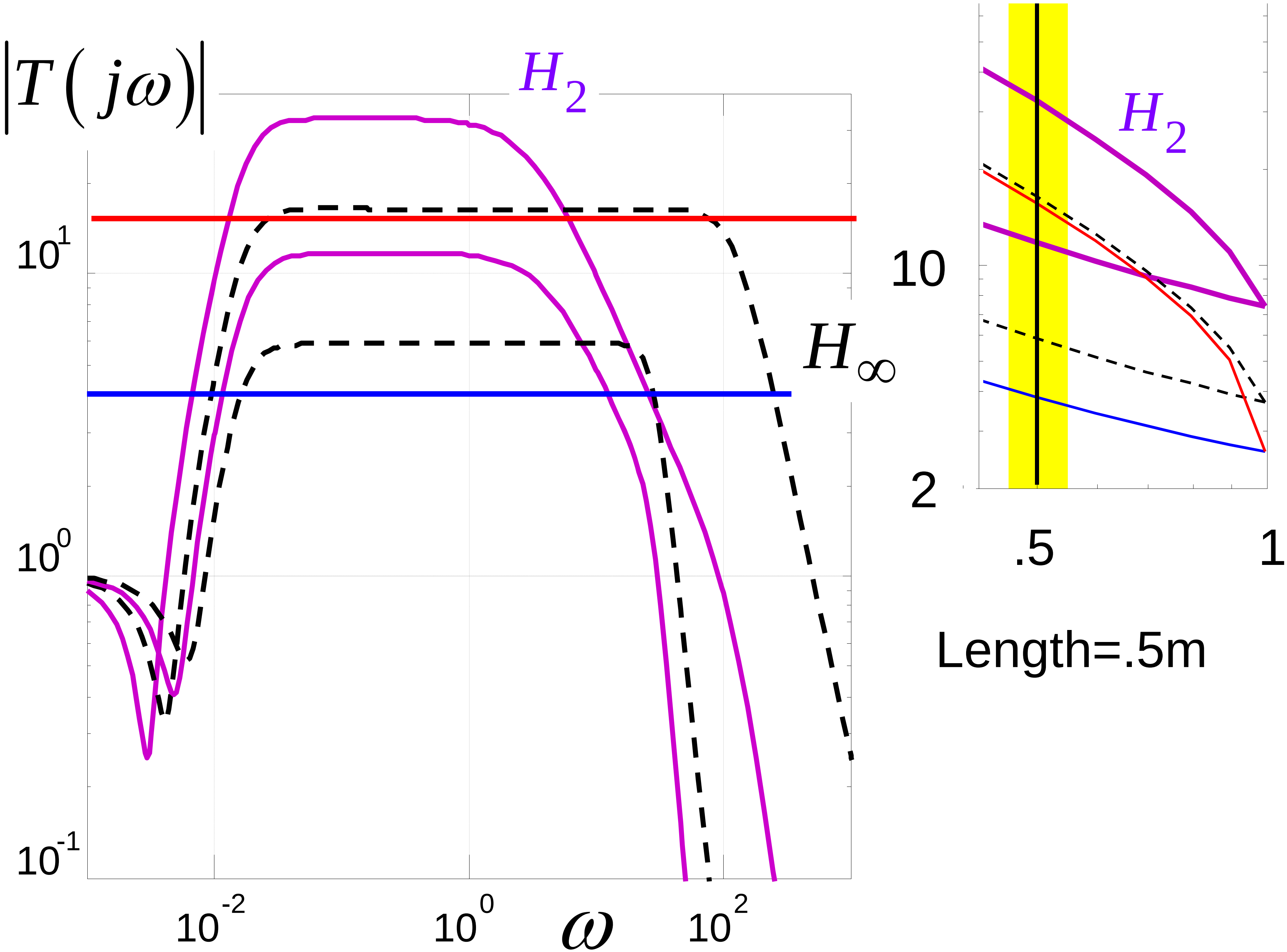




$H_2 > H_\infty >$ analytic
100
10
2
$\|T\|_\infty$
H_2
H_2
.1
.2
.5
1
Length, m


Length=.5m
100
10
2
$\|T\|_\infty$
H_2
H_∞
.1
.2
.5
1
Length, m

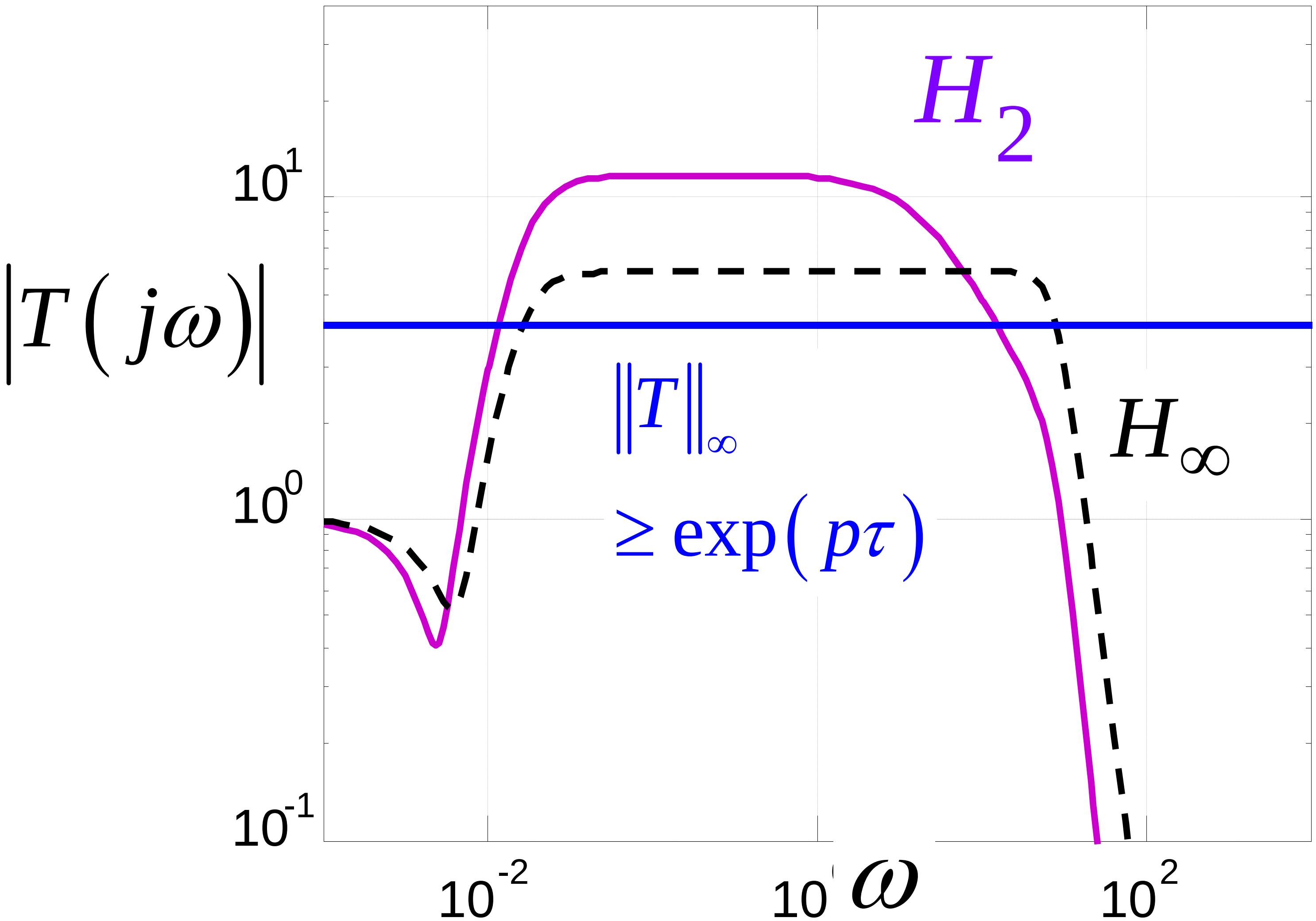

H_2
H_∞
10
2
.5
1
Length=.5m



Length=.5m

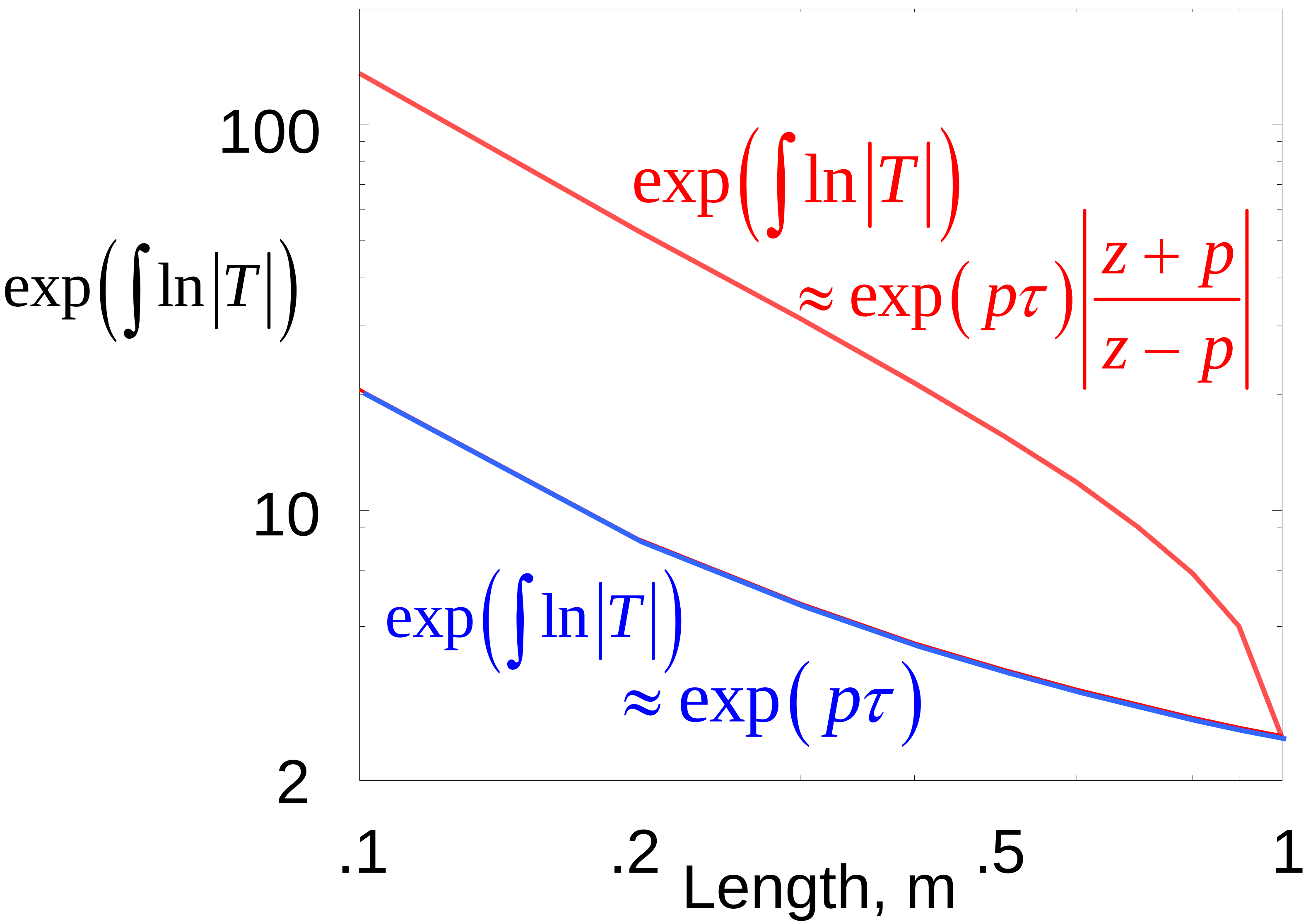
$$\exp\left(\int \ln|T|\right) \Rightarrow H_2 = H_\infty = \text{analytic}$$

$$\|T\|_\infty \Rightarrow H_2 > H_\infty > \text{analytic}$$



$$\exp\left(\int \ln|T|\right) \Rightarrow H_2 \approx H_\infty \approx \text{analytic}$$

$\approx$ is numerical errors



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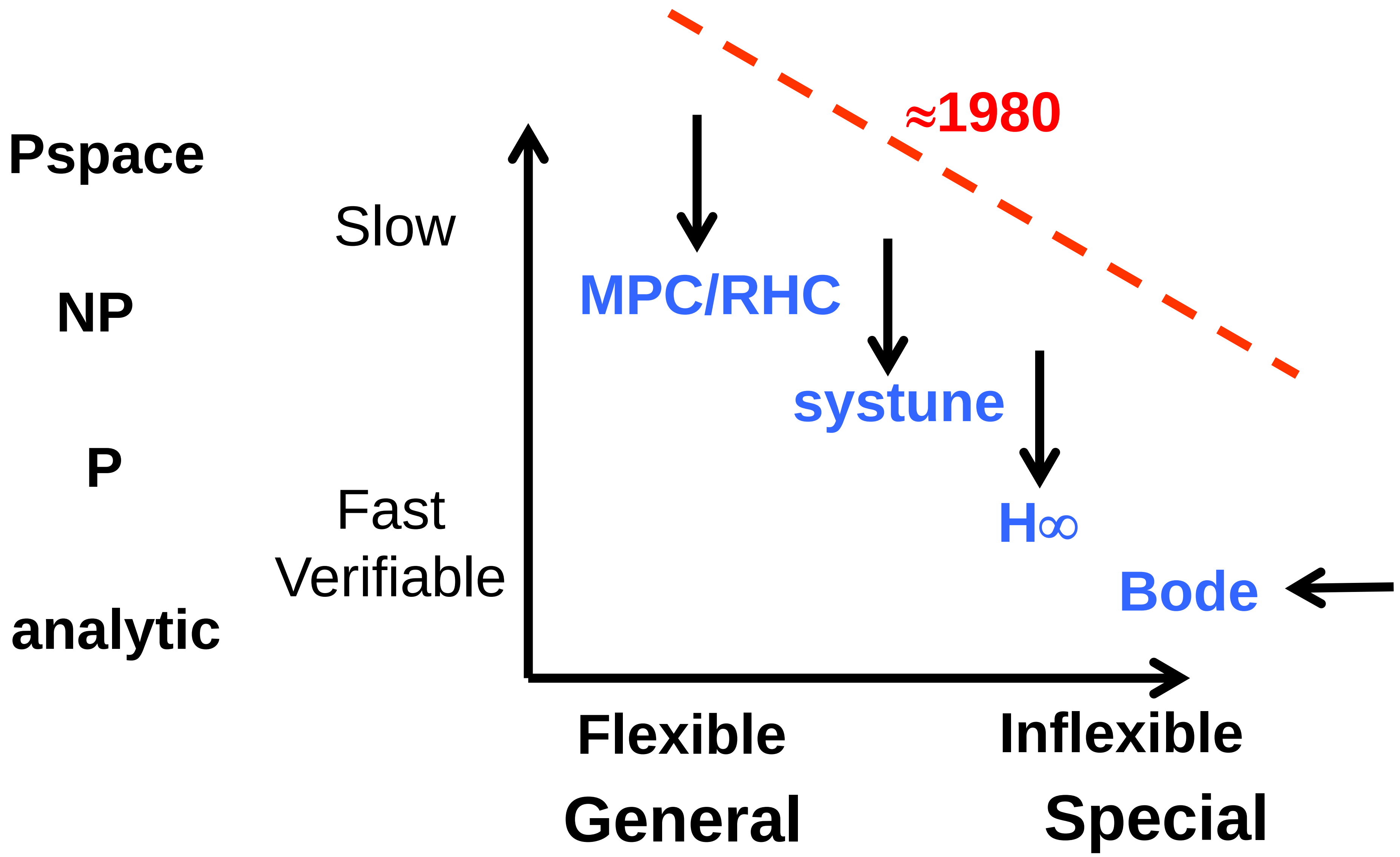
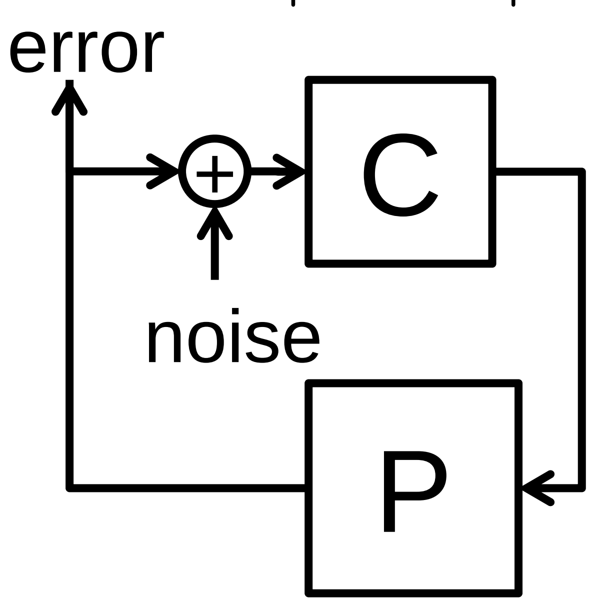
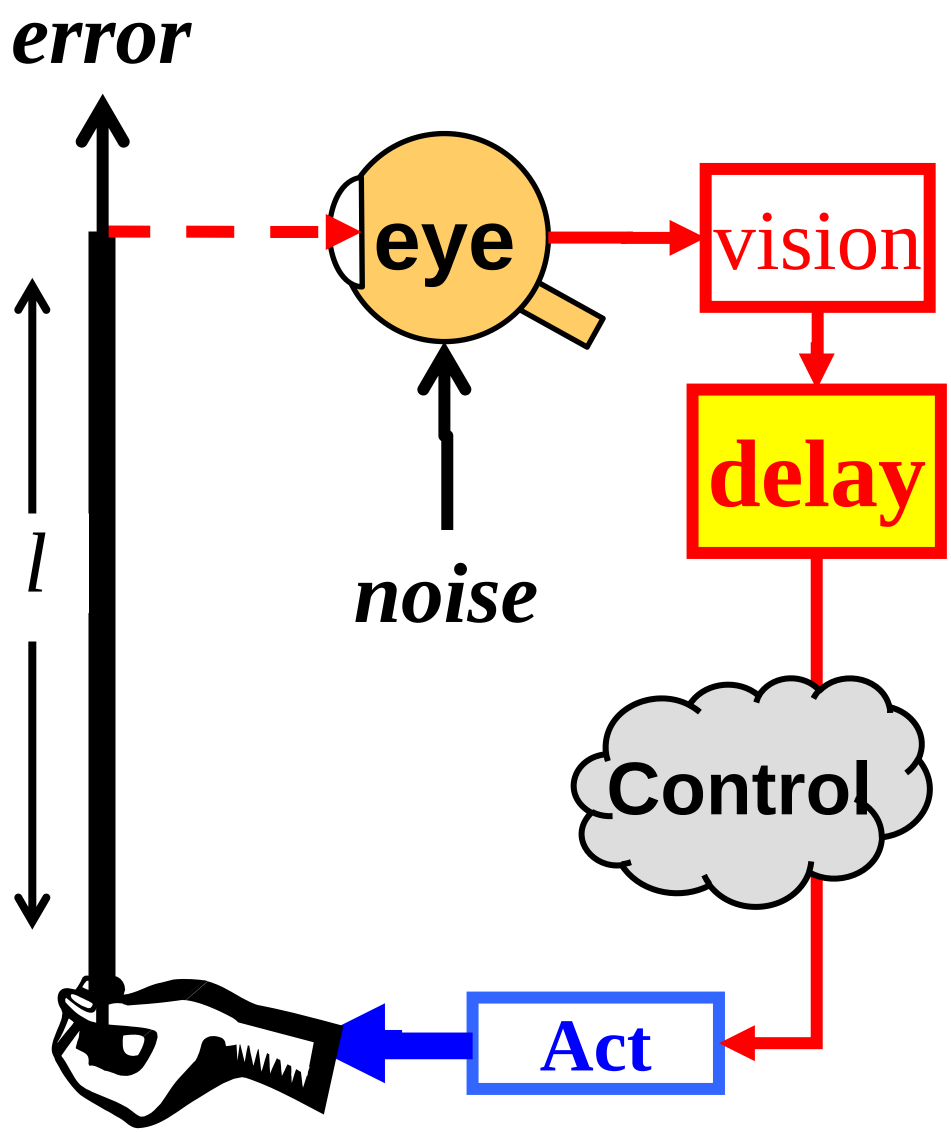
bound


≈1980
Pspace
Slow
MPC/RHC
NP
systune
P
Fast
H∞
Verifiable
Bode
analytic
Flexible
Inflexible
General
Special

What is we wanted to understand this more deeply?

$$\left|T(j\omega)\right| = \left|\frac{E}{N}\right|$$





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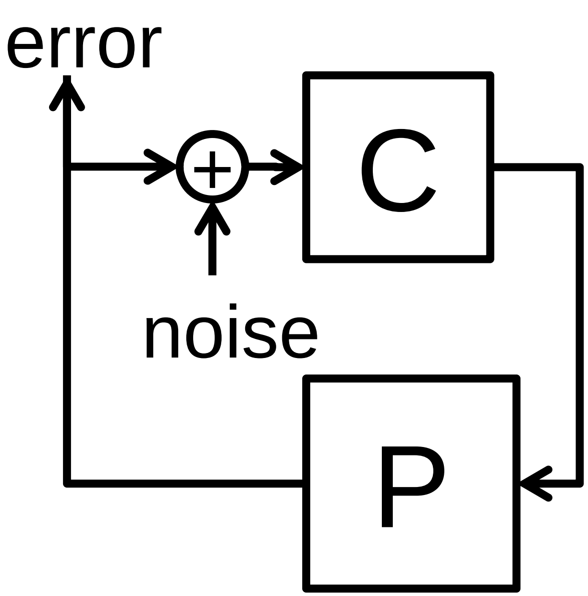
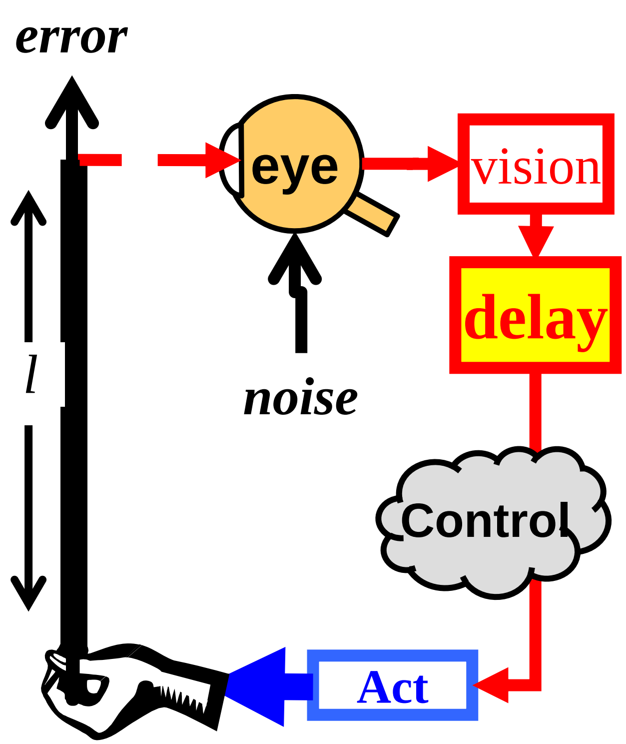
Mechanics+
Gravity +
Light +

+ Neuroscience

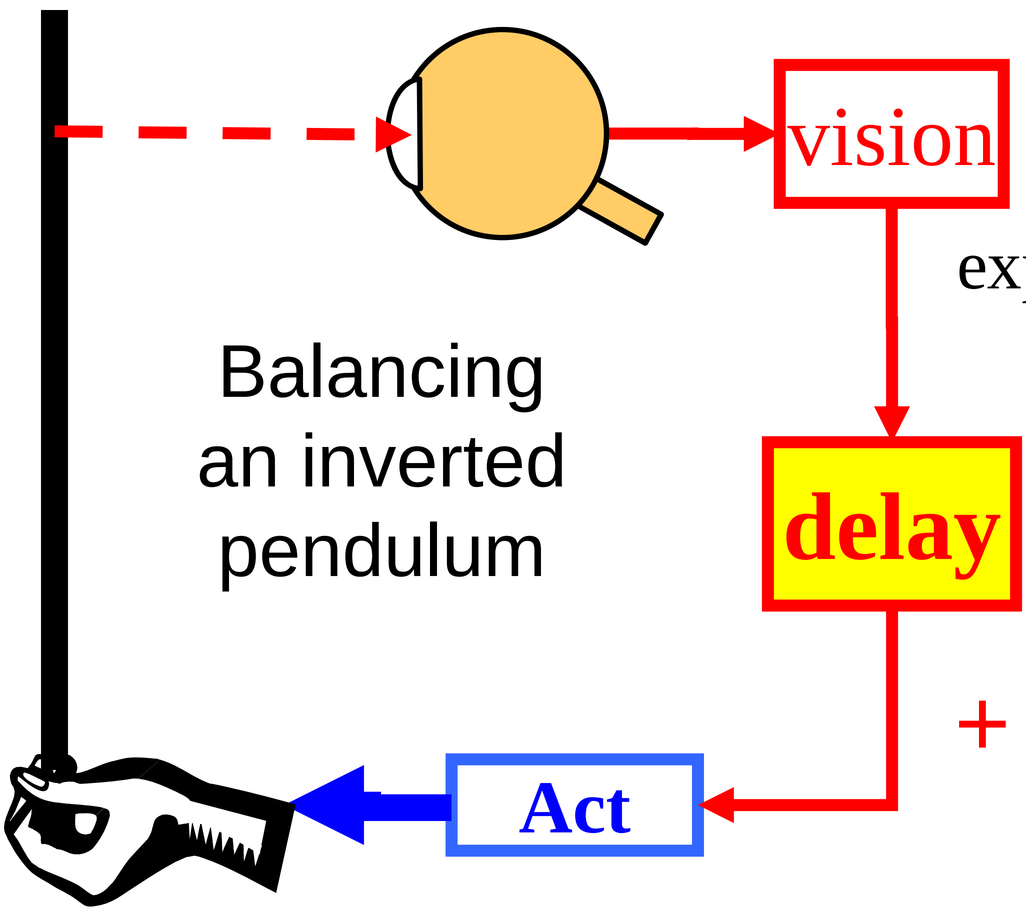
$$|T(j\omega)| = \left|\frac{E}{N}\right|$$





Universal laws

Mechanics+
Gravity +
Light +

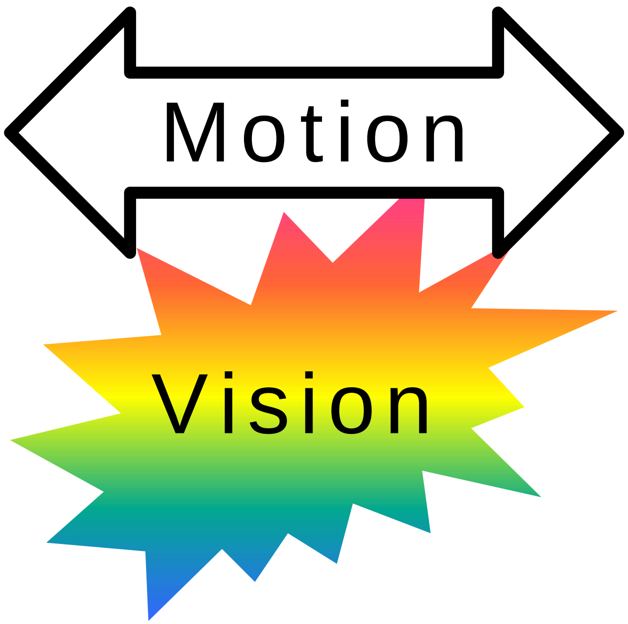


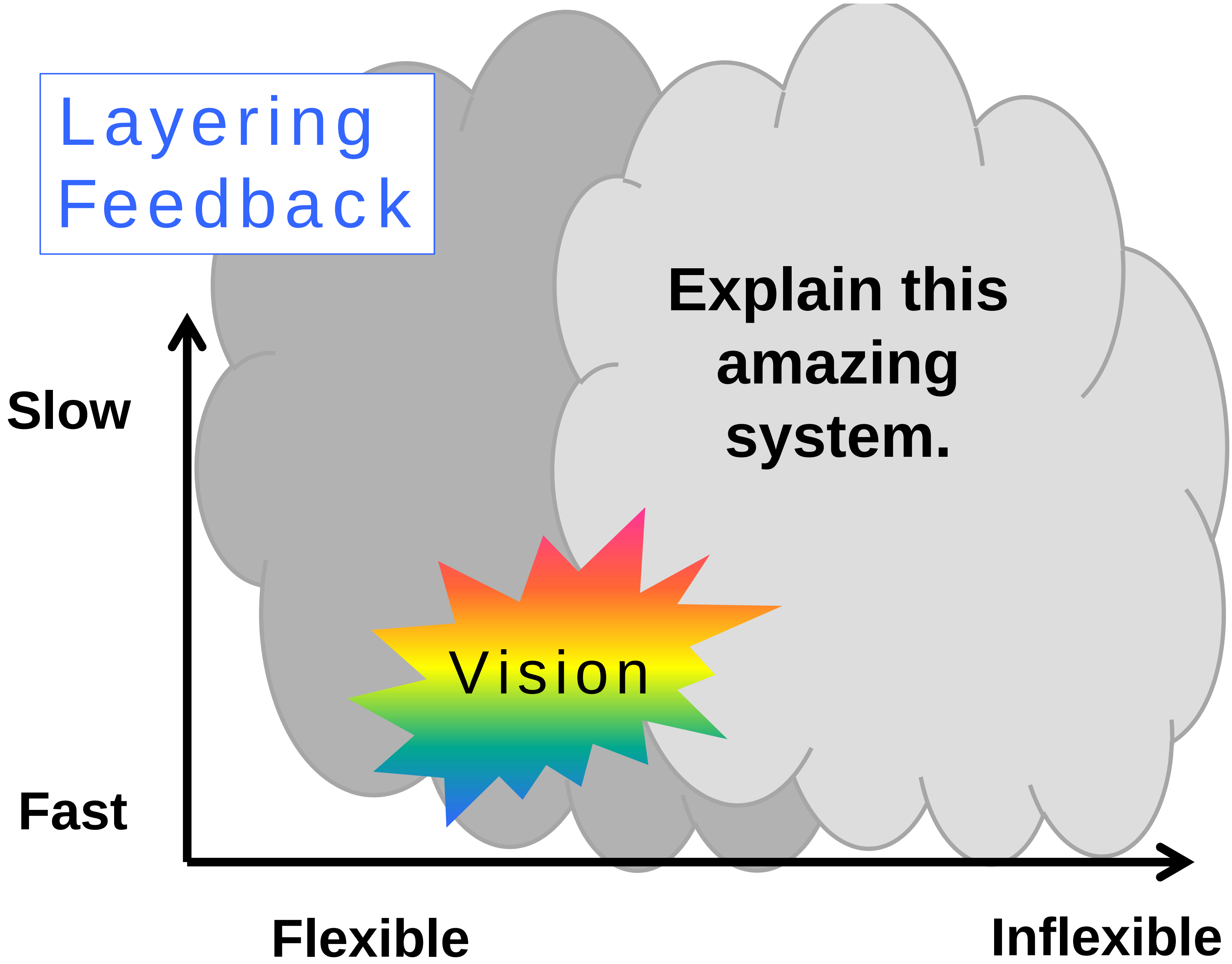
$$\left.\begin{matrix} \exp\left(\int \ln|T|\right) \\ \|T\|_\infty \end{matrix}\right\} \geq \exp(p\tau)\left|\frac{z+p}{z-p}\right|$$

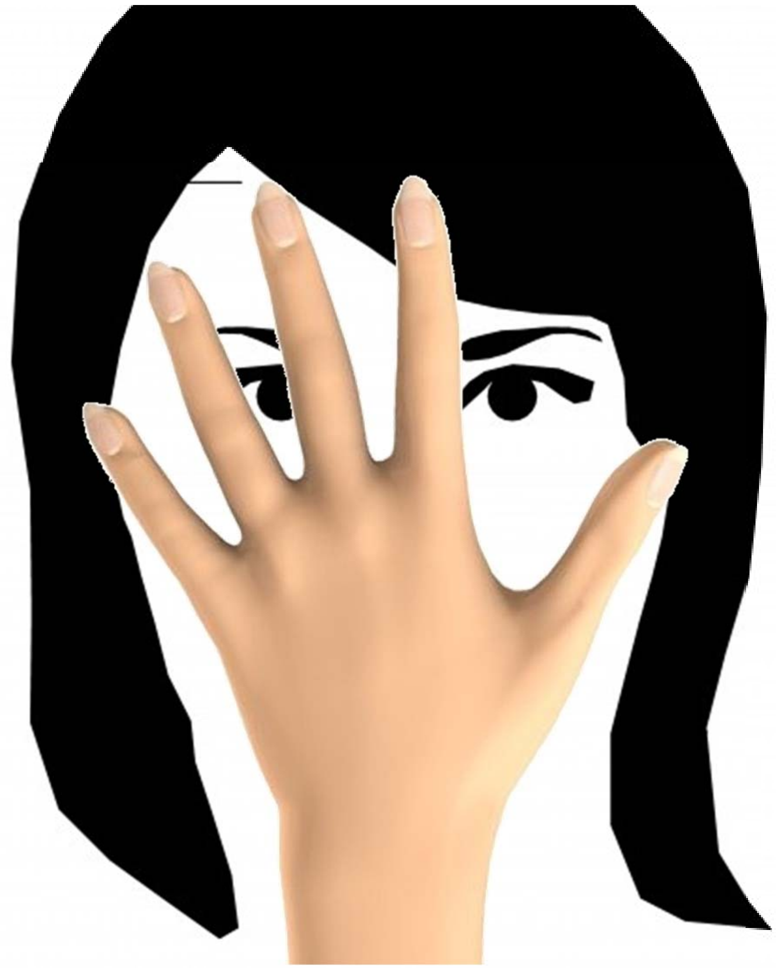
+ Neuroscience

Robust vision with motion

- Object motion
- Self motion




Layering
Feedback
Explain this amazing system.
Slow
Fast
Vision
Flexible
Inflexible



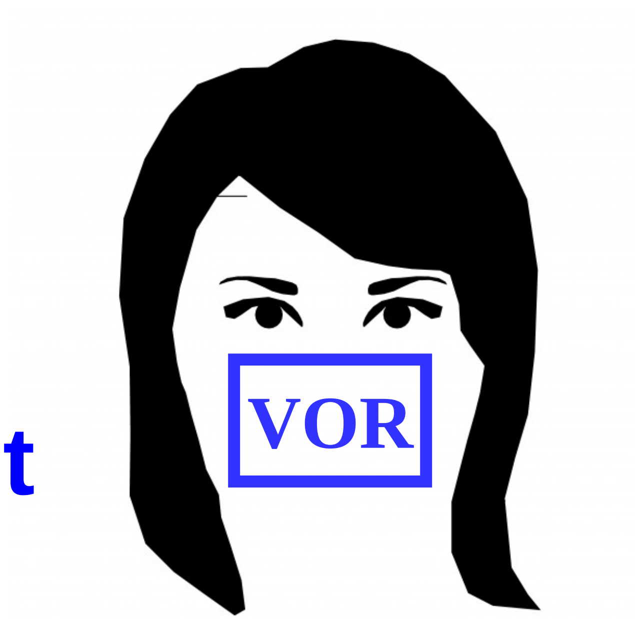
Robust vision with
- Hand motion
- Head motion

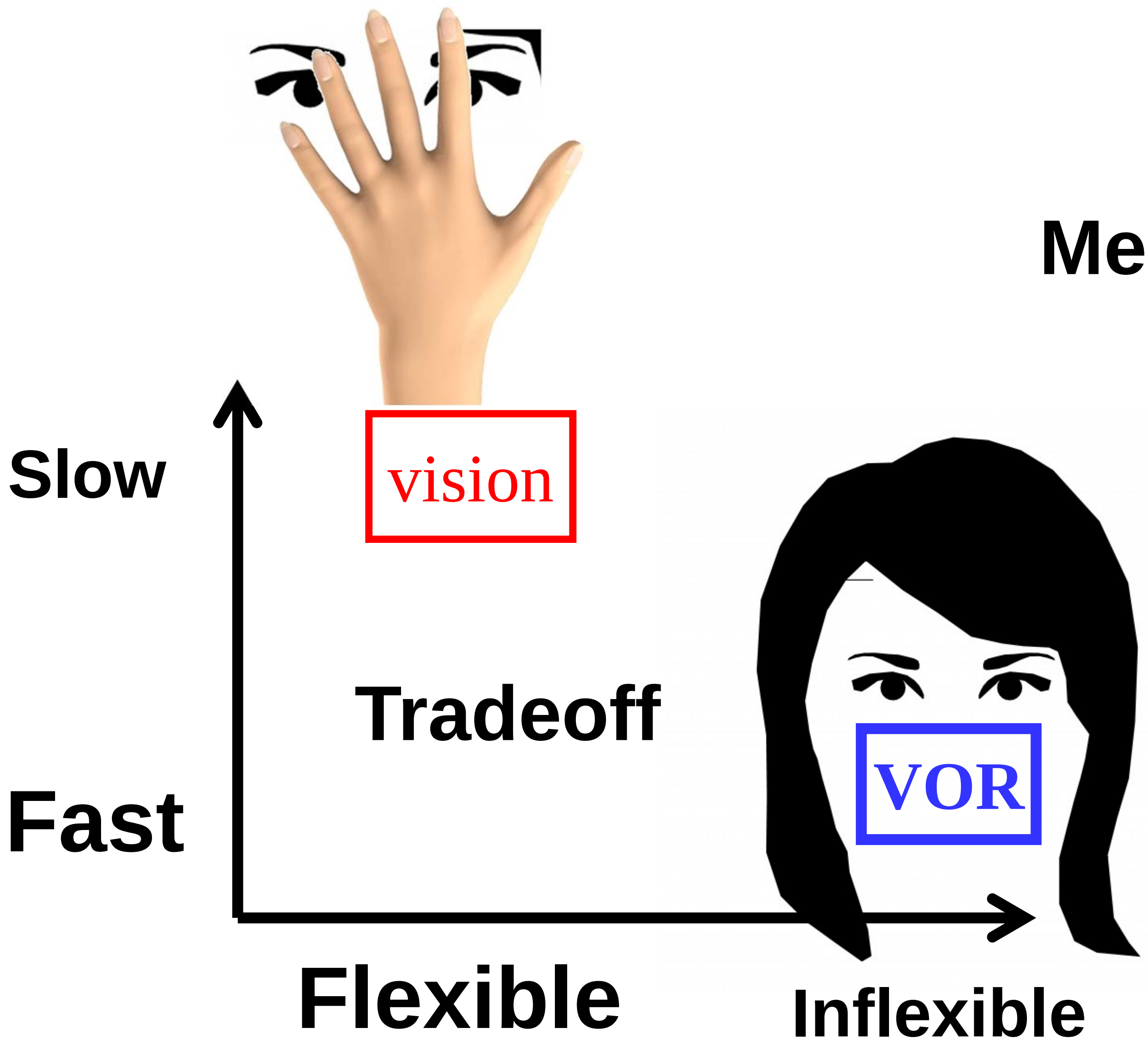
Experiment
- Motion/vision control without blurring
- Which is easier and faster?

Why?
- Mechanism
- Tradeoff

Slow

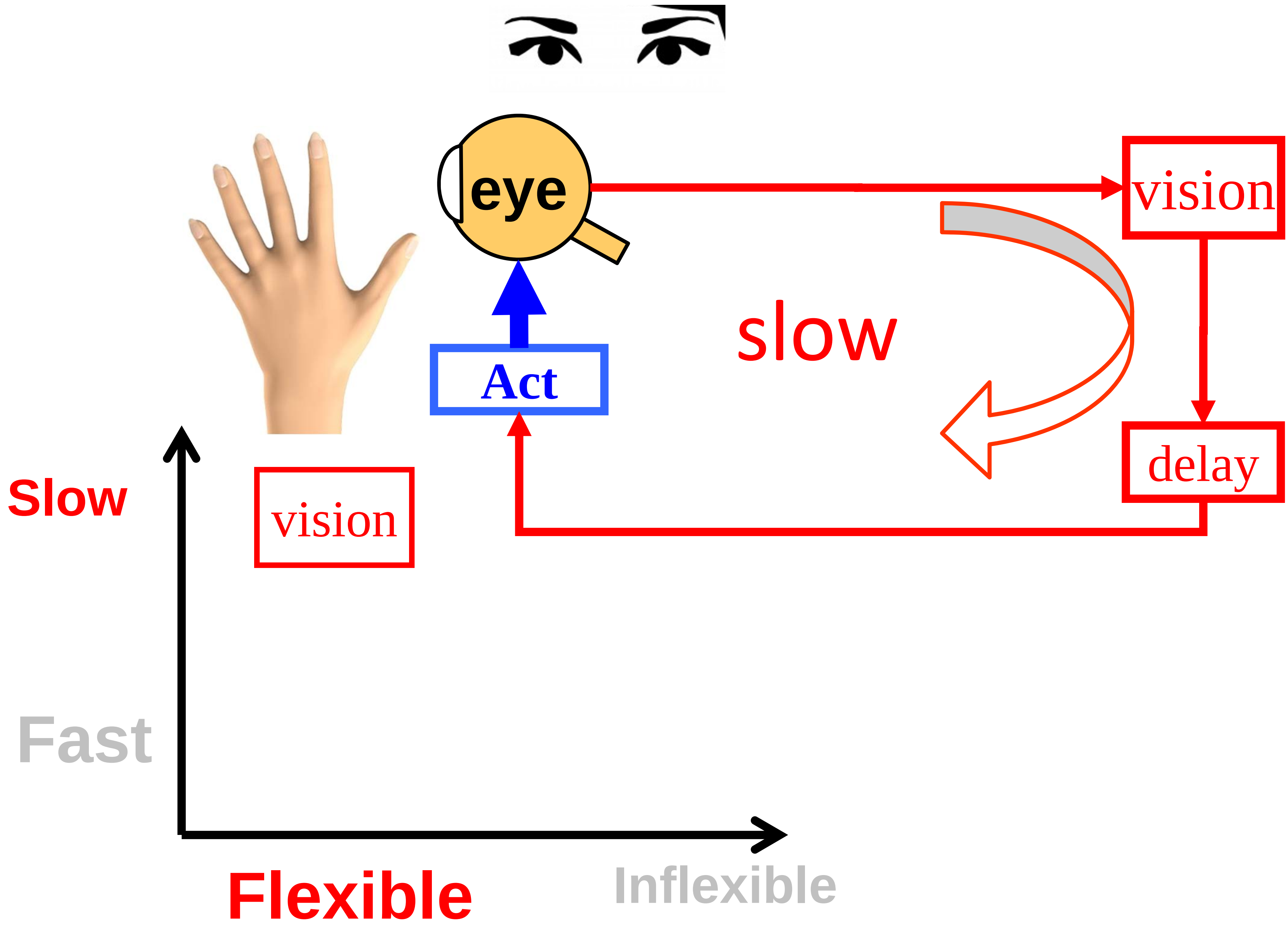
Fast

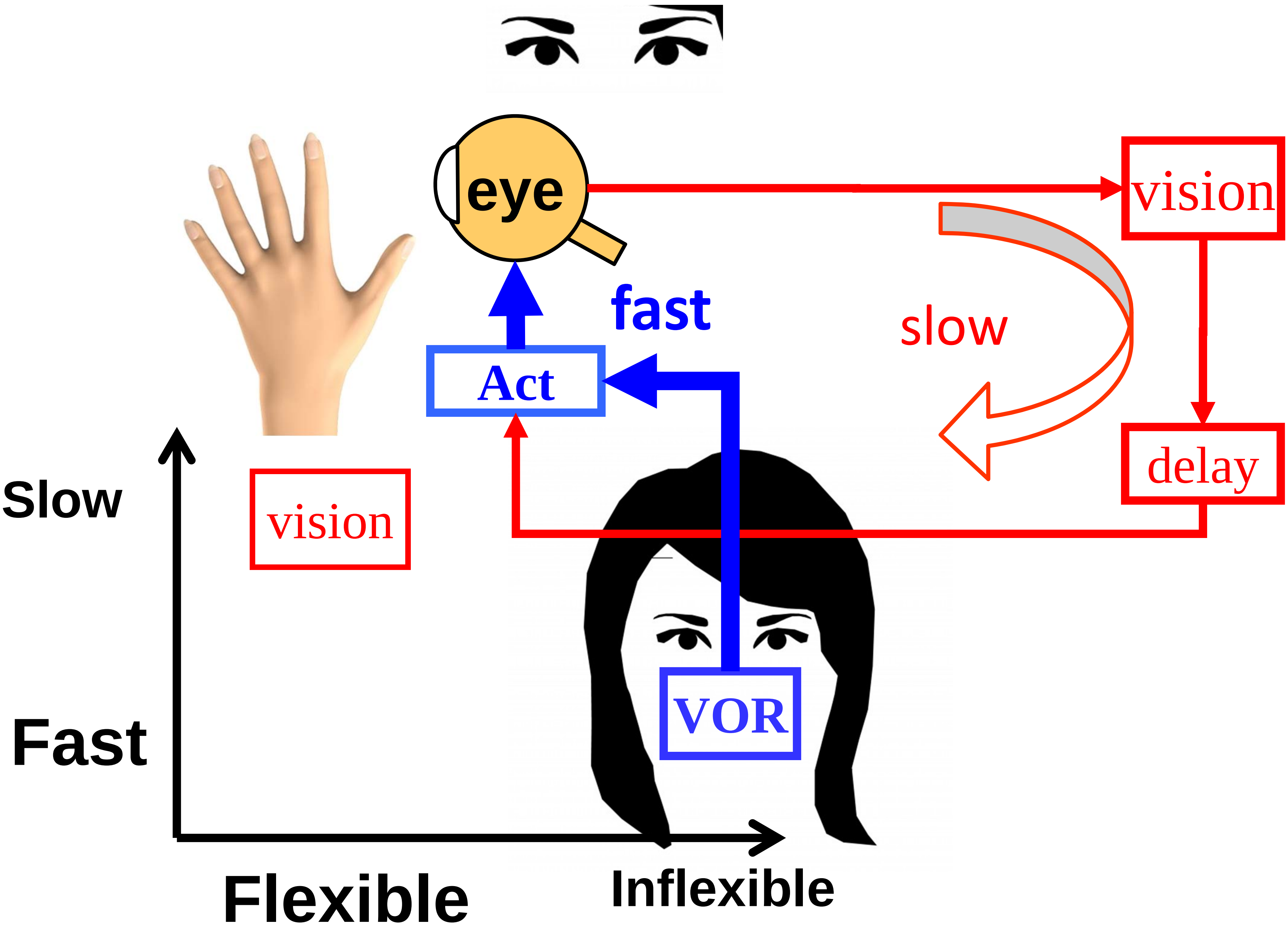


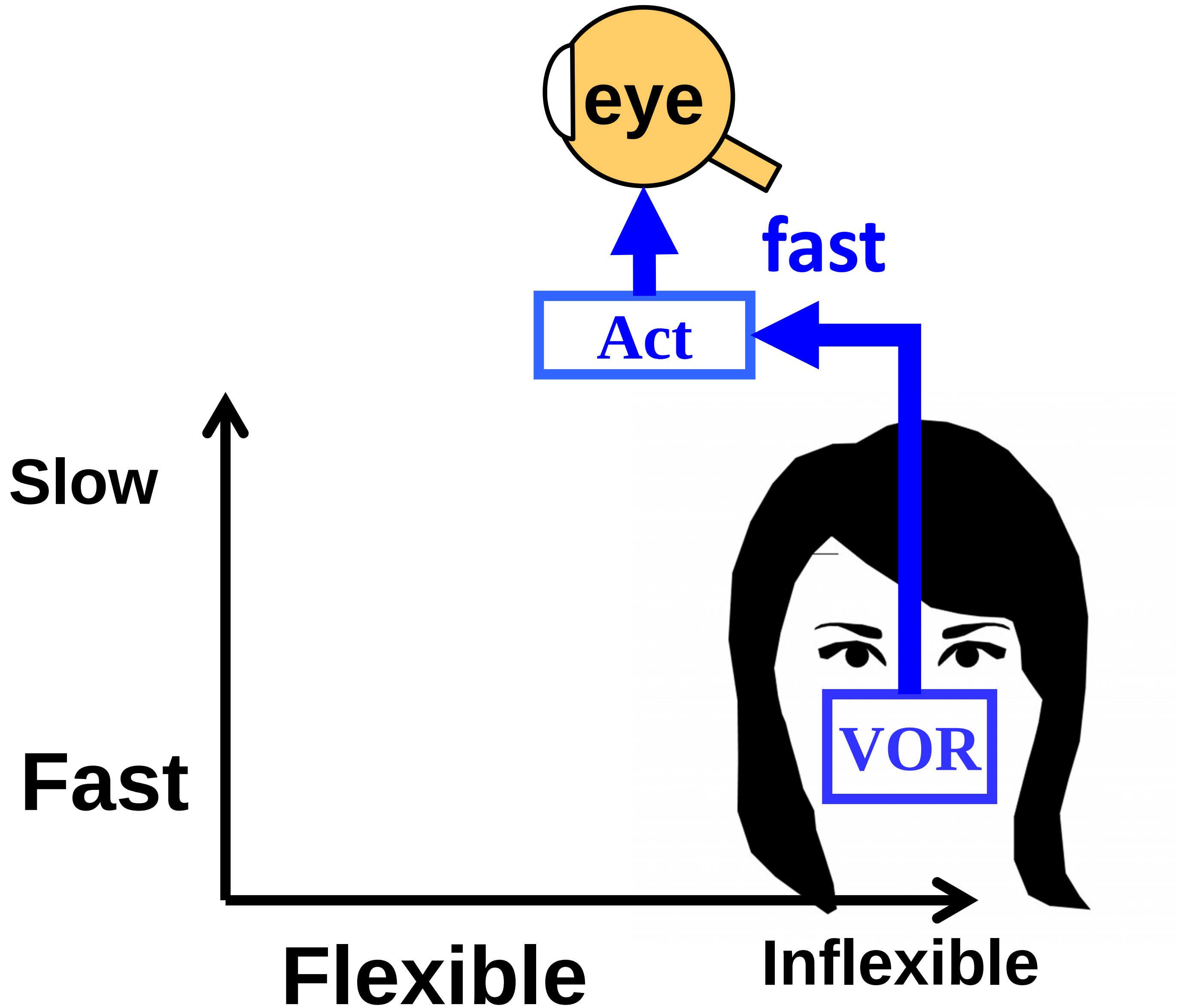


Mechanism

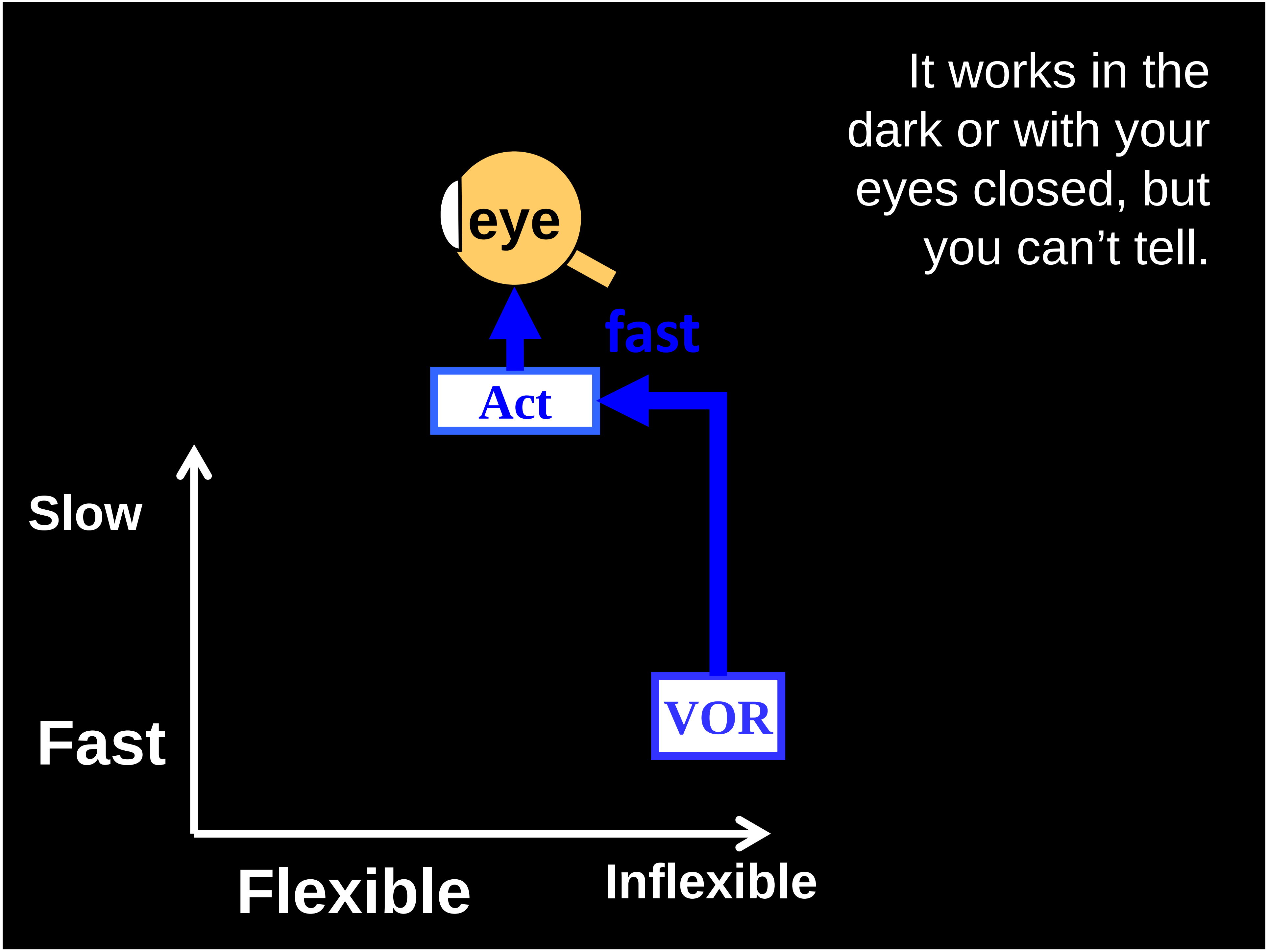
Vestibular
Ocular
Reflex
(VOR)

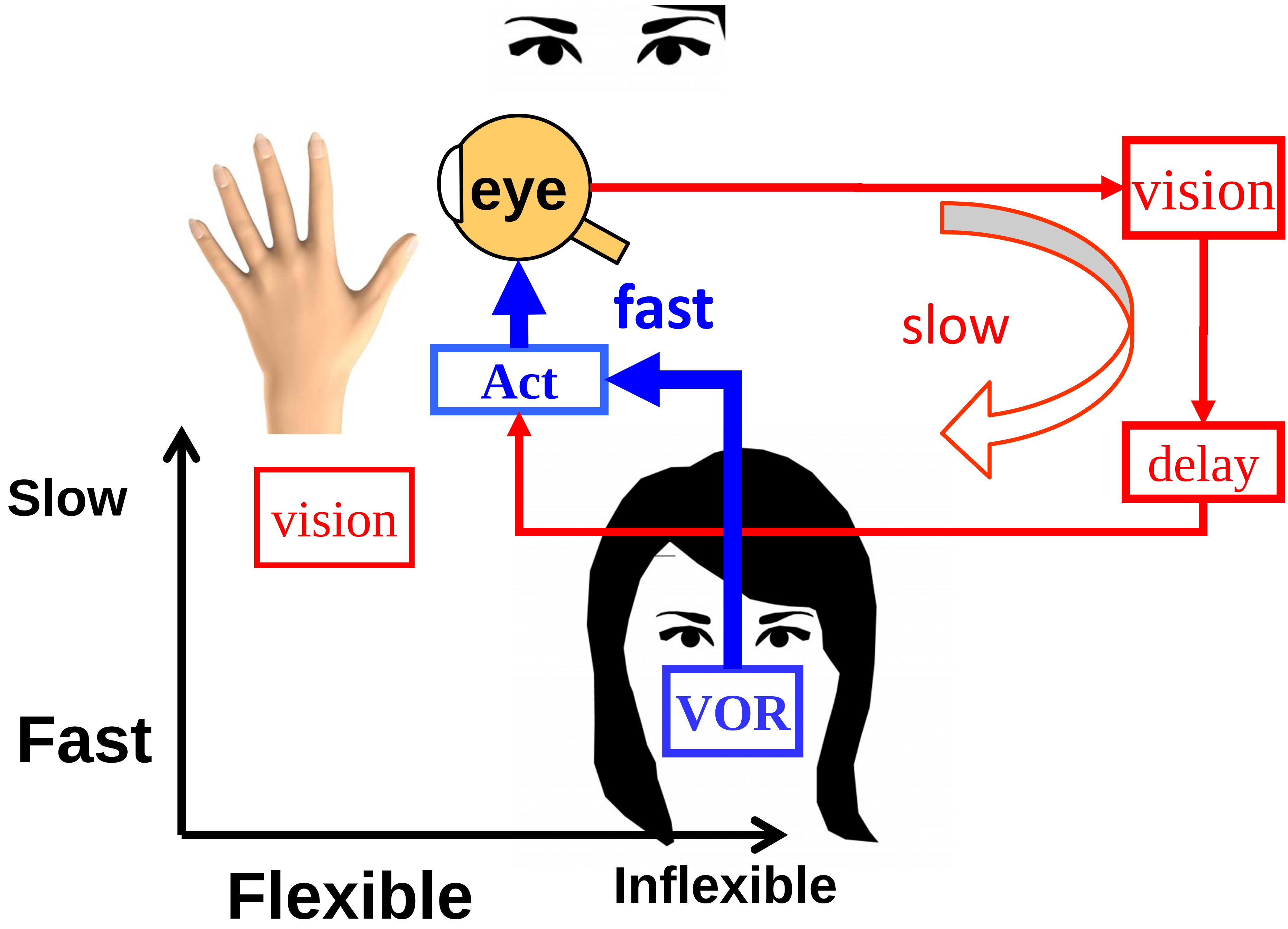

eye
vision
slow
Act
delay
Slow
vision
Fast
Flexible
Inflexible

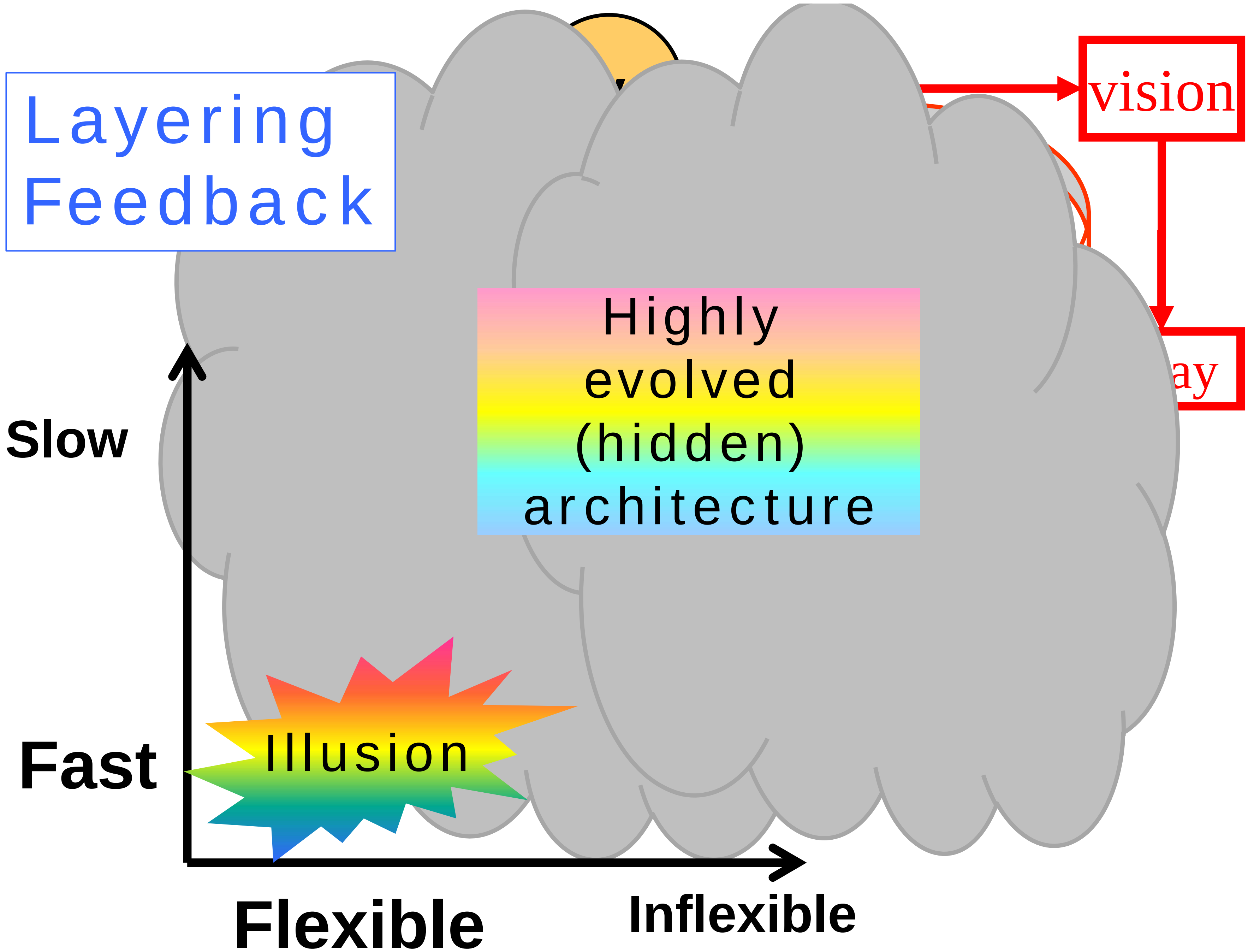

eye
vision
fast
slow
Act
delay
Slow
vision
VOR
Fast
Flexible
Inflexible



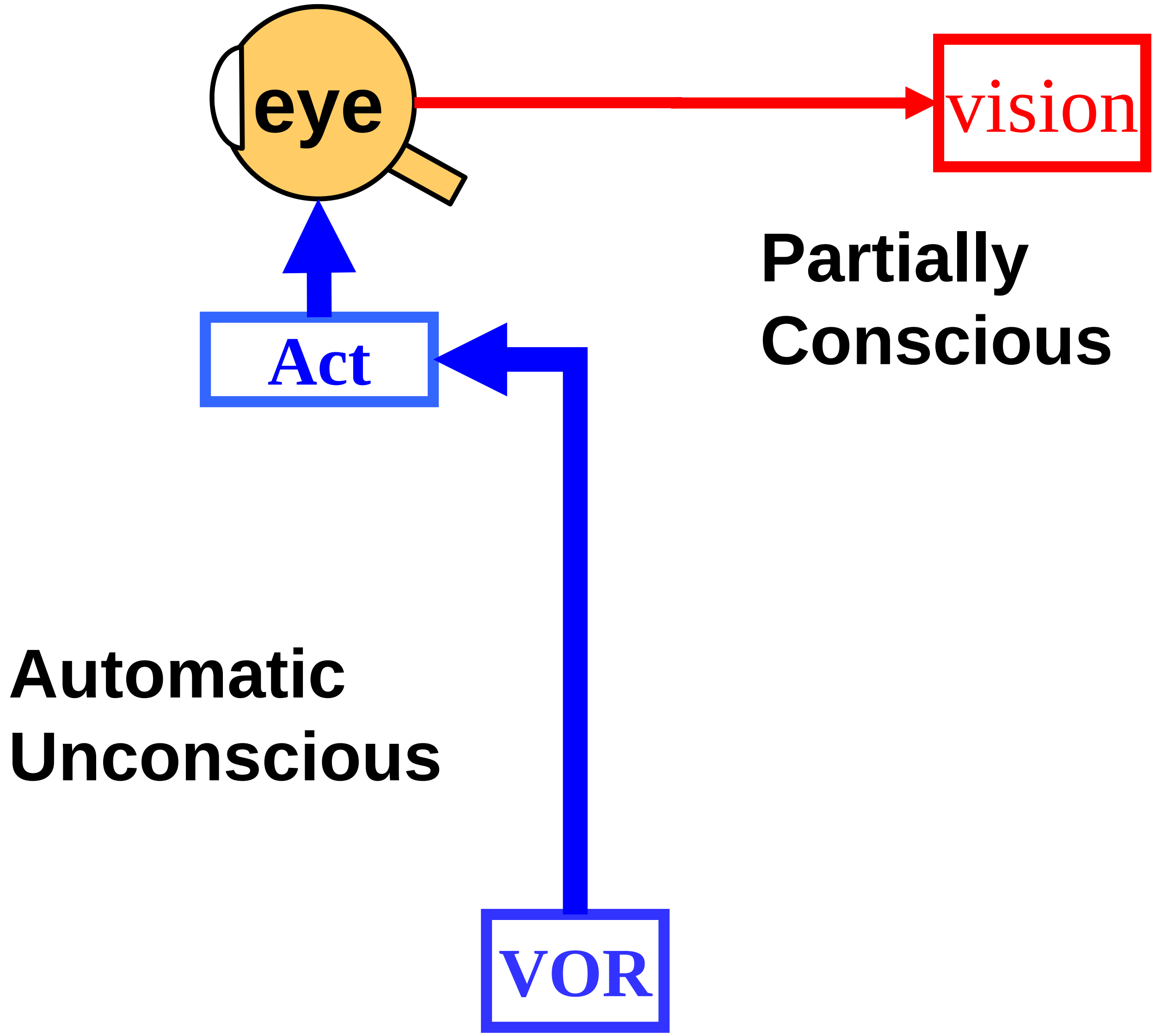
Vestibular
Ocular
Reflex
(VOR)


It works in the dark or with your eyes closed, but you can't tell.
eye
fast
Act
VOR
Slow
Fast
Flexible
Inflexible

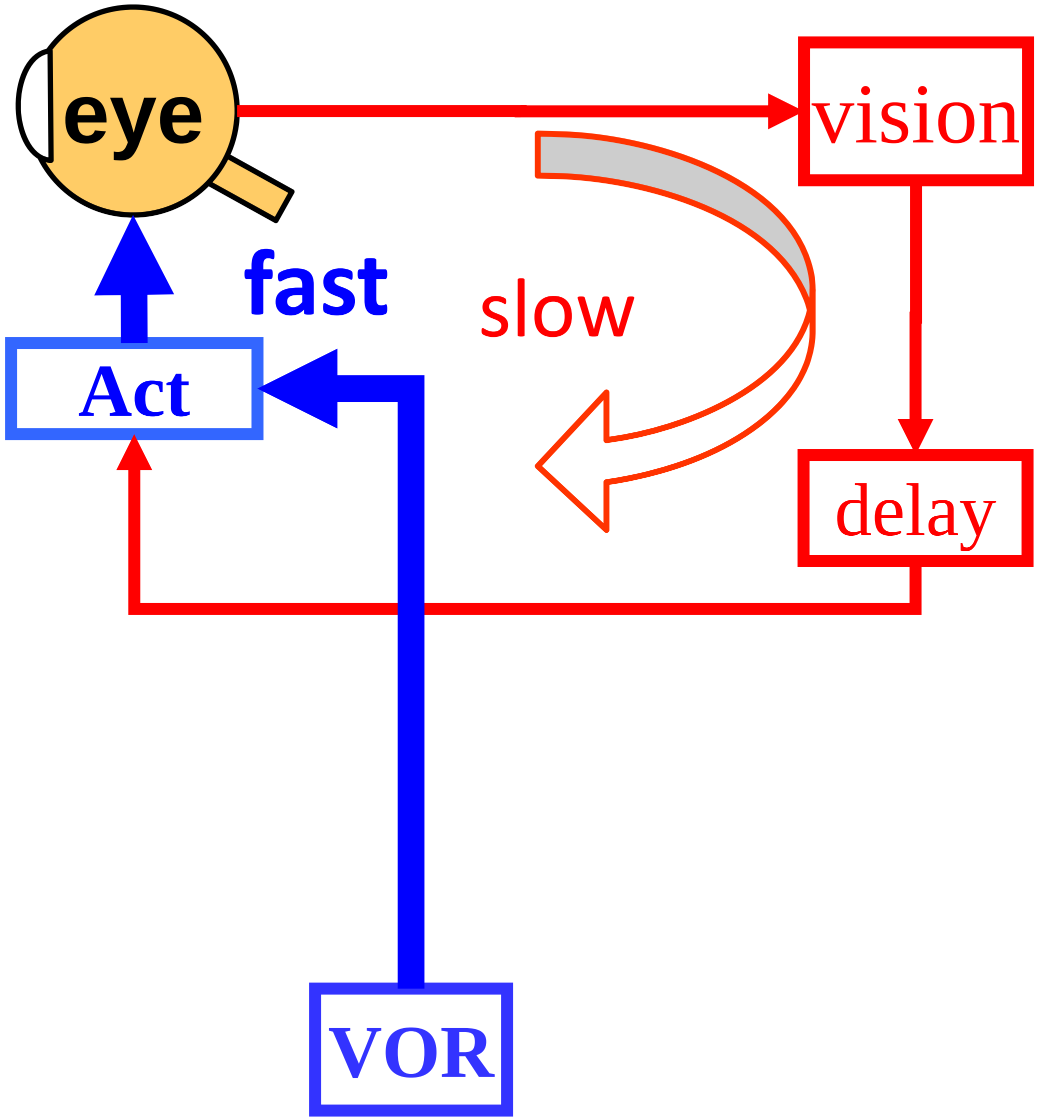

eye
vision
fast
slow
Act
delay
Slow
vision
VOR
Fast
Flexible
Inflexible


Layering
Feedback
vision
ay
Highly
evolved
(hidden)
architecture
Slow
Fast
Illusion
Flexible
Inflexible

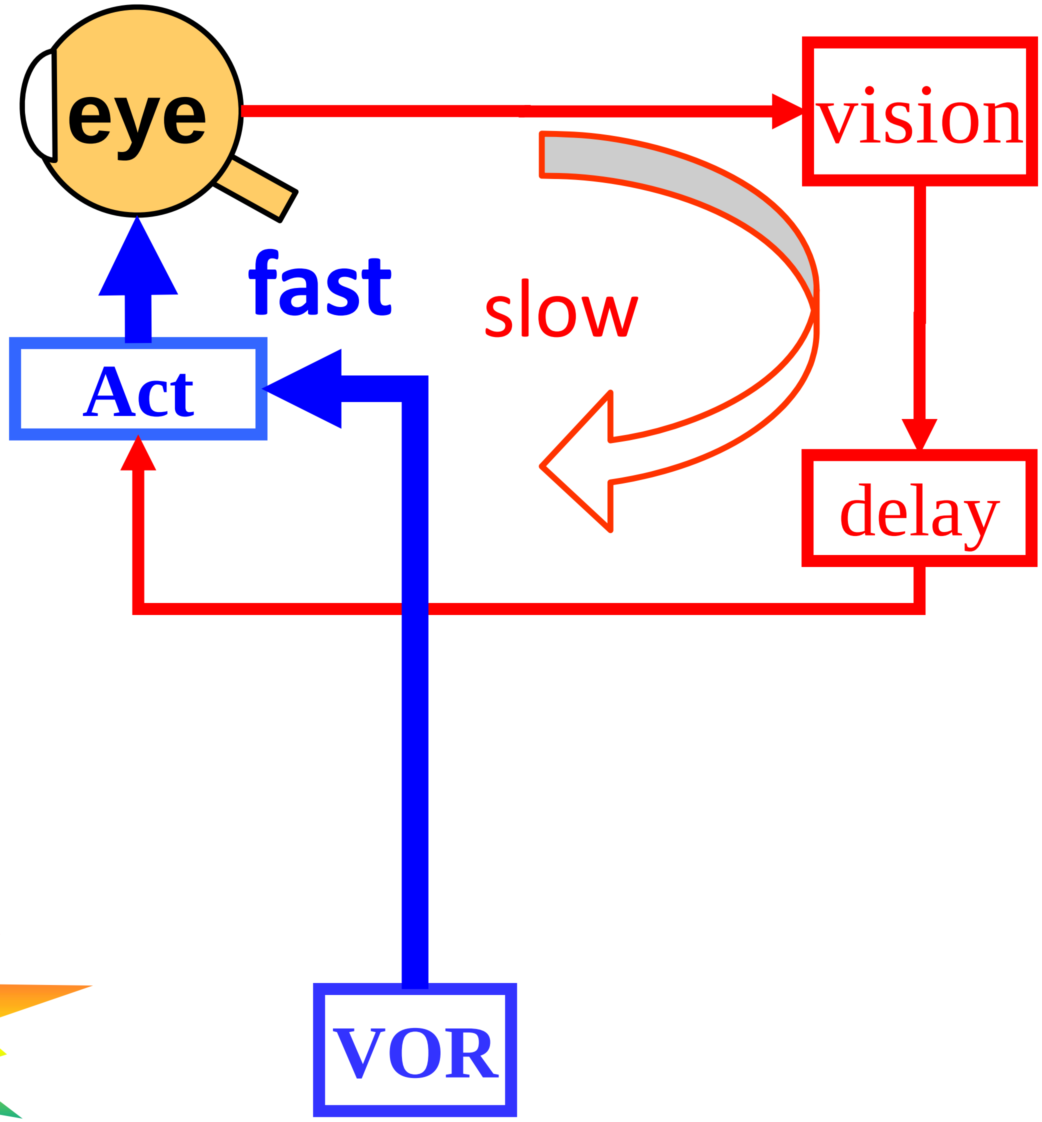
Layering

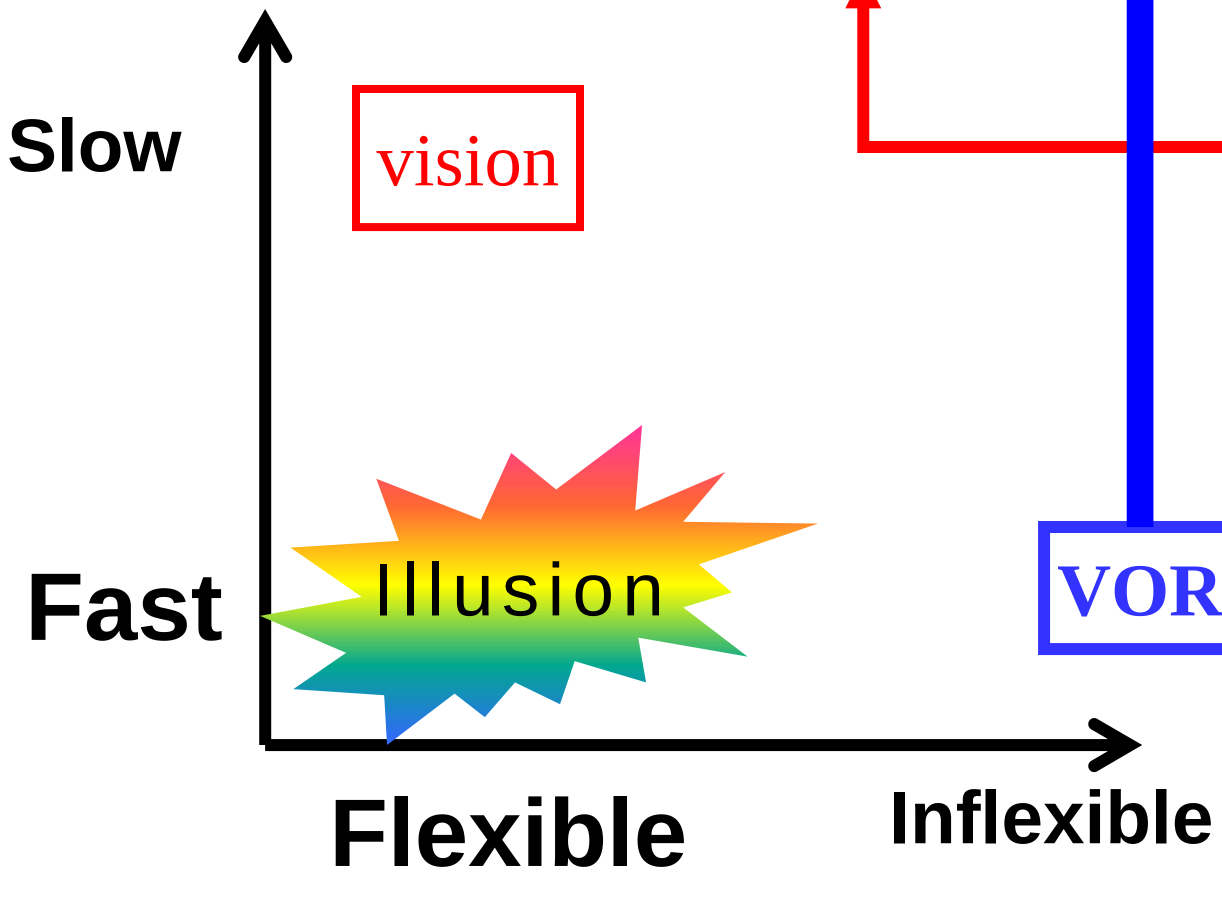

eye
vision
Partially
Conscious
Act
Automatic
Unconscious
VOR

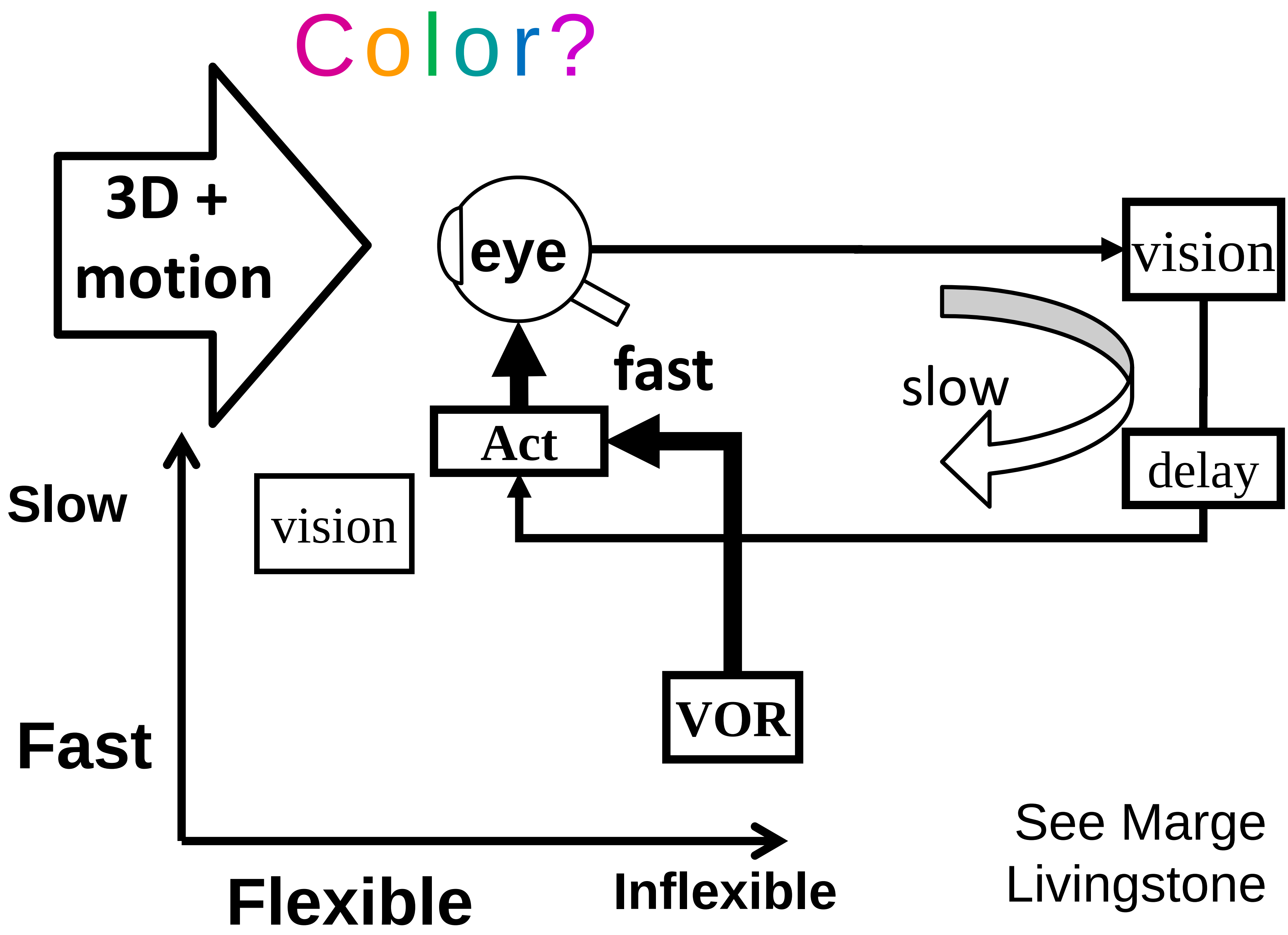
Layering
Feedback

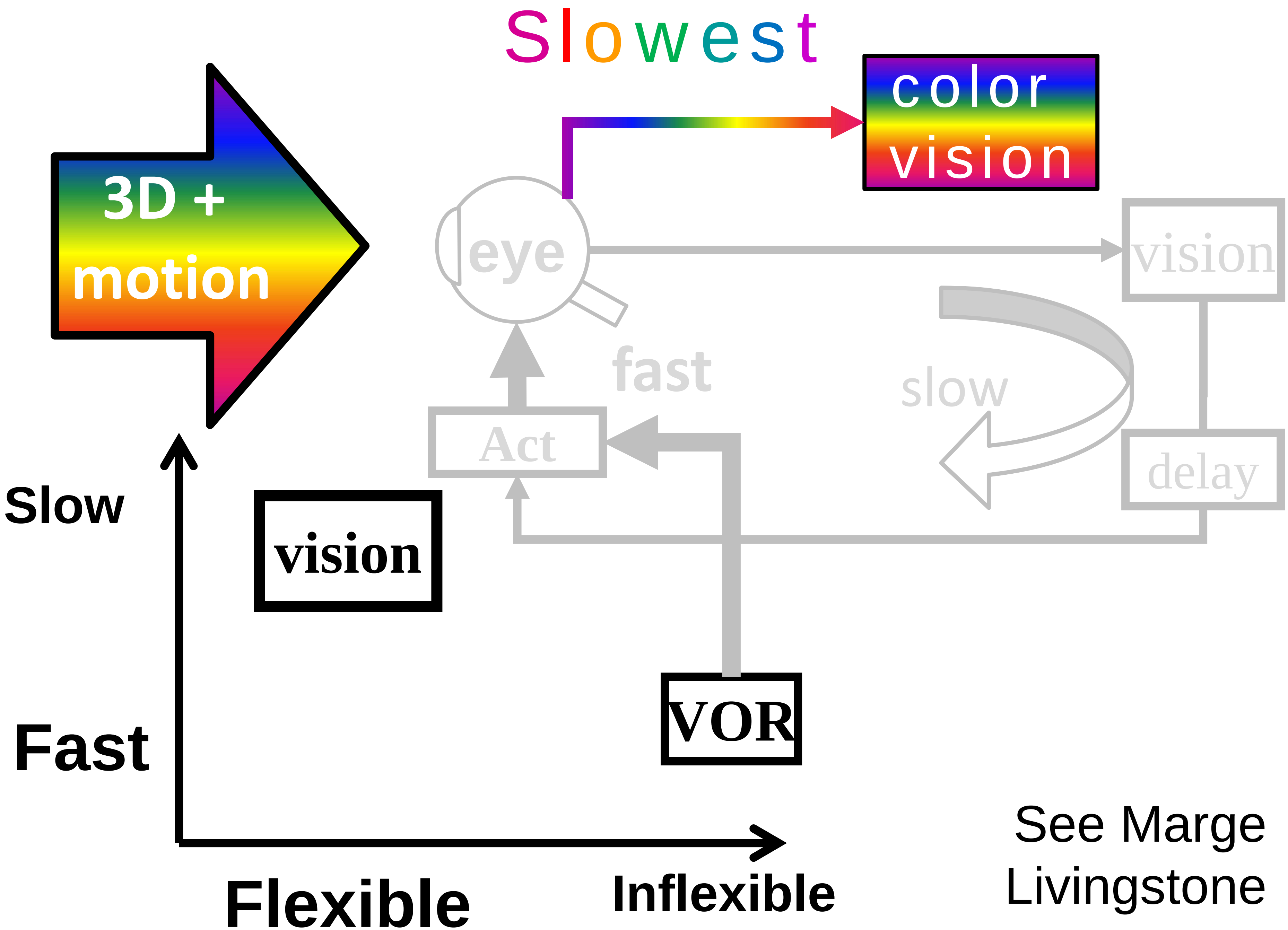


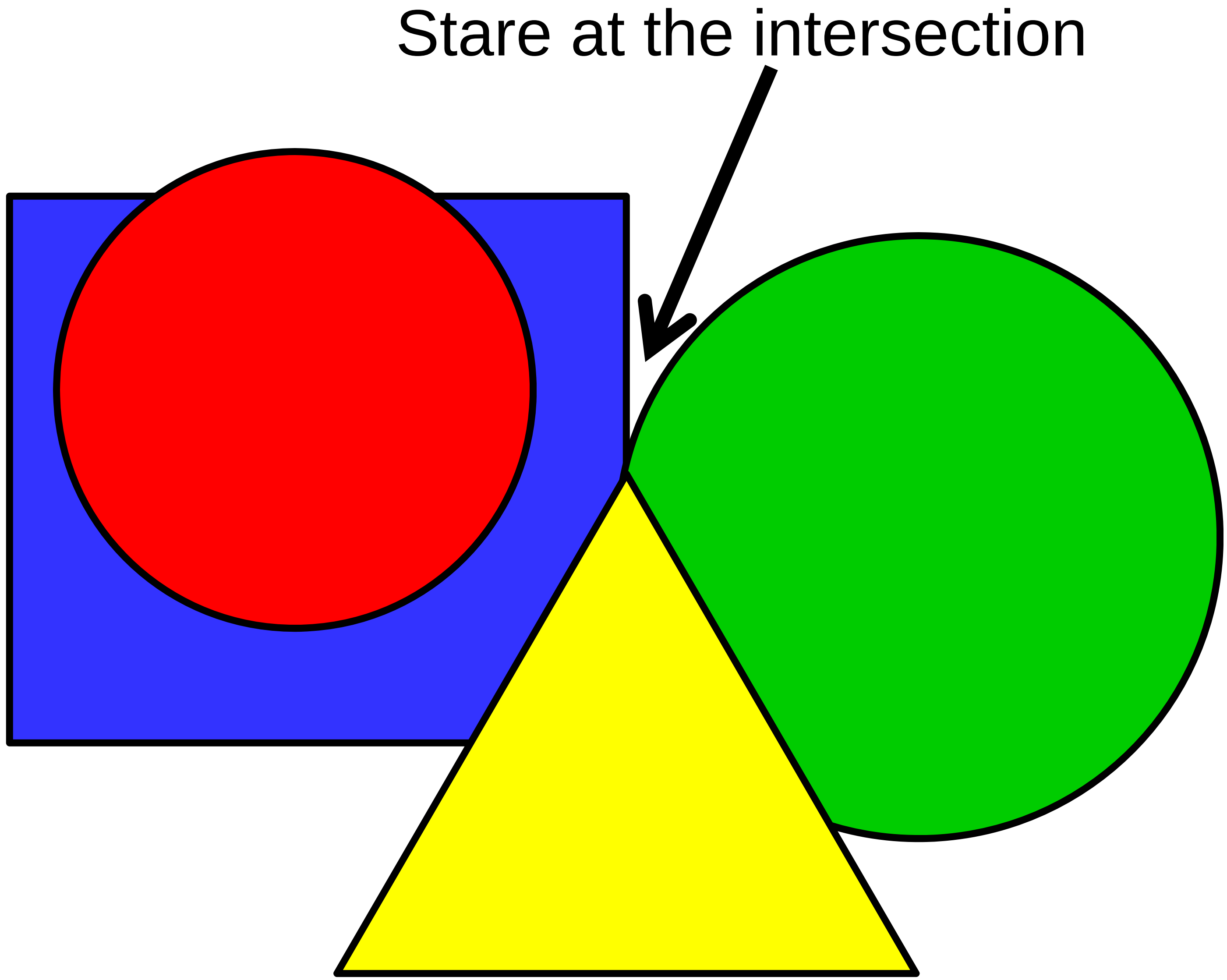
Architecture

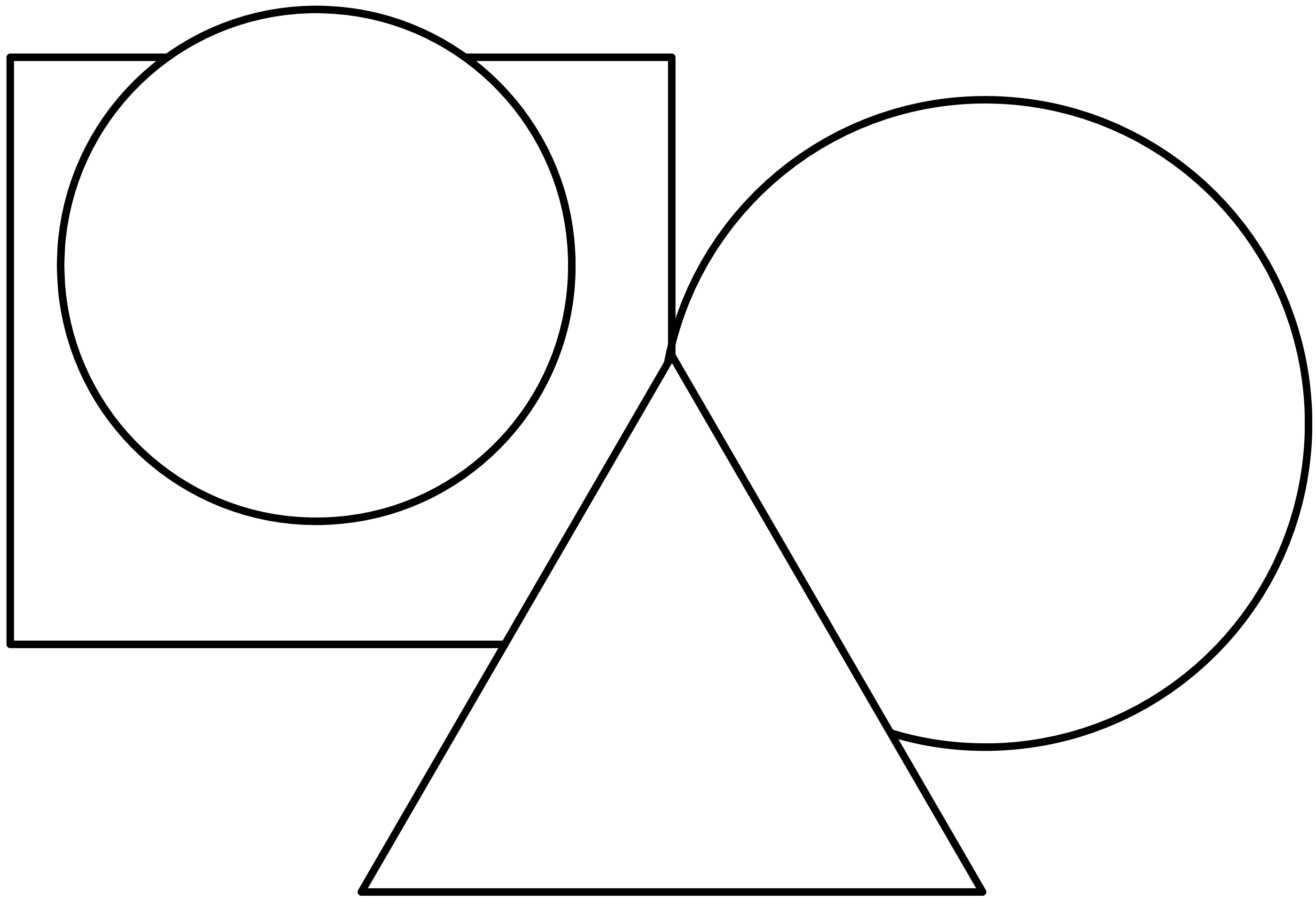




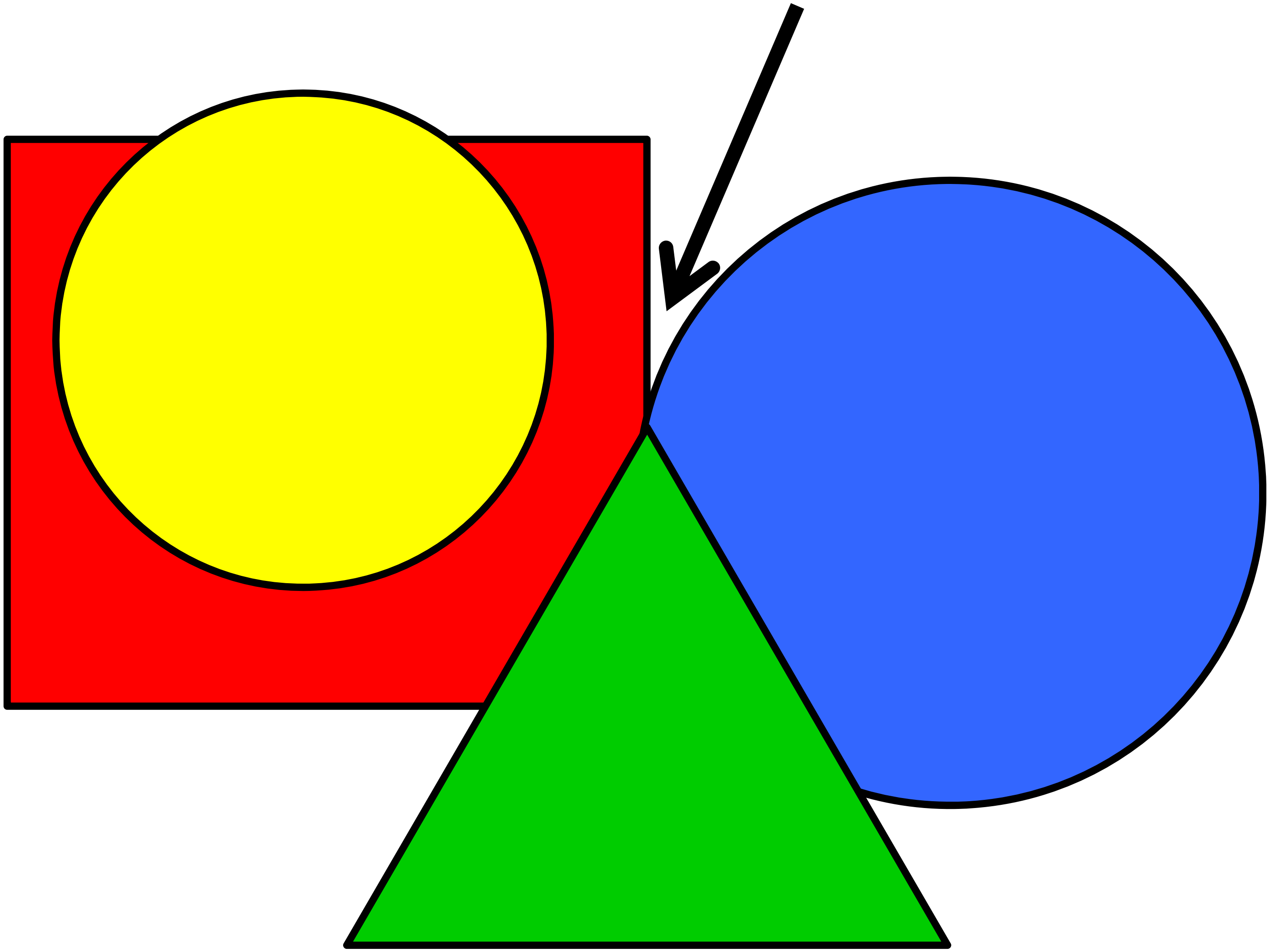

Color?
3D +
motion
eye
vision
fast
slow
Act
delay
vision
Slow
VOR
Fast
Flexible
Inflexible
See Marge
Livingstone

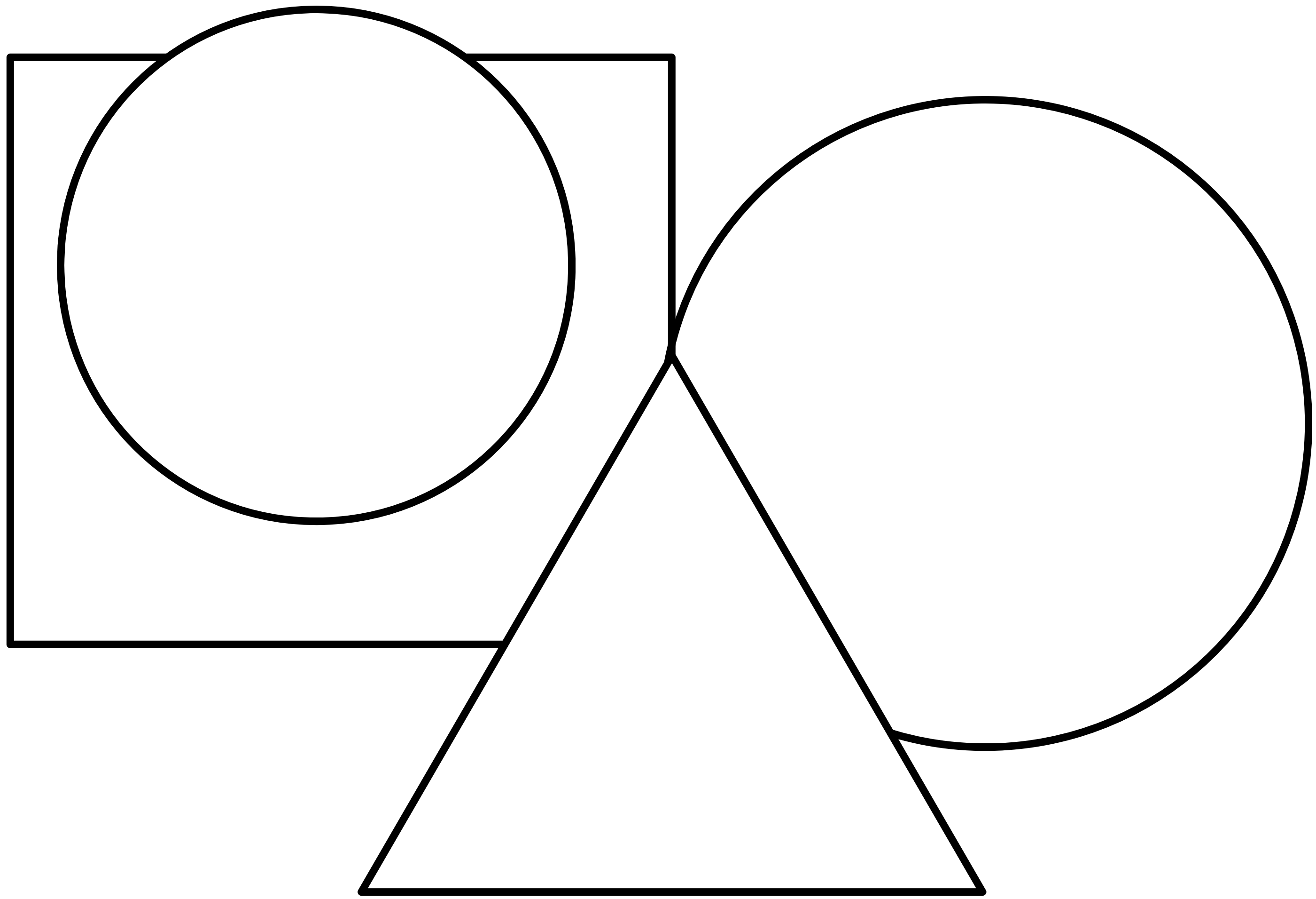

Slowest
color vision
3D + motion
eye
vision
fast
slow
Act
delay
Slow
vision
Fast
VOR
Flexible
Inflexible
See Marge Livingstone

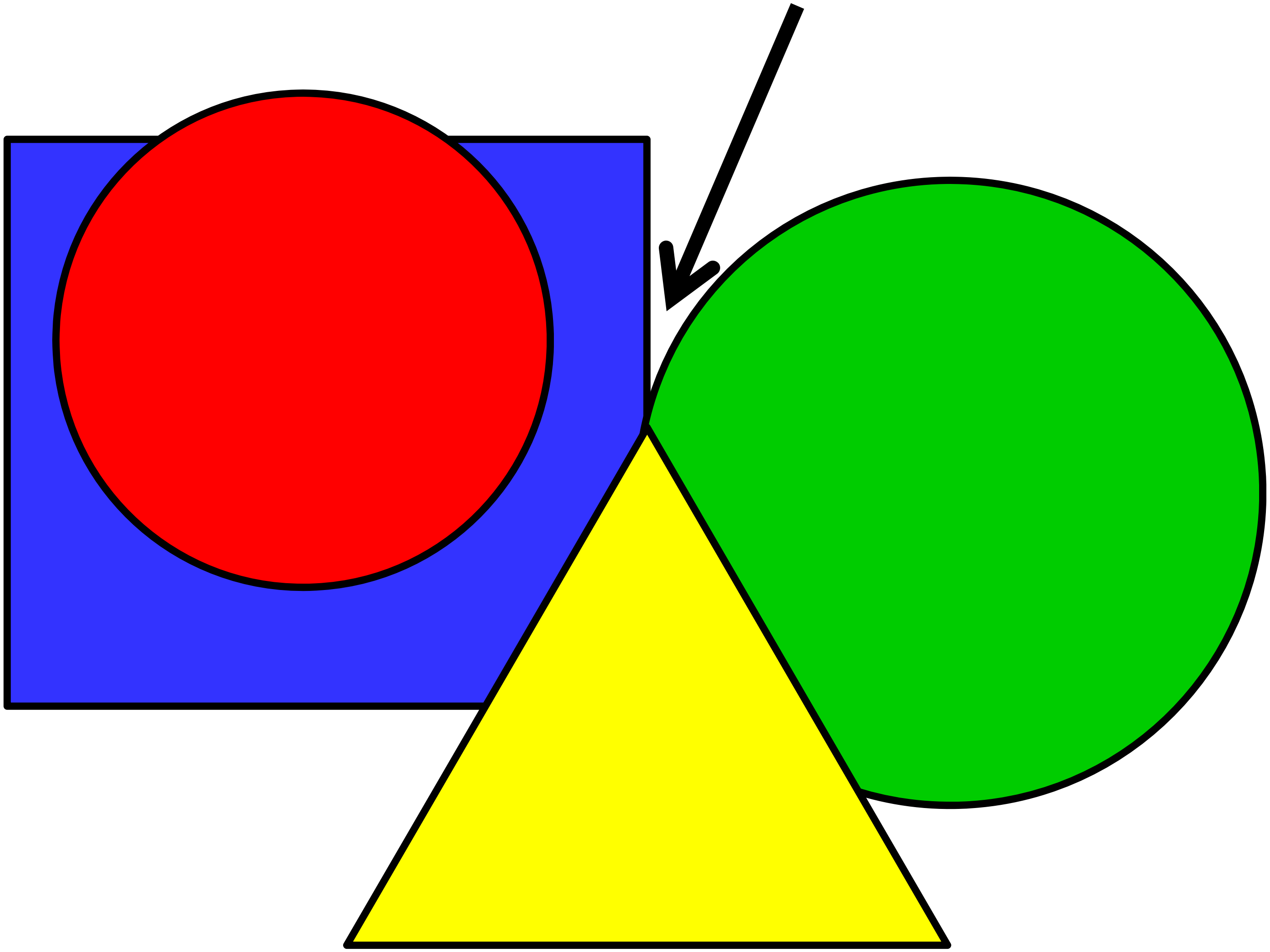

Stare at the intersection

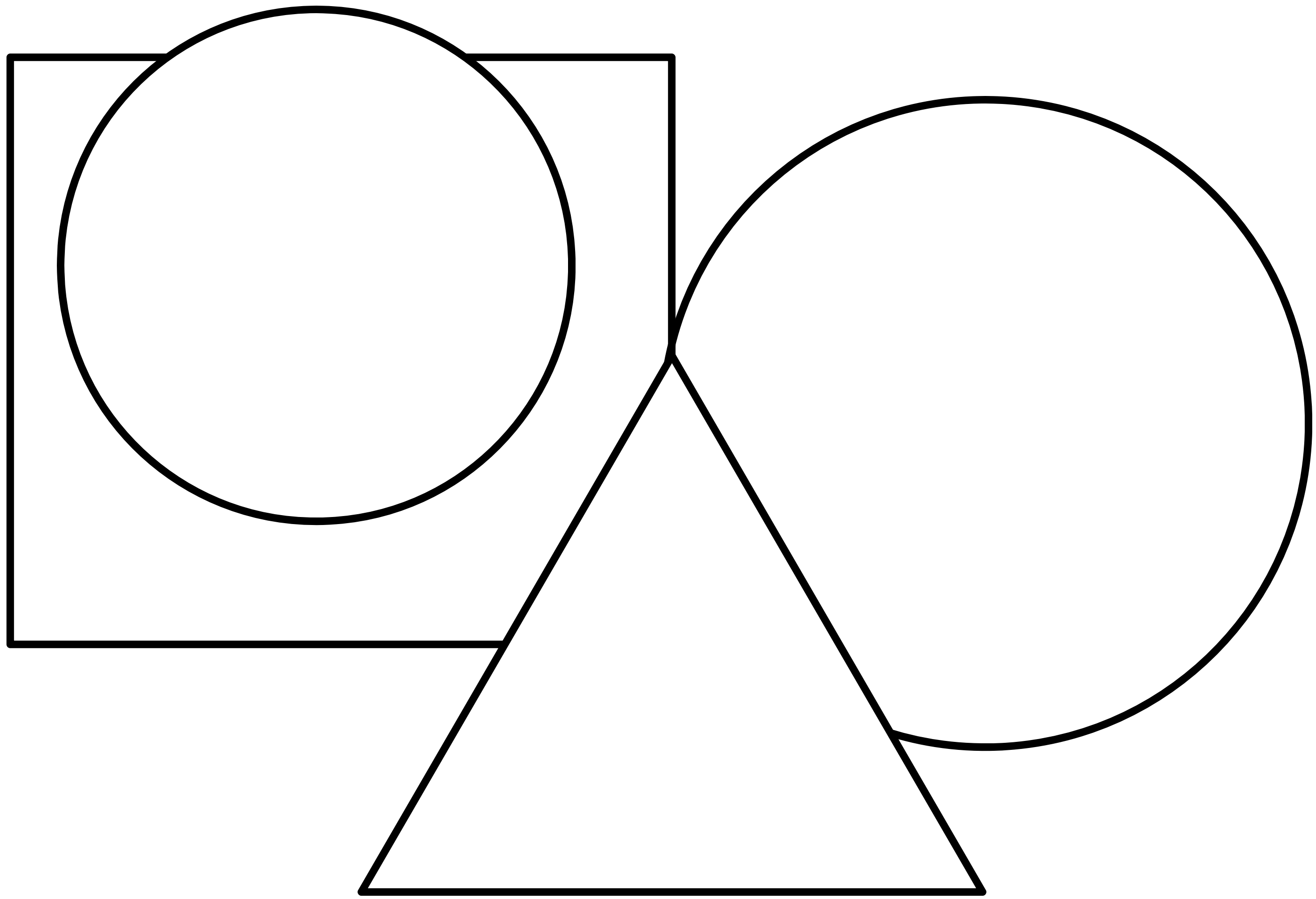


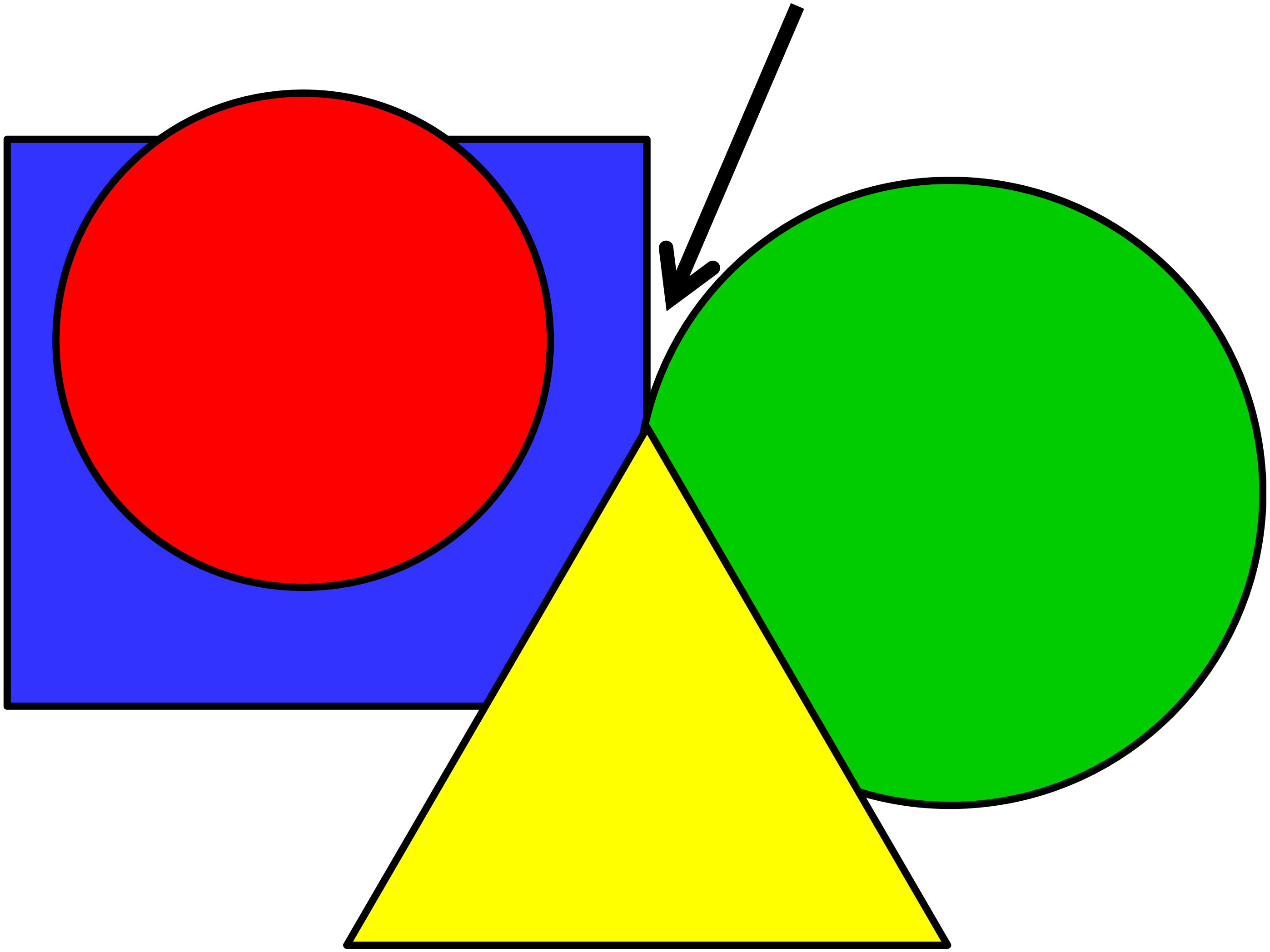
Stare at the intersection.

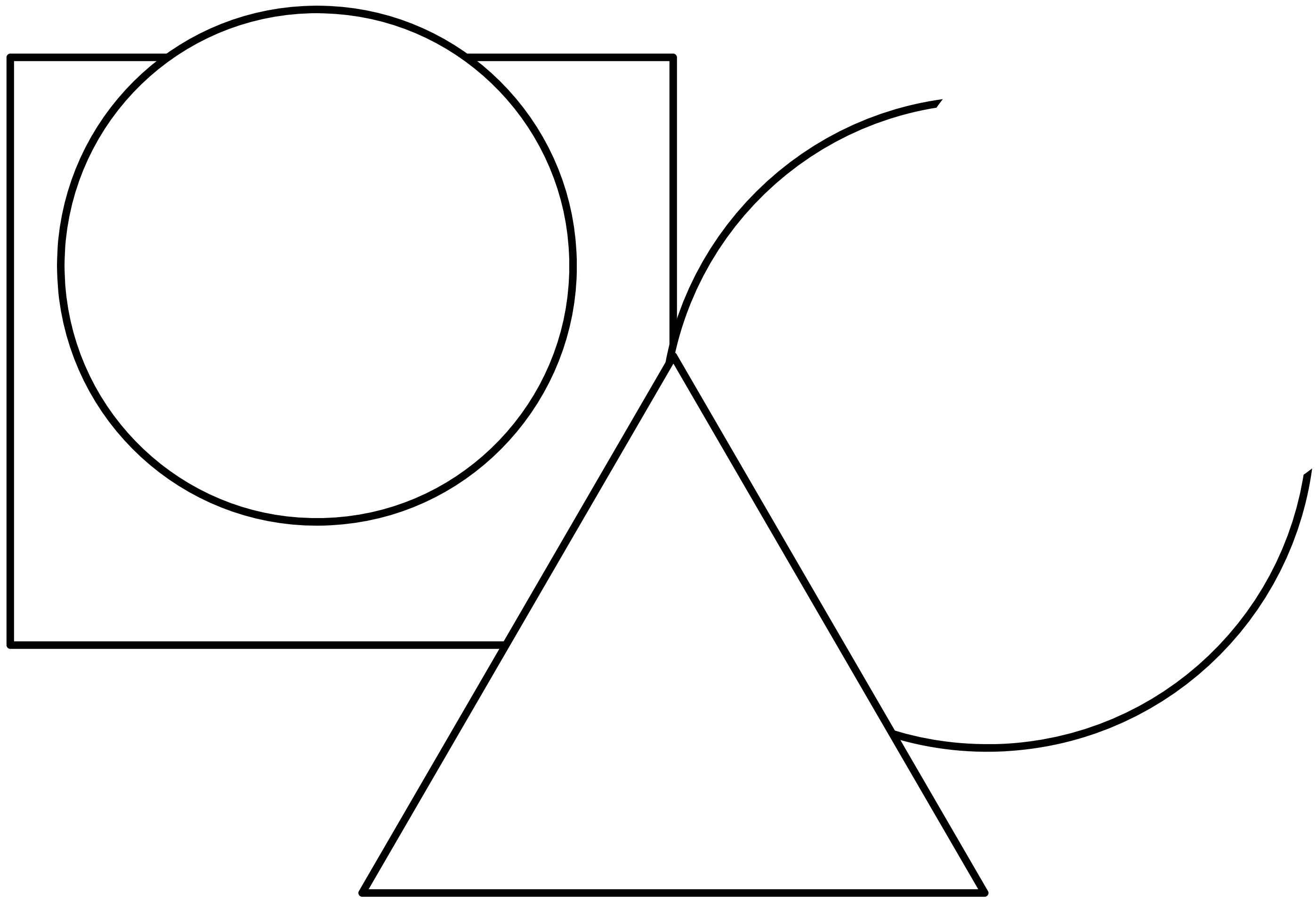




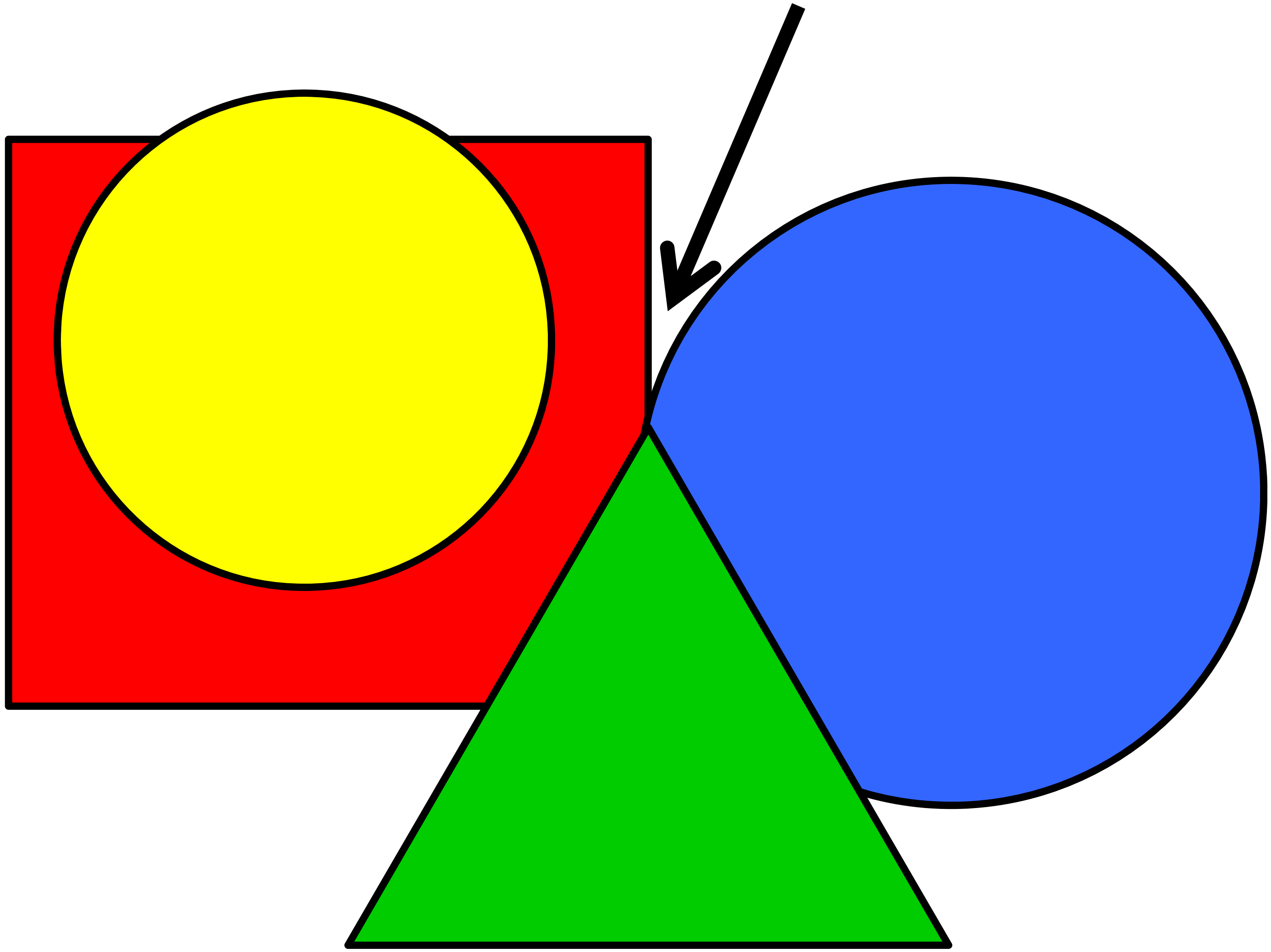


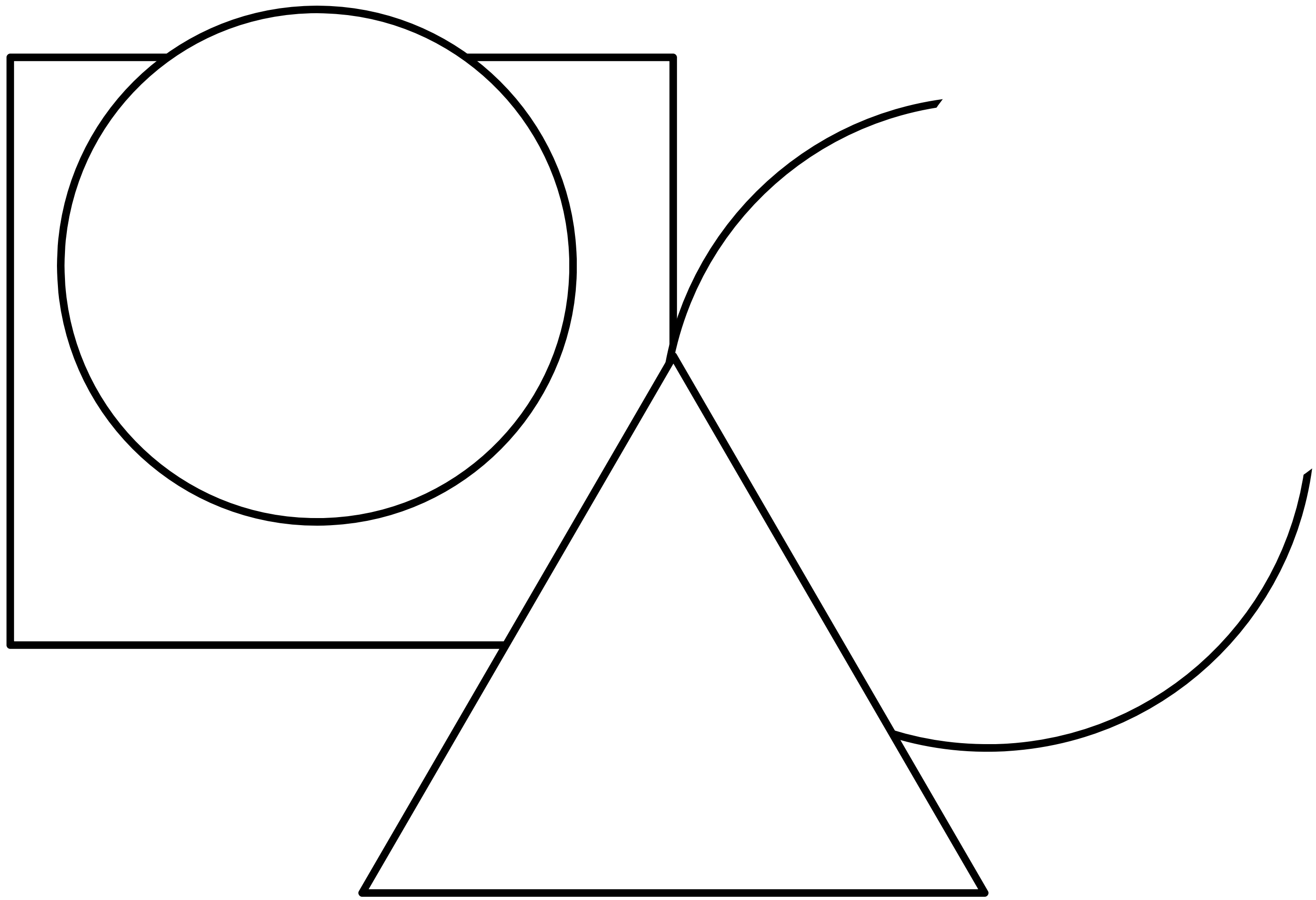


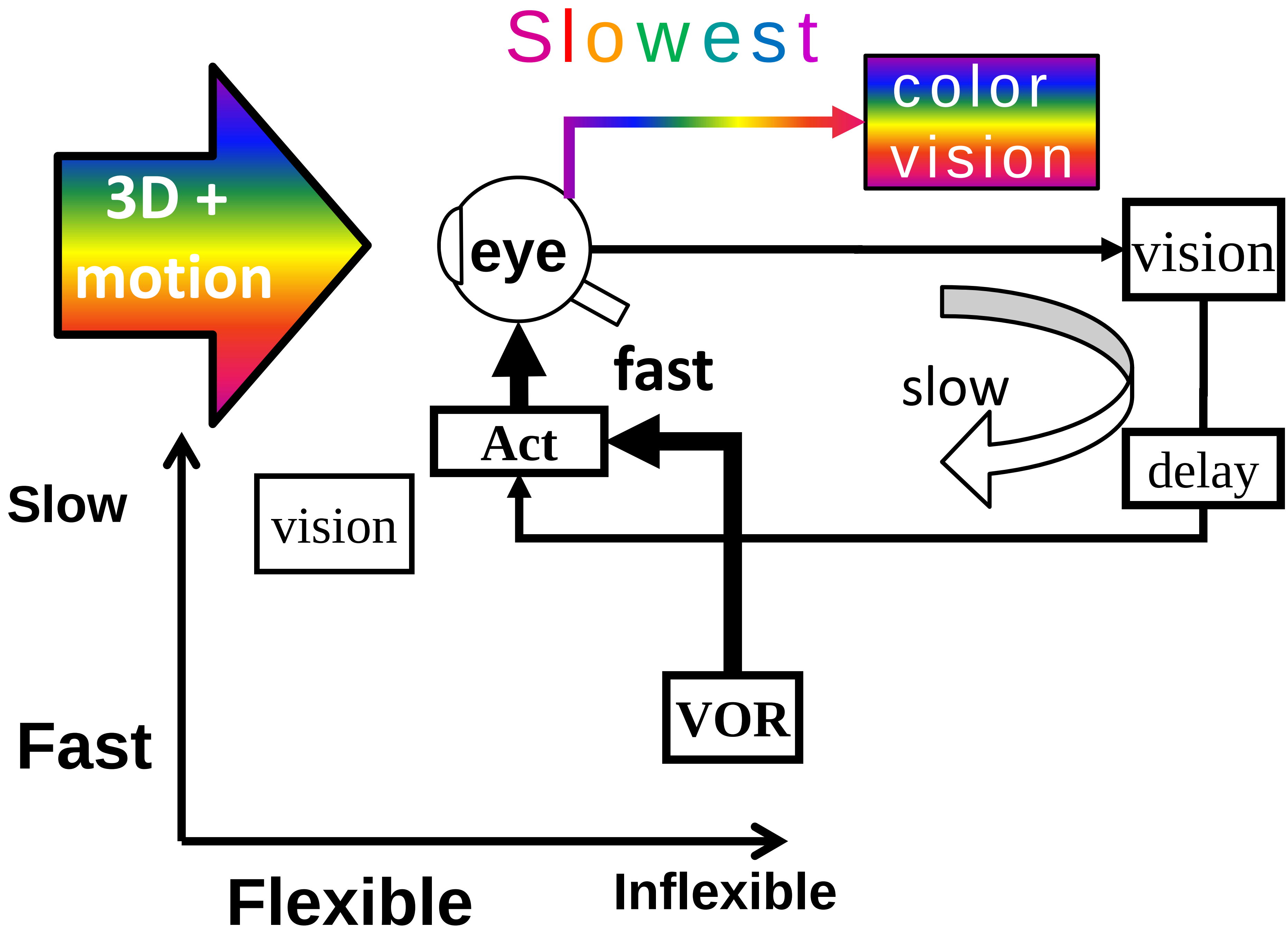




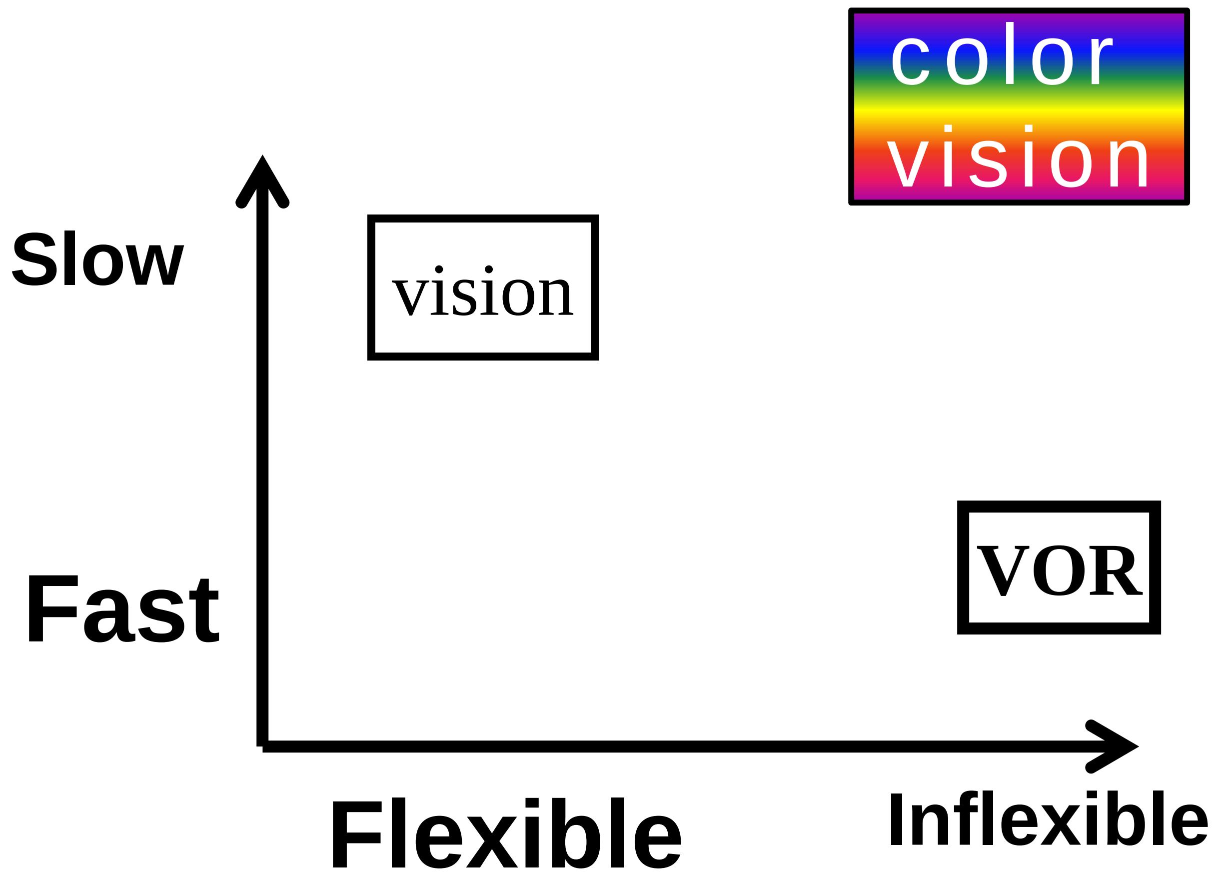
Stare at the intersection.

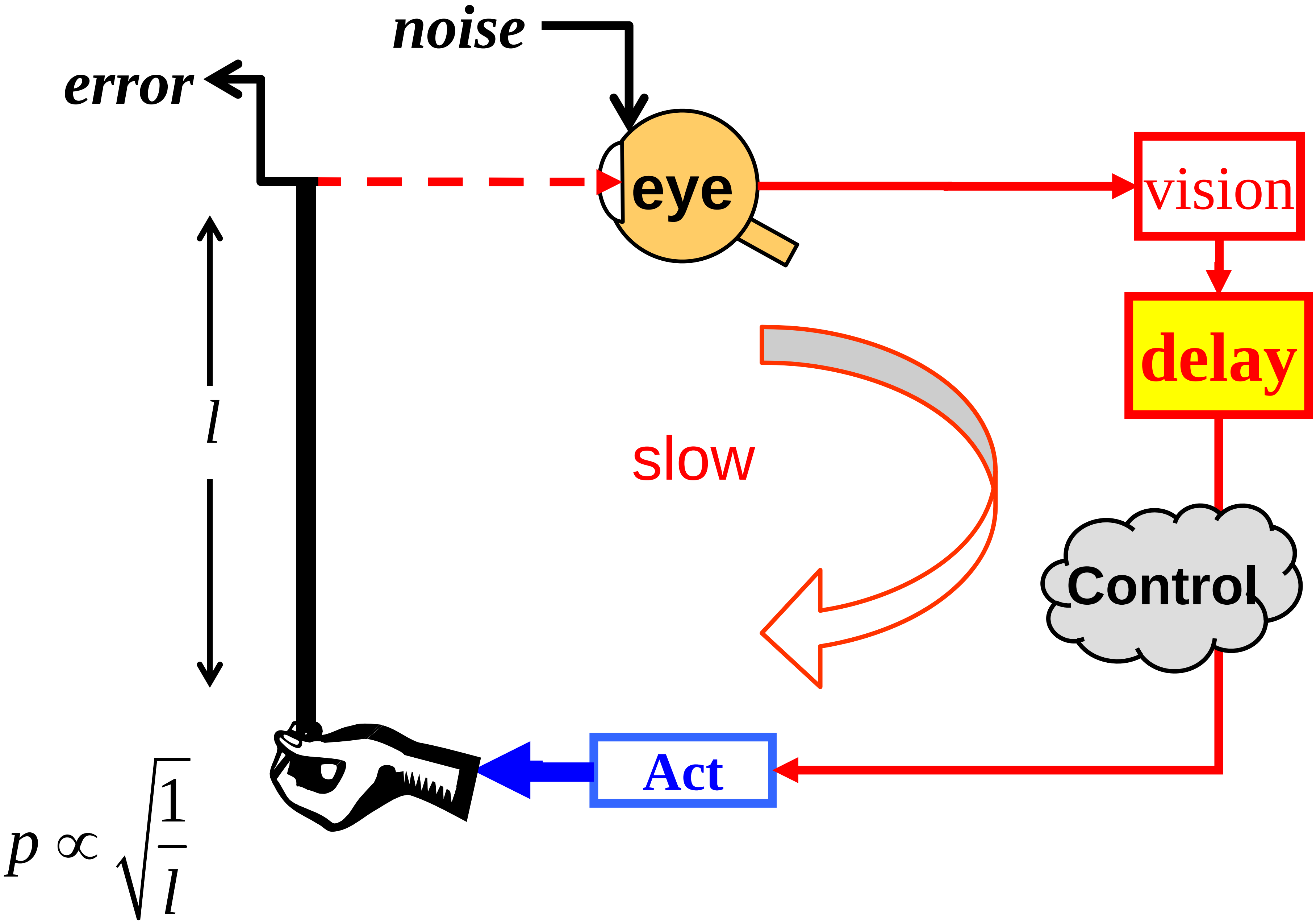




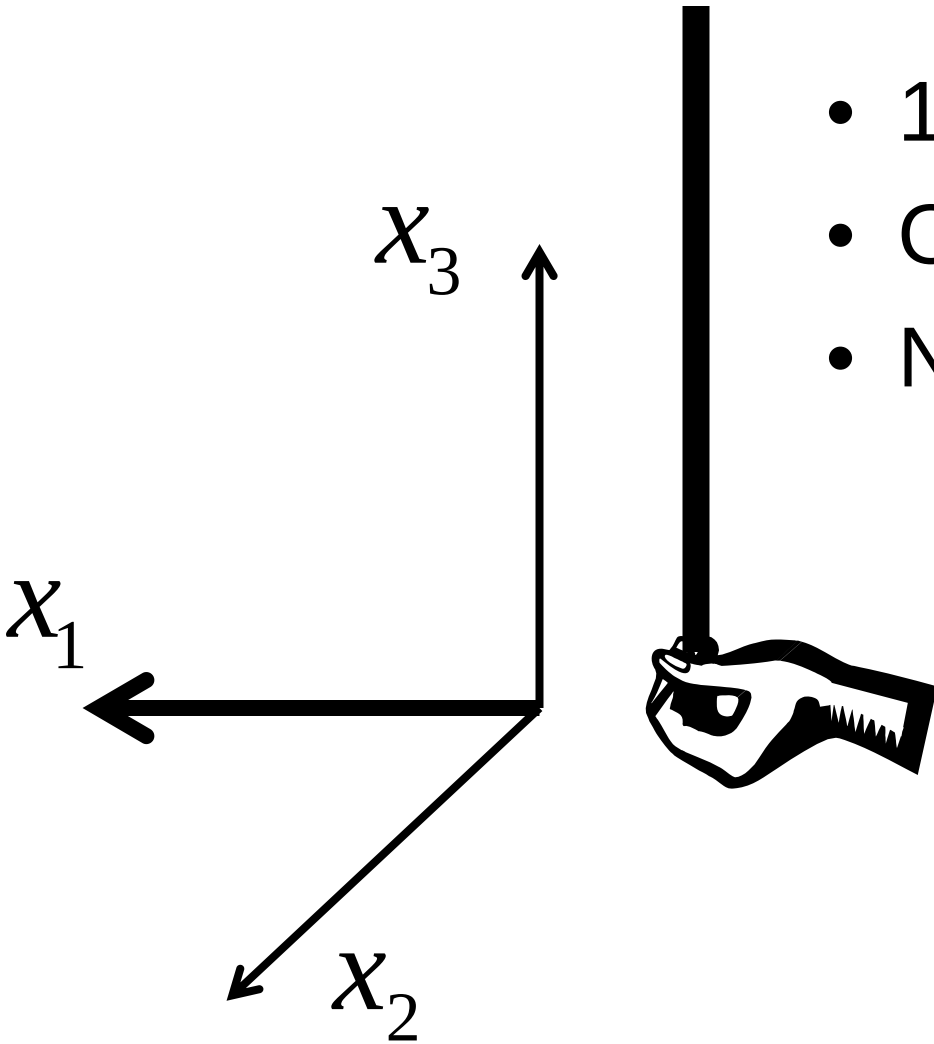

Slowest
color vision
3D + motion
eye
vision
fast
slow
Act
delay
Slow
vision
VOR
Fast
Flexible
Inflexible

Seeing is dreaming

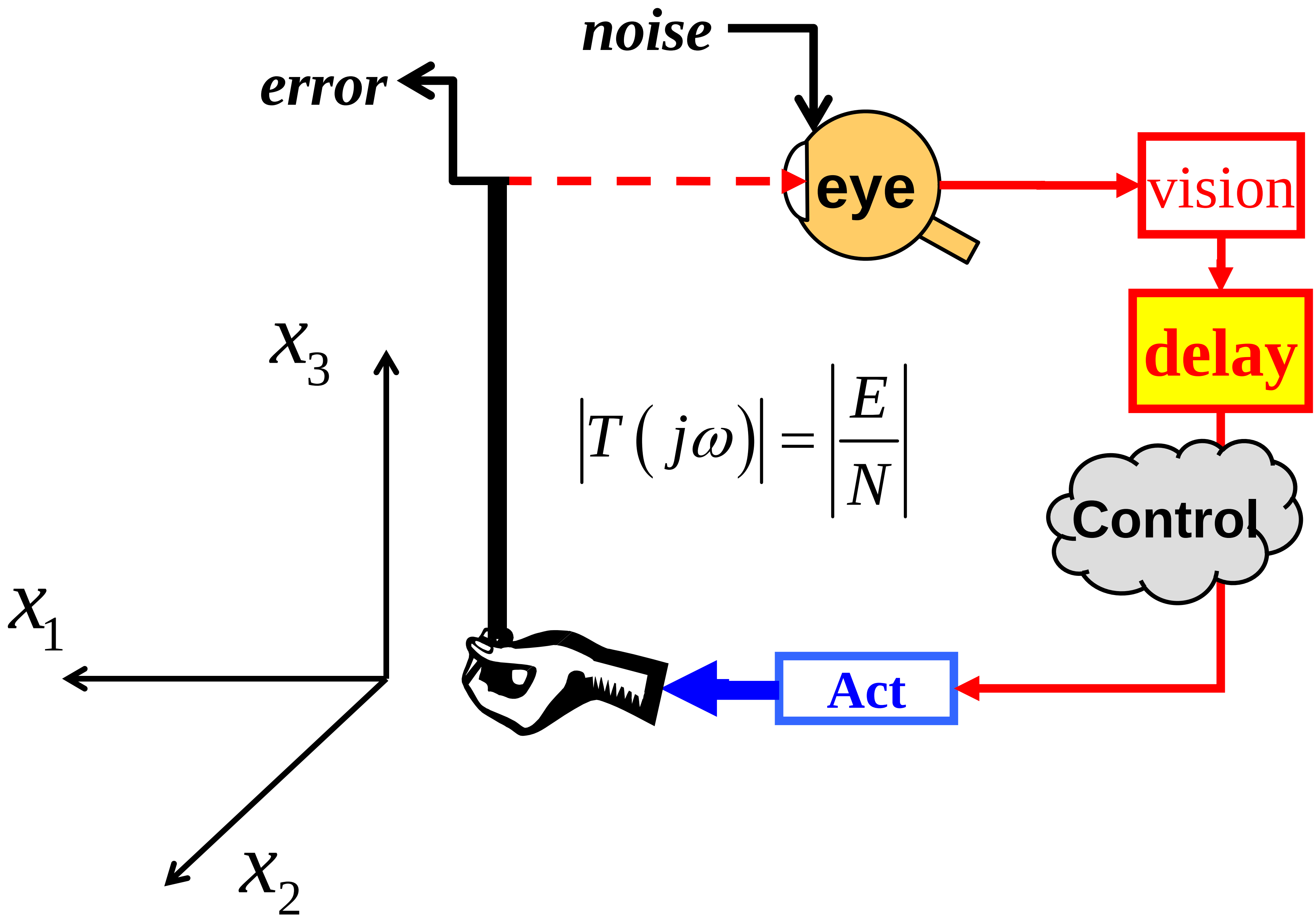


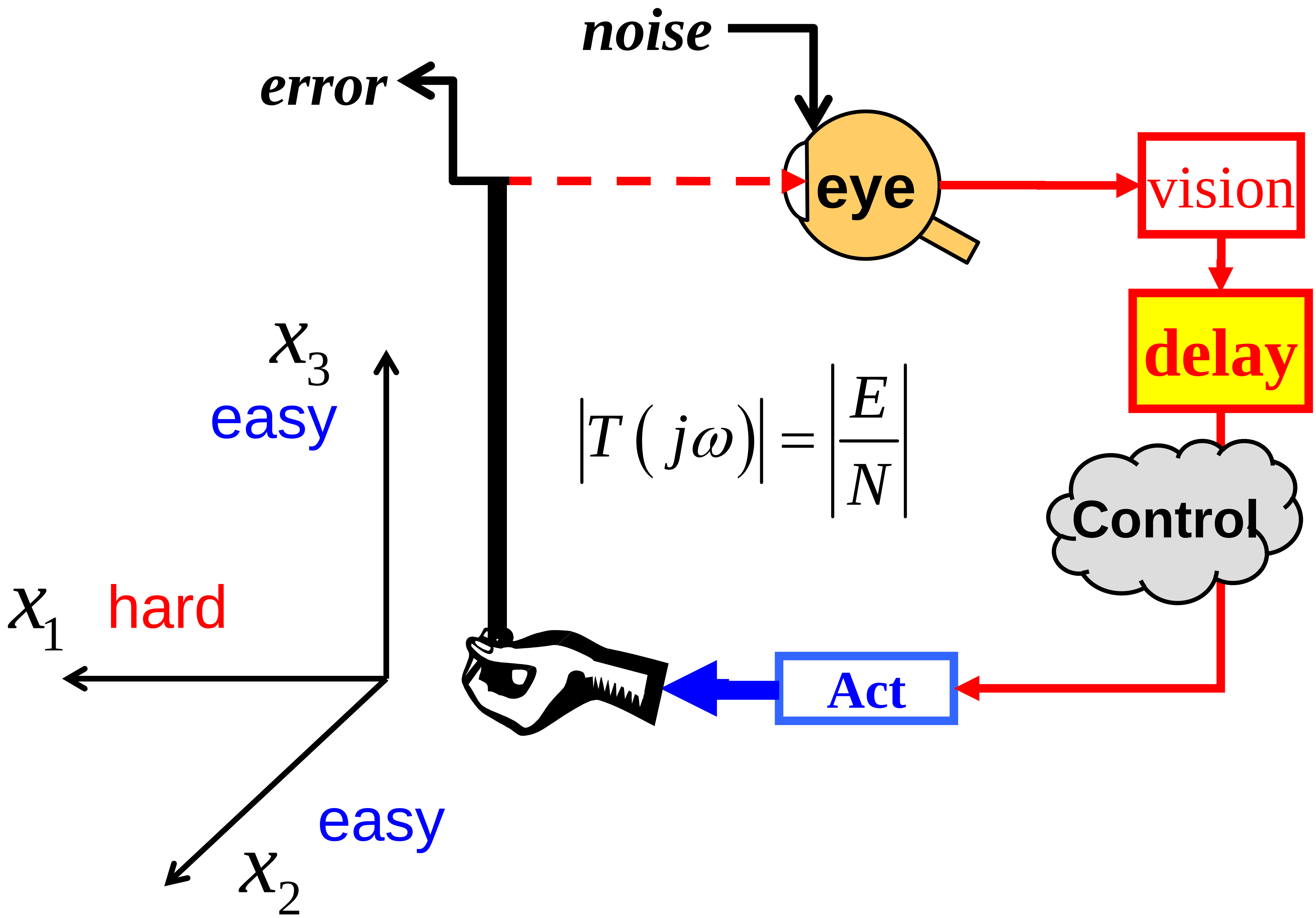

noise
error
eye
vision
delay
slow
Control
l
Act
$p \propto \sqrt{\frac{1}{l}}$

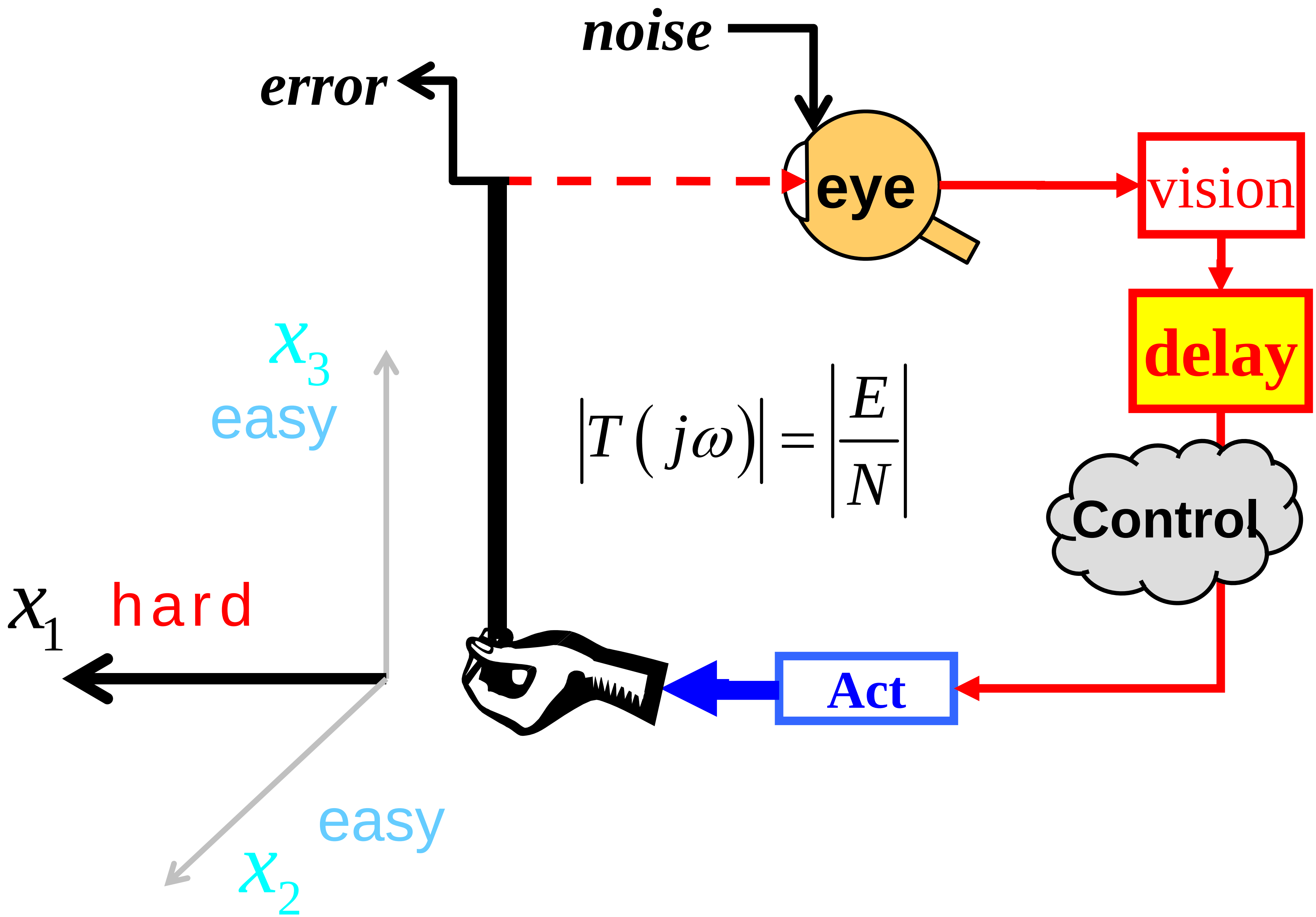
Model?



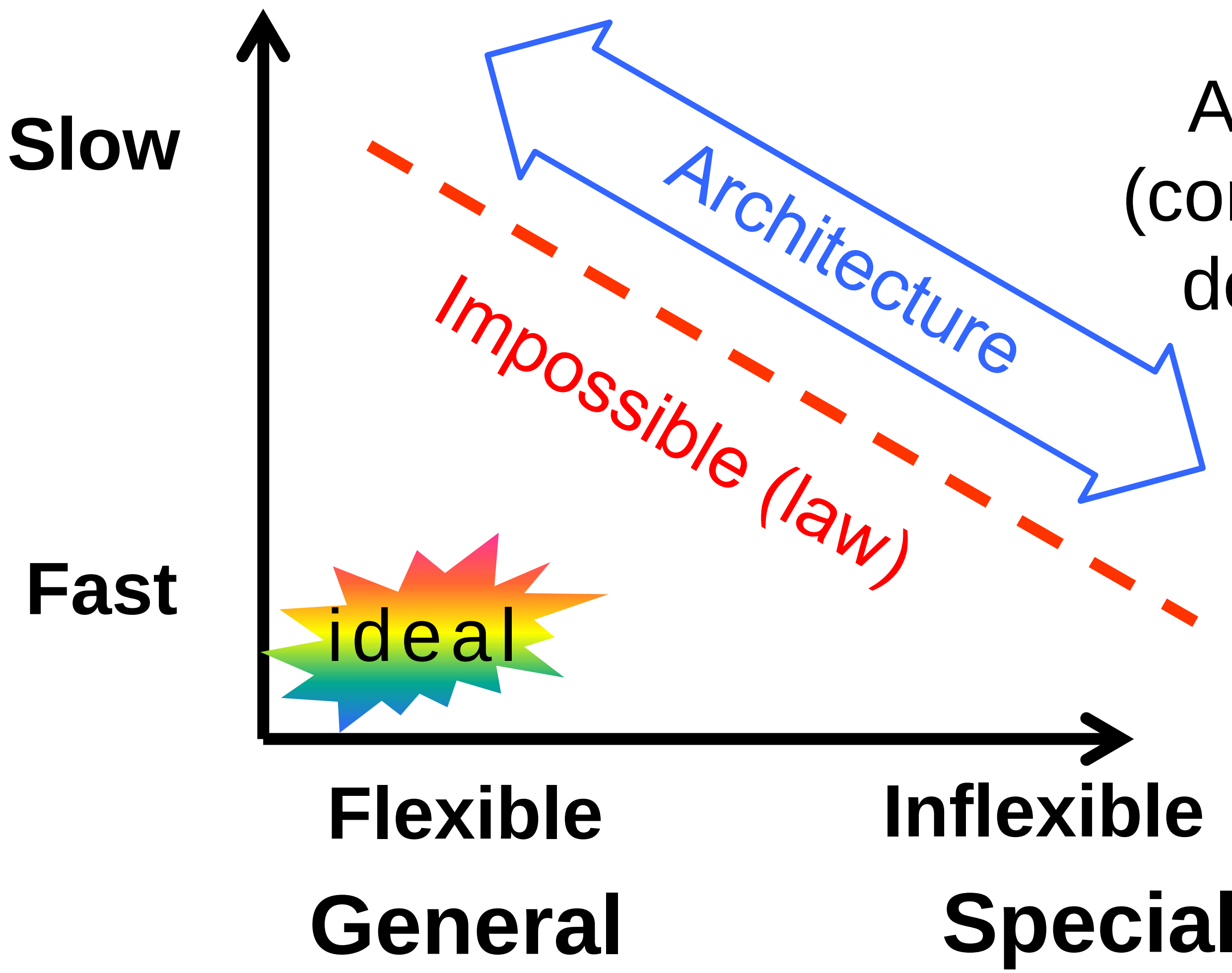
- 1 dimension, 4 states?
- Other 2 dimensions?
- New issues arise


noise
error
eye
vision
delay
Control
Act
x_3
x_1
x_2
$|T(j\omega)| = \left|\frac{E}{N}\right|$


noise
error
eye
vision
delay
Control
Act
x_3
easy
x_1
hard
easy
x_2
$|T(j\omega)| = \left|\frac{E}{N}\right|$

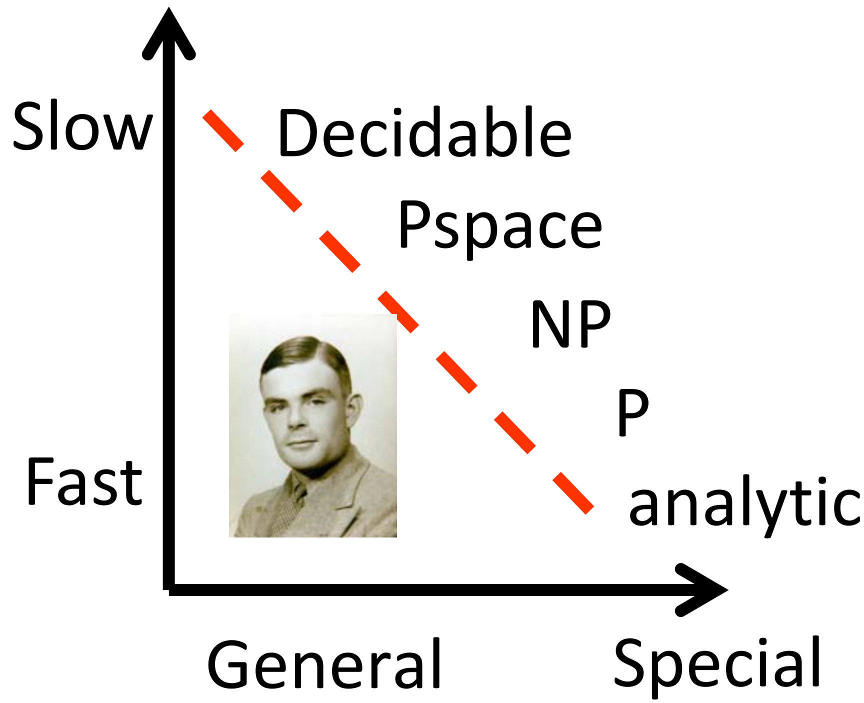

noise
error
eye
vision
delay
Control
Act
x_3
easy
x_1
hard
easy
x_2
$\left|T(j\omega)\right| = \left|\frac{E}{N}\right|$

Universal laws and architectures (Turing)

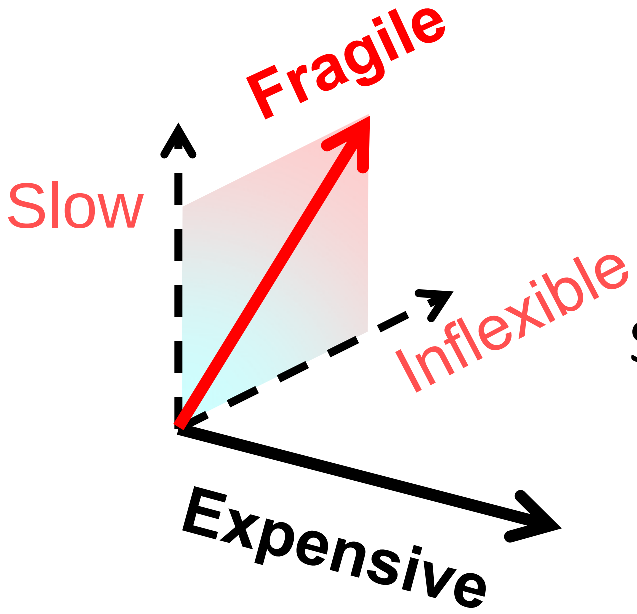


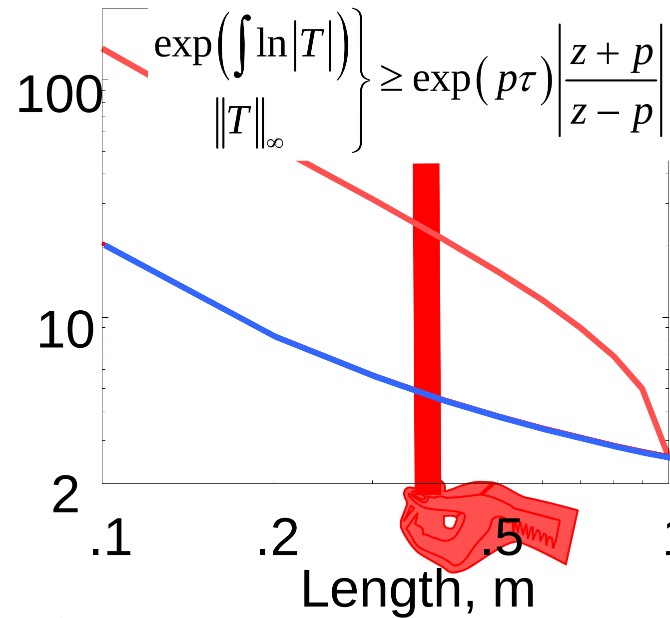
Architecture (constraints that deconstrain)

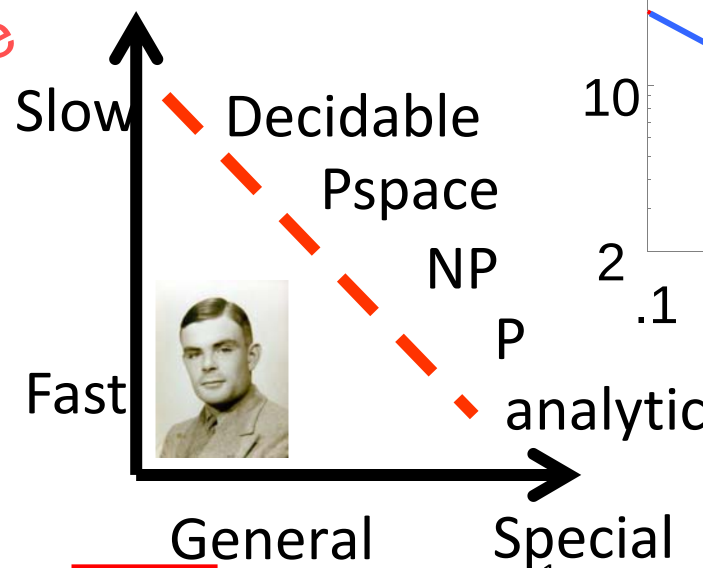
Universal laws?

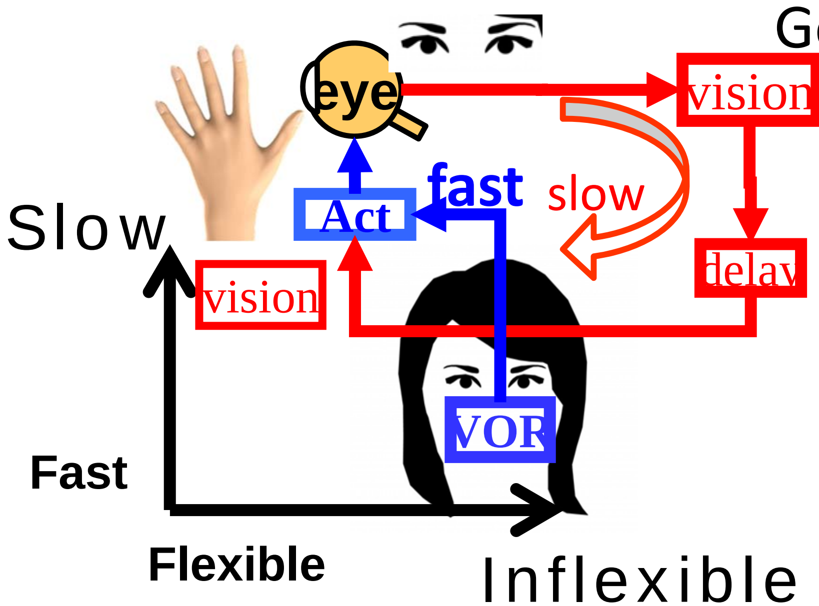


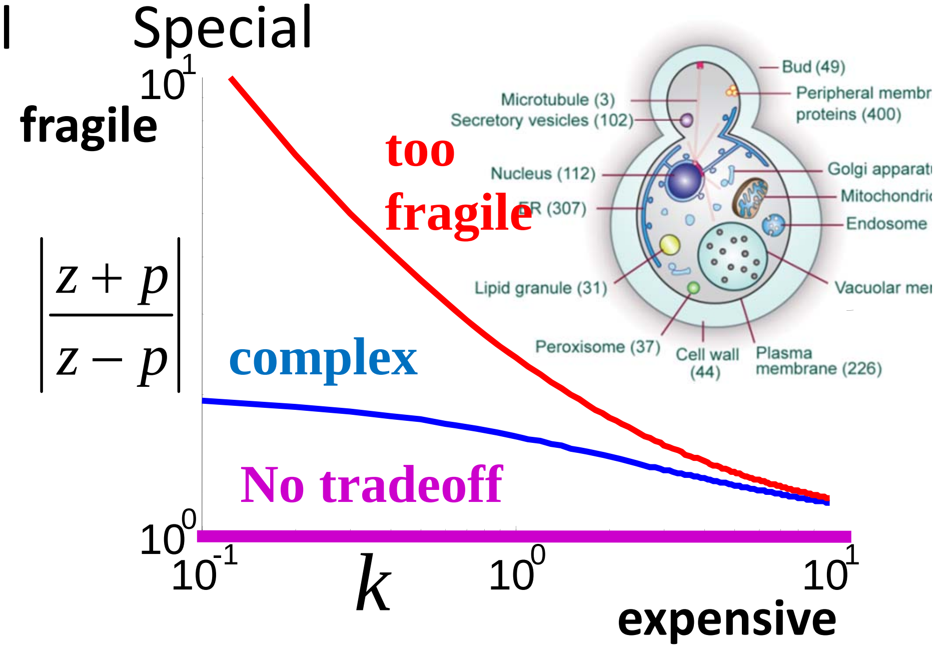
Universal laws?



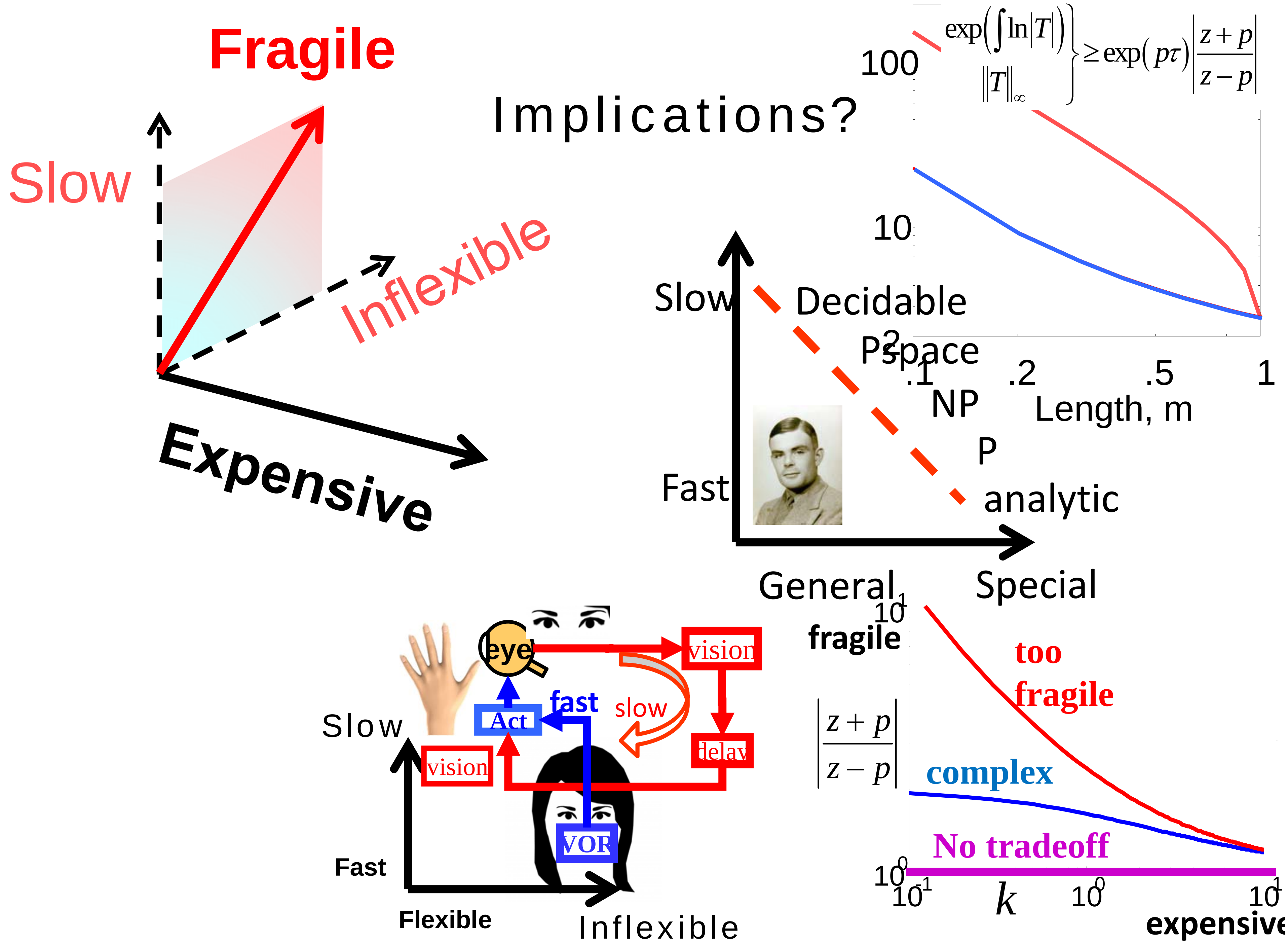








Implications?



Expensive tradeoffs

What is **costly** (and cheap) elements in:

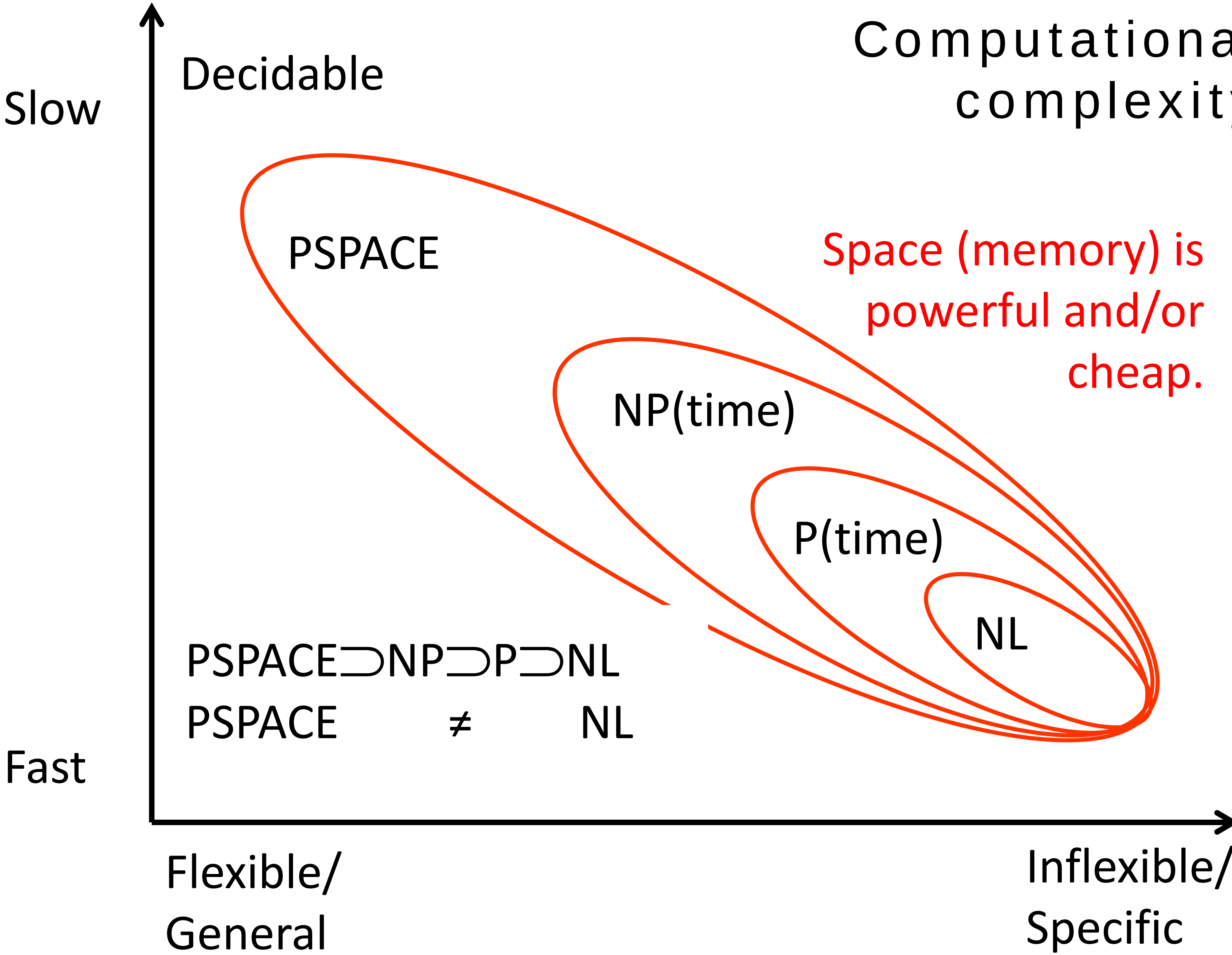
Physical: Both efficiency and stability

Control: Actuation (vs sensing)

Computing: Time (vs space)

Communication: Latency (vs bandwidth)

Computational complexity



Control of cyberphysical systems?

Physical: Efficient, therefore unstable

Computing: Distributed with delays

Communication: With latency

Therefore Control: Distributed

- with sparse actuation (but add sensing)
- with delays in computing
- and communications
- but “free” memory and bandwidth

How to make scalable?

Compute Gödel **Comms**

Turing Shannon

Von Neumann

Theory?

Deep, but fragmented, incoherent, incomplete

Nash Carnot

Bode Boltzmann

Heisenberg

Control, OR Einstein **Physics**

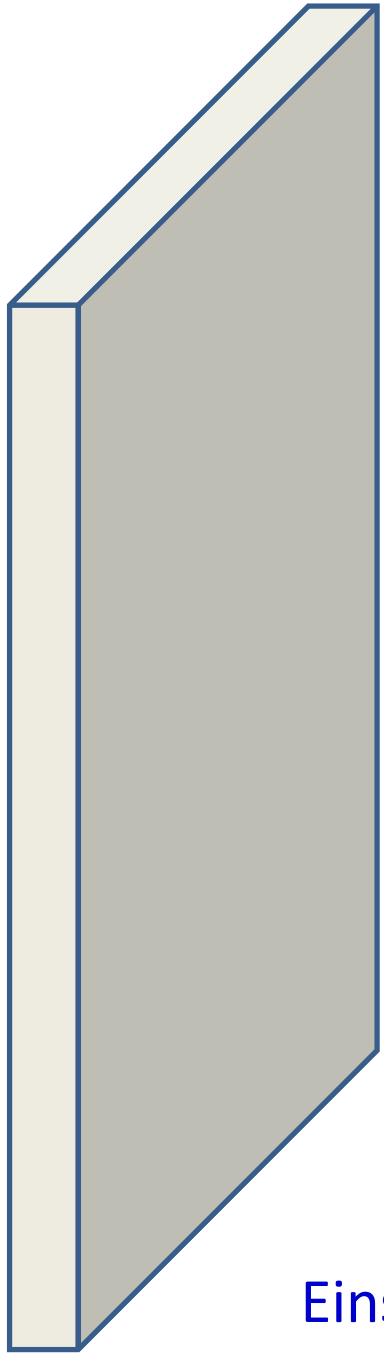
Compute

Turing

Delay and risk are *most* important

Bode

Control, OR



Communicate

Shannon

Delay and risk are *least* important

Carnot

Boltzmann

Heisenberg

Einstein

Physics

Compute

Turing

- Worst-case ("risk")
- Time complexity (delay)

Delay and risk are *most* important

Computation for control
- Off-line design
- On-line implementation
- Learning and adaptation

Bode

Control, OR

- Worst-case ("risk")
- Delay severely degrades robust performance

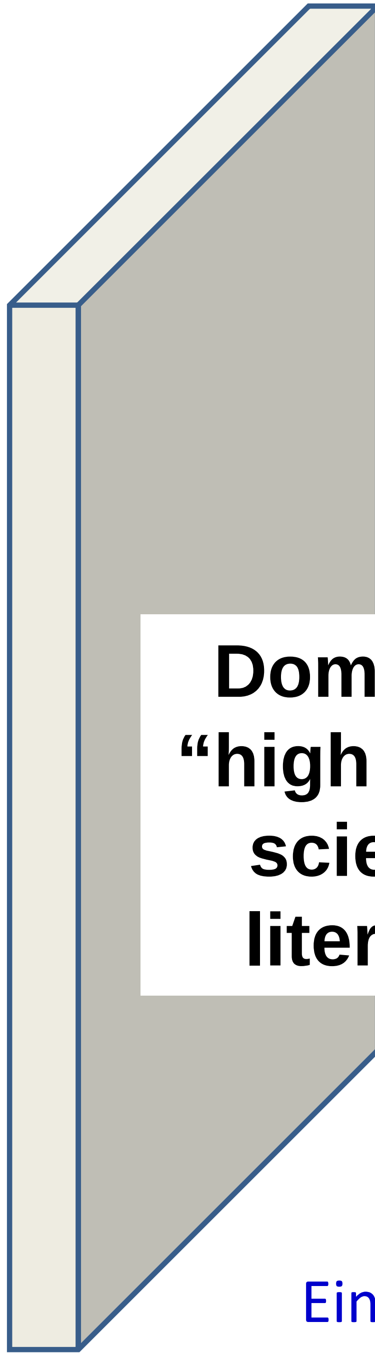
Compute

Turing

Delay and risk are *most* important

Bode

Control, OR



Dominates "high impact science" literature

Communicate

Shannon

Delay and risk are *least* important

Carnot

Boltzmann

Heisenberg

Einstein

Physics

Communicate

- Space complexity

Shannon

- Average case (risk neutral)
- Random ensembles
- Asymptotic (infinite delay)

- "Layering" by averaging

Carnot

Boltzmann

Heisenberg

Physics

Einstein

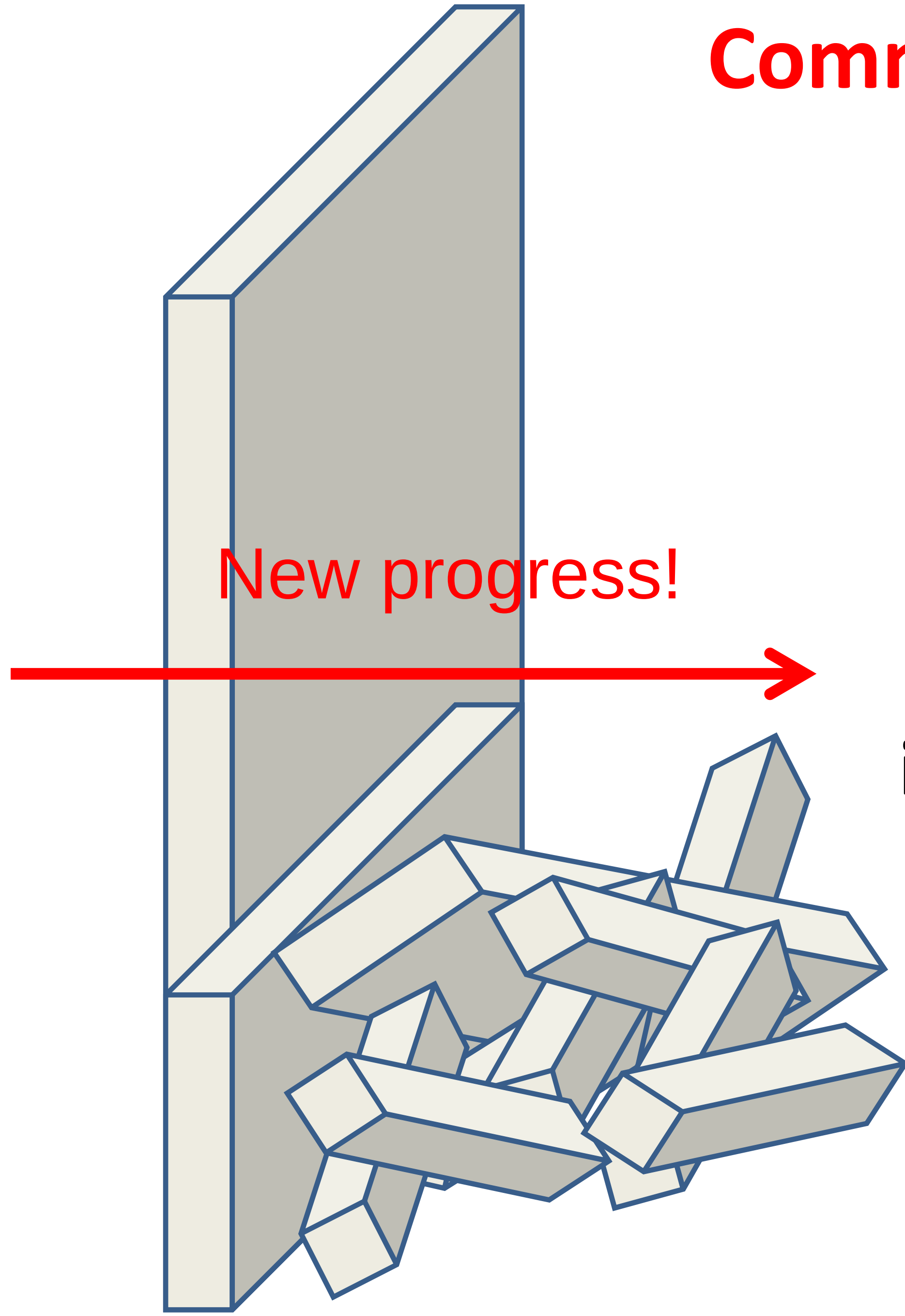
Compute

Turing

Communicate

Shannon

Delay and risk are *most* important



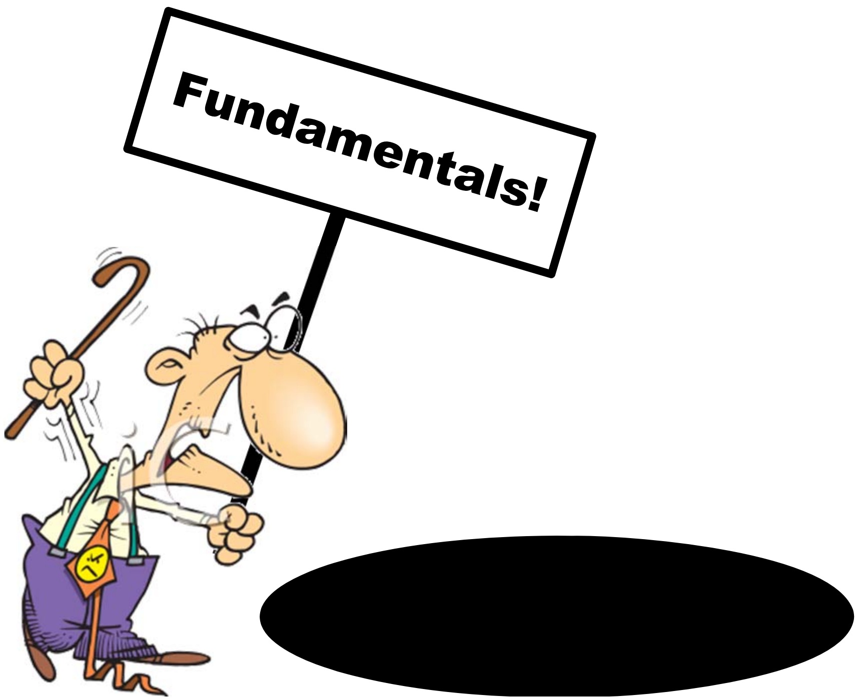
Delay and risk are ~~*least*~~ important

Bode

Control, OR

Physics

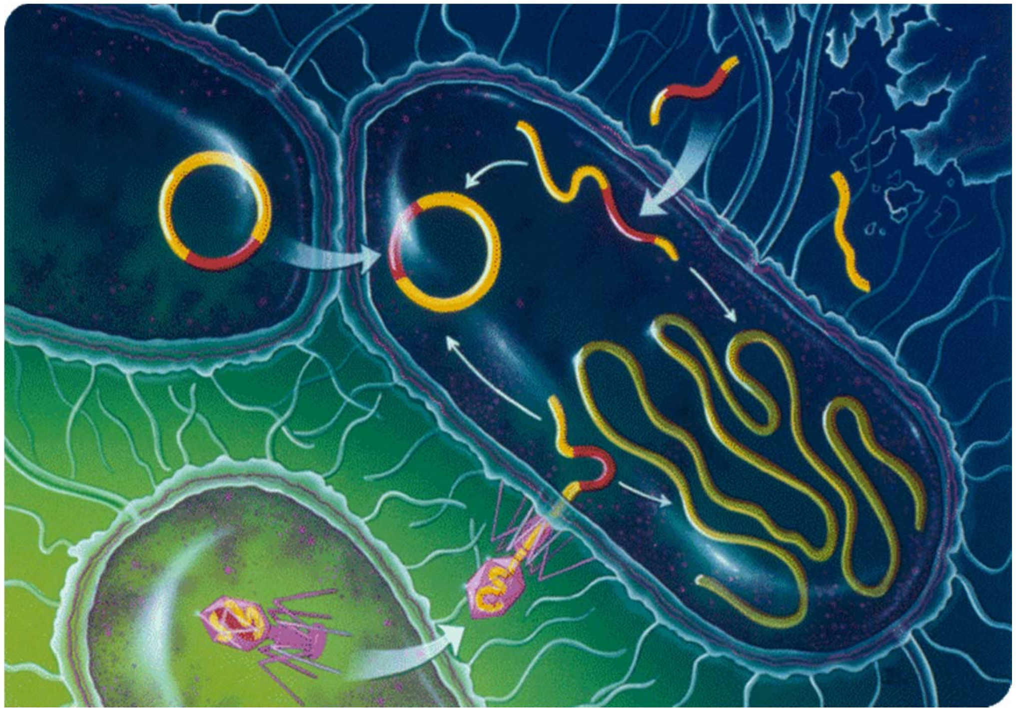
- **Brains**
- **Nets**
- **Grids (cyberphys)**
- Bugs (microbes, ants)
- Medical physiology



$\bigcap$ {Case Study}

- Lots of aerospace
- Wildfire ecology
- Earthquakes
- Physics:
  - turbulence,
  - stat mech (QM?)
- “Toy”:
  - Lego
  - clothing, fashion
- Buildings, cities
- Synesthesia

- Neuroscience
  - + People care
  - + Live demos
- Internet (& Cyber-Phys)
  - **+** Understand the details
  - **-** Flawed designs
  - **-** Everything you've read is wrong (in science)*
- Cell biology (bacteria)
  - **+** Perfection
  - ± Some people care



* this comment is for scientists

- **Neuroscience**
  + People care
  +Live demos!

1. experiments
2. data
3. theory
4. universals

Multivariable Stability Robustness

Doyle/Stein, 1981

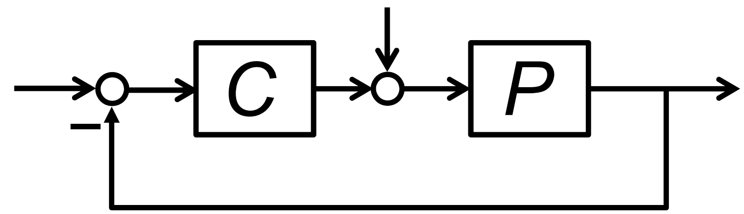
2014 American Control Conference

40 years of robust control: 1978-2018

Effect of uncertainty at plant input

Plant, *P*

– Linear, time invariant



Controller, *C*

– Linear, time-invariant

– Stabilizes *P*

Nyquist plot of det(*I*+*PC*)

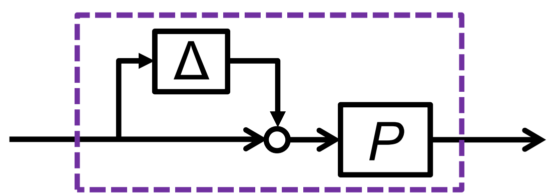
– does not pass through 0

– encircles 0 the correct number of times, CCW

Uncertainty in *P*

– Uncertain gain (complex matrix) at input

$$\mathcal{P}_\beta := \left\{ P(I_{n_u} + \Delta) : \Delta \in \mathbf{C}^{n_u \times n_u}, \bar{\sigma}(\Delta) \leq \beta \right\}$$



$$A_P \to A_P$$
$$B_P \to B_P(I + \Delta)$$
$$C_P \to C_P$$

<u>Question</u>: What is the smallest $\Delta \in \mathbf{C}^{n_u \times n_u}$ such that feedback interconnection of $P(I+\Delta)$ and C is unstable?

Effect of uncertainty at plant input

What is smallest Δ such that

- Nyquist plot of det(*I*+*P*(*I*+Δ)*C*) passes through 0?
- Solve independently at each frequency

$$
\begin{aligned}
N(\tilde{P}, C, \omega) &:= \det(I + P(j\omega)(I+\Delta)C(j\omega)) \\
&= \det(I+PC)\det(I + \underbrace{C(I+PC)^{-1}P}_{M}\Delta)
\end{aligned}
$$

$$
\frac{1}{\bar{\sigma}(M)} = \min_{\Delta \in \mathbf{C}^{m\times n}} \bar{\sigma}(\Delta) \quad \text{s.t. } \det(I_n + M\Delta) = 0
$$

- Find "worst" frequency (with smallest such Δ)

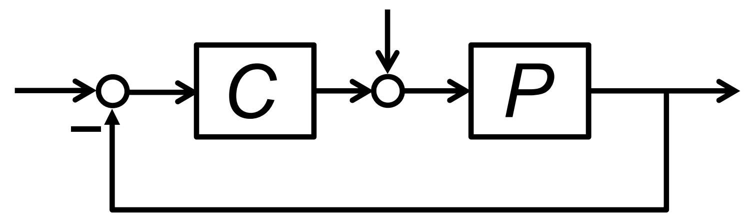
$$
\min_{\omega\in\mathbf{R}} \frac{1}{\bar{\sigma}\left[C(j\omega)\left(I_{n_u} + P(j\omega)C(j\omega)\right)^{-1} P(j\omega)\right]}
$$

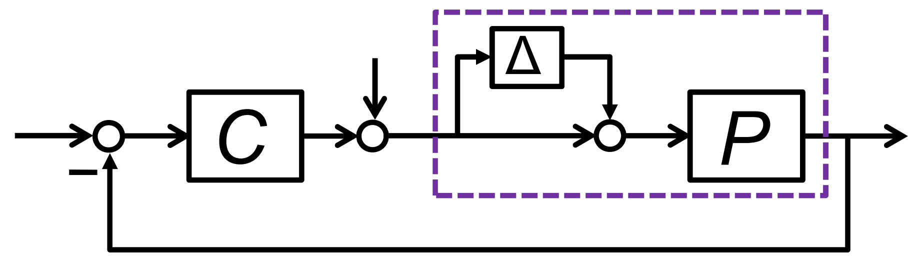
"Easiest" location for pole to migrate from *stable* to *unstable* is at this frequency

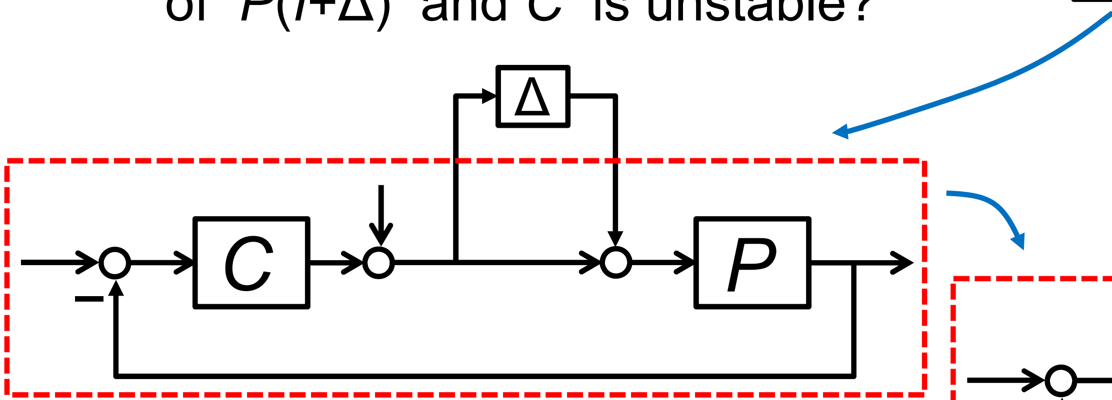
$$
\frac{1}{\max_{\omega\in\mathbf{R}} \bar{\sigma}\left[\underbrace{C(j\omega)\left(I_{n_u} + P(j\omega)C(j\omega)\right)^{-1} P(j\omega)}_{T_I}\right]} = \frac{1}{\|T_I\|_\infty}
$$

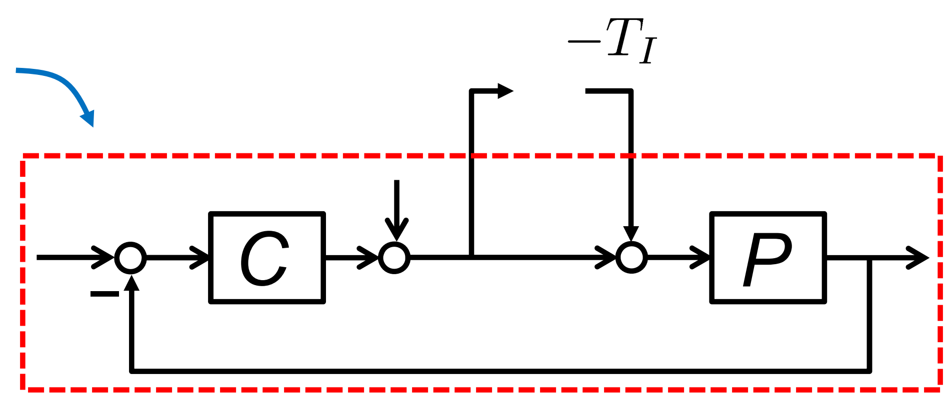
Δ is in feedback with T_I

- Plant, Controller, *P*, *C*
  - Linear, time invariant
  - *C* Stabilizes *P*
- Robustness of Stability
  - What is the smallest (complex matrix) Δ such that feedback of *P*(*I*+Δ) and *C* is unstable?



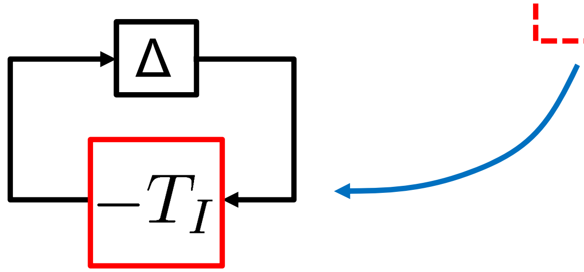






Remember this picture

Uncertain closed-loop system represented as feedback between <u>known</u> and <u>unknown</u> part



Relation to *distance of* $\det(I+PC)$ *to 0*

Are these the same idea?

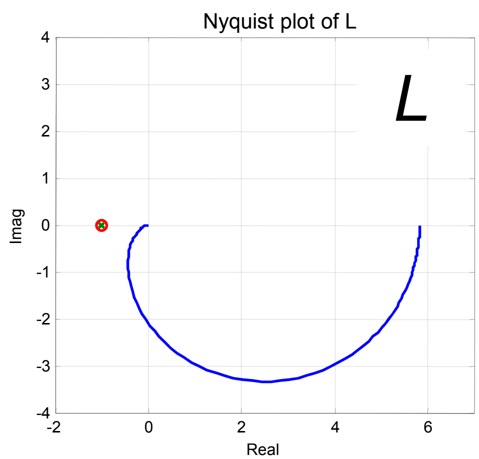
?

$$\det(I+PC) \approx 0$$

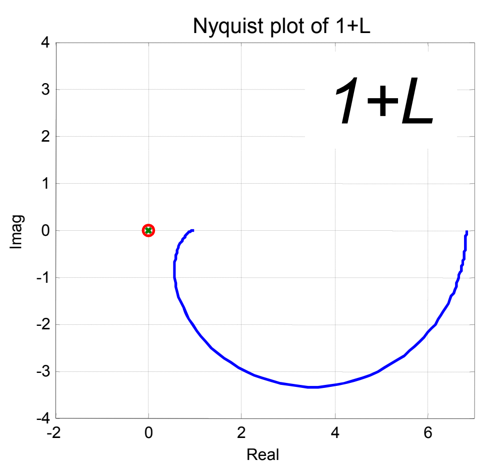
Multivariable Nyquist plot
passes close to 0

There exists small Δ to make
$\det(I+P(I+\Delta)C)=0$

No!

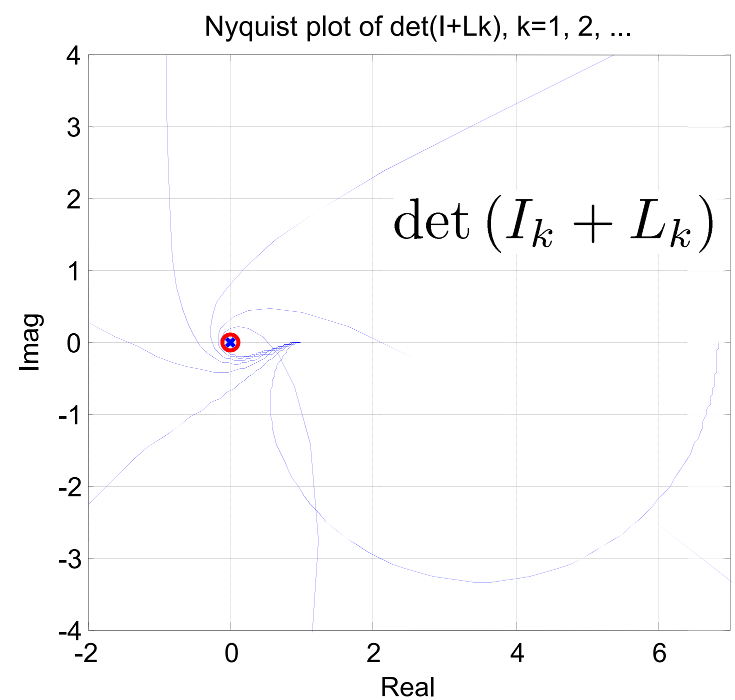


$$\frac{1}{\max_{\omega\in\mathbf{R}} \left|\frac{L(j\omega)}{1+L(j\omega)}\right|} = \frac{1}{\max_{\omega\in\mathbf{R}} \bar{\sigma}\left[L_k(j\omega)\left(I+L_k(j\omega)\right)^{-1}\right]}$$



$$\det\left(I_k + L_k(I+\Delta)\right)$$

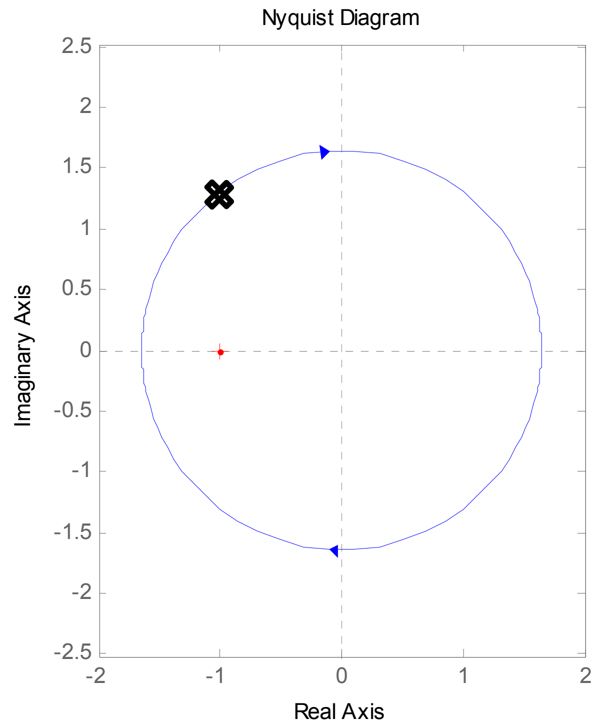
$$L_k := \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{bmatrix}$$

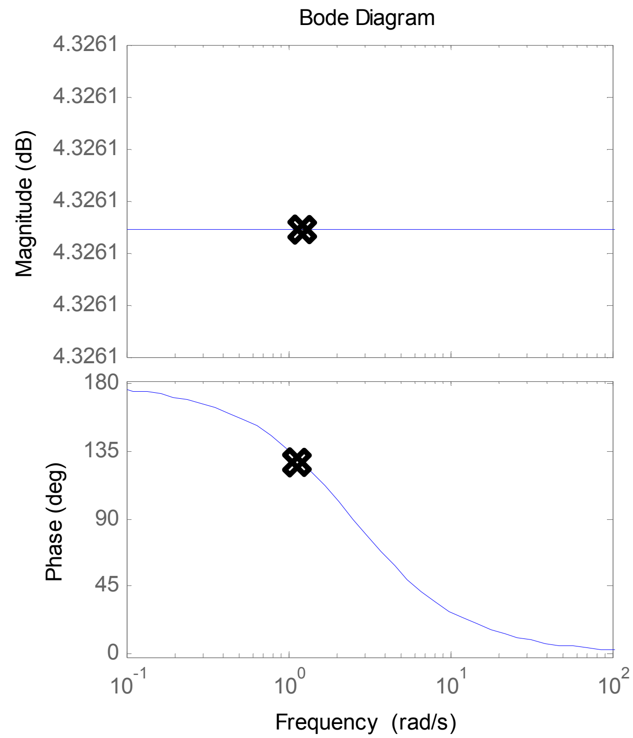


Real Dynamic models mimicking Complex numbers

Theorem: Given a positive $\bar{\omega} > 0$, and a complex number δ, with $\text{Imag}\,(\delta) \neq 0$, there is a $\beta > 0$ such that by proper choice of sign

$$\pm |\delta| \left. \frac{s-\beta}{s+\beta} \right|_{s=j\bar{\omega}} = \delta$$





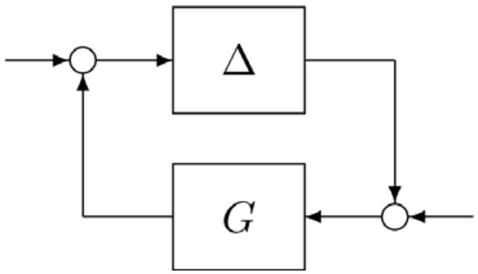
Relation between complex and real-rational uncertainty

For linear, uncertain systems, an "equivalence" between

–Constant, complex, uncertainty, and

–Linear, dynamic (with real coefficients)

can be established.



Given stable, SISO *G*(*s*) and constants $\beta, \bar{\omega}$

- there exists a complex scalar Δ with $|\Delta| \leq \beta$ such that feedback connection of (*G*,Δ) has a pole at $j\bar{\omega}$

if and only if

- there exists a stable linear system (with real coefficients) $\hat{\Delta}$ satisfying

$$\|\hat{\Delta}\|_\infty \leq \beta$$

and the feedback connection of $(G, \hat{\Delta})$ has a pole at $j\bar{\omega}$

$$|\Delta| \leq \beta, 1 - G(j\bar{\omega})\Delta = 0$$

$$\|\hat{\Delta}\|_\infty \leq \beta, 1 - G(j\bar{\omega})\hat{\Delta}(j\bar{\omega}) = 0$$

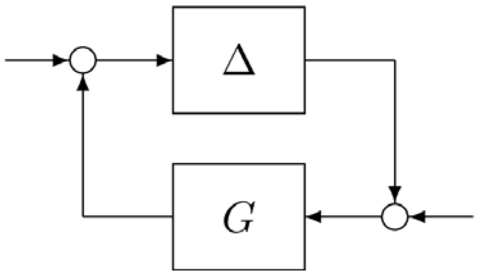
Relation between complex and real-rational uncertainty

For linear, uncertain systems, an "equivalence" between

–Constant, complex, uncertainty, and

–Linear, dynamic (with real coefficients)

can be established.



Given stable, MIMO $G(s)$ and constants $\beta, \bar{\omega}$

– there exists a complex matrix Δ with $\bar{\sigma}(\Delta) \leq \beta$ such that feedback connection of (G,Δ) has a pole at $j\bar{\omega}$

if and only if

– there exists a <u>linear system</u> (with real coefficients) $\hat{\Delta}$ satisfying

$$\|\hat{\Delta}\|_\infty \leq \beta$$

and the feedback connection of $(G, \hat{\Delta})$ has a pole at $j\bar{\omega}$

$$\bar{\sigma}\left(\Delta\right) \leq \beta, \det(I - G(j\bar{\omega})\Delta) = 0$$

$$\|\hat{\Delta}\|_\infty \leq \beta, \det(I - G(j\bar{\omega})\hat{\Delta}(j\bar{\omega})) = 0$$

LFT uncertainty modeling and stability of Uncertain Interconnections

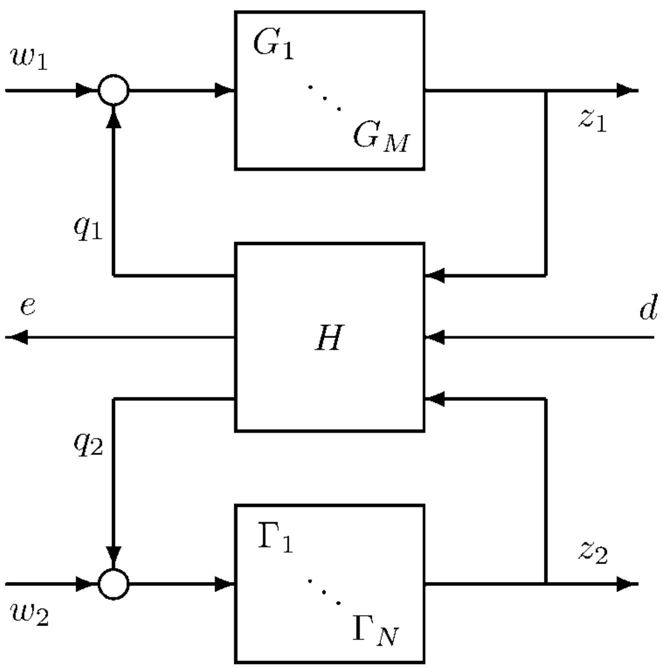
Doyle 1982

2014 American Control Conference

40 years of robust control: 1978-2018

General linear interconnection: known G_k unknown Γ_k

- each is FDLTI, with proper transfer function, and stabilizable and detectable internal state-space description.
- constant interconnection matrix $H \in \mathbf{R}^{\bullet \times \bullet}$
- *well-posed* if for any initial conditions and any piecewise-continuous inputs w_1, w_2, d, there exist unique solutions to the interconnection equations.
  - For a well-posed interconnection, a state-space model or proper transfer function description for the map from (d, w) to (e, z) can be derived.
- *stable* if the resultant state-space model is internally stable – the eigenvalues of it's "A" matrix are in the open, left-half plane.



Well-posed if and only if

$$\det\left(I - \begin{bmatrix} H_{11} & H_{13} \\ H_{31} & H_{33} \end{bmatrix} \begin{bmatrix} G(\infty) & 0 \\ 0 & \Gamma(\infty) \end{bmatrix}\right) \neq 0$$

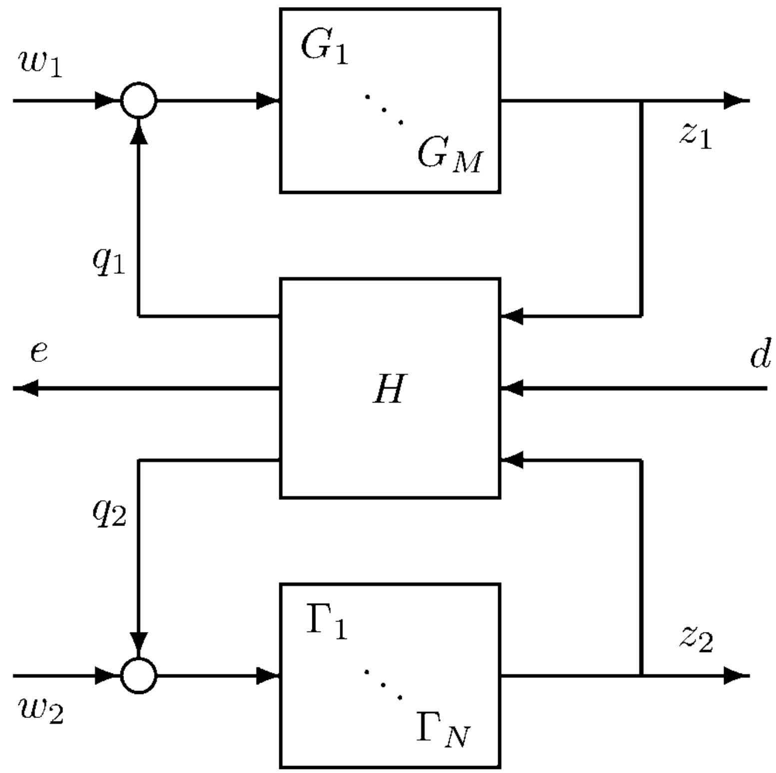
Stable if and only if $T_{wz} \in \mathcal{S}^{\bullet \times \bullet}$

Different assumptions on unknown components Γ_k

1. Γ_k is a stable linear system, known only to satisfy $\|\Gamma_k\|_\infty < 1$;
2. Γ_k is a stable linear system of the form $\gamma_k I$, where the scalar linear system γ_k is known to satisfy $\|\gamma_k\|_\infty < 1$;
3. Γ_k is a constant gain, of the form $\gamma_k I$, where the scalar $\gamma_k \in \mathbf{R}$ is known to satisfy $-1 < \gamma_k < 1$.

Is the interconnection well-posed and stable for all possible values of Γ?

If so, is the $\|\cdot\|_\infty$ gain from $d \to e \leq 1$ for all possible values of Γ?

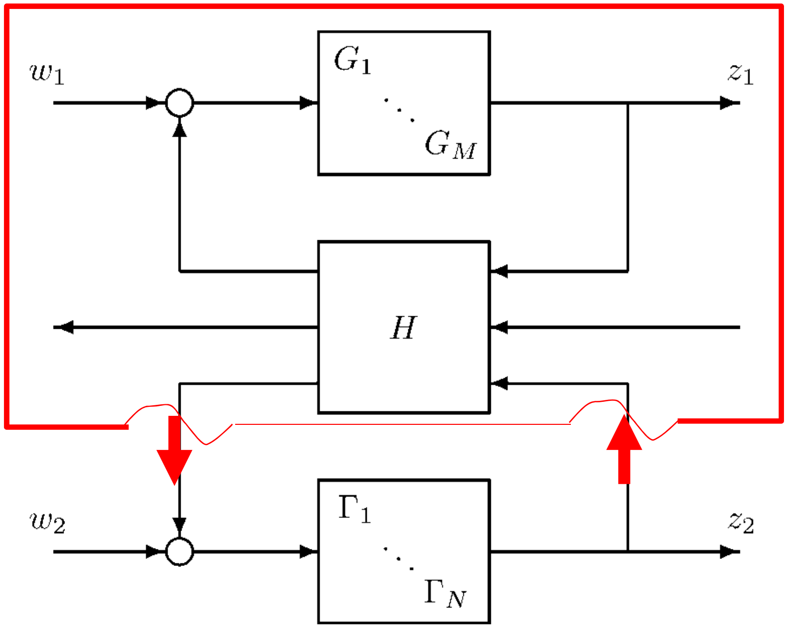


Interconnection: robust well-posedness and stability

Interconnection is well-posed at $\Gamma = 0$

$$\det\left(I - G(\infty)H_{11}\right) \neq 0$$

$$V := G(s)(I - H_{11}G(s))^{-1} \in \mathcal{R}^{p_1 \times n_1}$$



Interconnection is stable at $\Gamma = 0$

$$V := G(s)(I - H_{11}G(s))^{-1} \in \mathcal{S}^{p_1 \times n_1}$$

$$M := H_{33} + H_{31}VH_{13} \in \mathcal{S}^{\bullet \times \bullet}$$

$$X := I - \Gamma M$$

Interconnection is well-posed at Γ

$$\det\left(I - \Gamma(\infty)M(\infty)\right) = \det(X(\infty)) \neq 0.$$

X^{-1} is proper

Non-vanishing determinant conditions

Interconnection is stable at Γ

$$\det\left(I - \Gamma(s_0)M(s_0)\right) = \det\left(X(s_0)\right) \neq 0 \quad \forall s_0 \in \mathbf{C}_+$$

$$X^{-1} \in \mathcal{S}^{\bullet \times \bullet}$$

SSV (μ)

Doyle, 1982
Doyle, Wall, Stein 1982

2014 American Control Conference

40 years of robust control: 1978-2018

Structured Singular Value

Importance of the nonvanishing determinant condition

- new definition to formalize,
- separate arithmetic from system theory.

For example, consider a problem-specific set of block diagonal matrices, say,

$$\mathbf{\Delta} := \left\{ \mathrm{diag}\left[\delta_1 I_2, \delta_2, \delta_3, \Delta_4\right] : \delta_1, \delta_2 \in \mathbf{R}, \delta_3 \in \mathbf{C}, \Delta_4 \in \mathbf{C}^{2\times 2} \right\} \subseteq \mathbf{C}^{6\times 6}$$

$$\begin{bmatrix} \delta_1 I_2 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & \Delta_4 \end{bmatrix}$$

Given a single matrix $M \in \mathbf{C}^{6\times 6}$ define the quantity

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\left\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\right\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular, then $\mu_{\mathbf{\Delta}}(M) := 0$.

J. Doyle, “Analysis of feedback systems with structured uncertainties,” *IEE Proceedings*, part D, vol. 129, no. 6, pp. 242-250, 1982.

General form of Uncertain Element: Structured Singular Value

In general, the set $\mathbf{\Delta} \subseteq \mathbf{C}^{n\times n}$ will be of the form

$$\mathbf{\Delta} = \{\text{diag}\ [\delta_1^r I_{t_1}, \ldots, \delta_V^r I_{t_V}, \delta_1^c I_{r_1}, \ldots, \delta_S^c I_{r_S}, \Delta_1, \ldots, \Delta_F] : \delta_k^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_j \in \mathbf{C}^{n_j \times n_j}\}$$

which just includes many instances of the 3 "blocks" considered.

Given a matrix $M \in \mathbf{C}^{n\times n}$

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}$$

- $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n\times n} \to \mathbf{R}$
- Smallest (measured in $\bar{\sigma}(\cdot)$) root, drawn from $\mathbf{\Delta}$, of the polynomial equation
$$\det(I - M\Delta) = 0$$
- For any $\alpha \in \mathbf{R}$, $\mu(\alpha M) = |\alpha|\mu(M)$

Alternate form

Claim: Manipulation of the definition gives

$$\max_{\Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) \leq 1} \rho_R(M\Delta) = \mu_{\mathbf{\Delta}}(M).$$

Proof: If $\Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) \leq 1$ and $\rho_R(M\Delta) = \beta$ then (w/ correct sign)

$$\det(I \pm M\beta^{-1}\Delta) = 0$$

with $\beta^{-1}\Delta \in \mathbf{\Delta}, \bar{\sigma}\left(\beta^{-1}\Delta\right) \leq \beta^{-1}$. Hence $\mu_{\mathbf{\Delta}}(M) \geq \beta$, including the largest such β.

Conversely, if $\Delta \in \mathbf{\Delta}$ has $\det(I - M\Delta) = 0$, then $\rho_R(M\Delta) \geq 1$. Define

$$\Delta_N := \frac{1}{\bar{\sigma}(\Delta)}\Delta$$

Then $\Delta_N \in \mathbf{\Delta}, \bar{\sigma}(\Delta_N) = 1$ and $\rho_R(M\Delta_N) \geq \frac{1}{\bar{\sigma}(\Delta)}$, including the smallest such Δ.

Alternate form with only Complex Blocks

In general

$$\max_{\Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) \leq 1} \rho_R(M\Delta) = \mu_{\mathbf{\Delta}}(M).$$

If there are no real parameter blocks, the problem is simpler

$$\max_{\Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) \leq 1} \rho(M\Delta) = \mu_{\mathbf{\Delta}}(M)$$

involving the spectral radius since $\mathbf{\Delta}$ is unaltered by complex scalar multiplication. **Moreover, in both cases, the maximizing complex blocks are unitary, not just norm-bounded.**

Lower bound algorithms for $\mu_{\mathbf{\Delta}}(M)$ are based on these equalities.

- The left-hand-side optimization is not easy to solve, though constrained optimization can be attempted.
- Any algorithm generally finds a number smaller than the maximum, so the obtained value is a lower bound for $\mu_{\mathbf{\Delta}}(M)$.

Properties of $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n \times n} \to \mathbf{R}$

In general, μ always satisfies a maximum-modulus property. For $M(s)$, stable, and any block-structure $\mathbf{\Delta}$,

$$\max \left\{ \sup_{\mathrm{Re}(s) \geq 0} \mu_{\mathbf{\Delta}}(M(s)) \ , \ \mu_{\mathbf{\Delta}}(M_\infty) \right\} = \max \left\{ \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \ , \ \mu_{\mathbf{\Delta}}(M_\infty) \right\}.$$

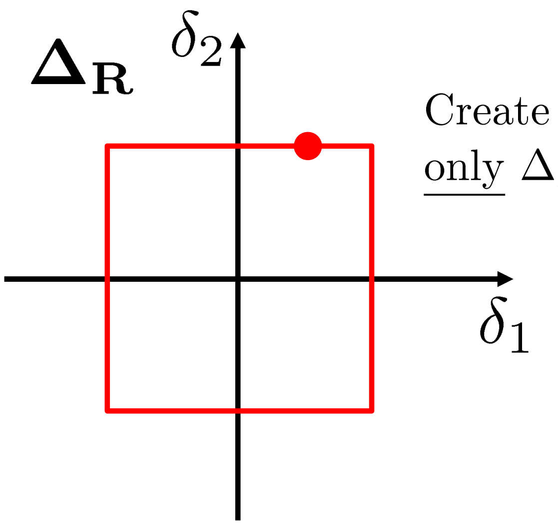
In general, $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n \times n} \to \mathbf{R}$ is upper-semicontinuous, but not continuous

- If $\mathbf{\Delta}$ only consists of complex blocks, then $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n \times n} \to \mathbf{R}$ is continuous
- Suppose that $\mathbf{\Delta}$ consists of a diagonal concatenation of two uncertainty sets, one with only real blocks, and one with only complex blocks. Denote these as $\mathbf{\Delta_R}$ and $\mathbf{\Delta_C}$. So

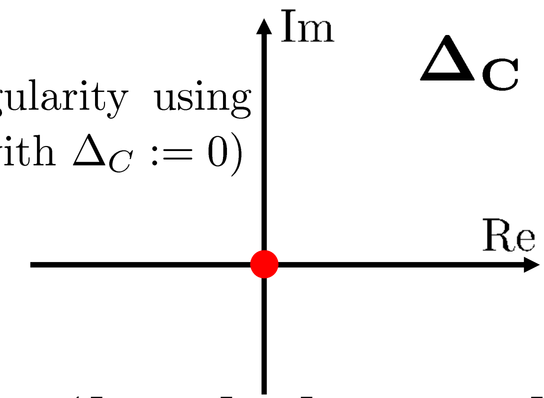
$$\mathbf{\Delta} = \{\mathrm{diag}\,[\Delta_R, \Delta_C] : \Delta_R \in \mathbf{\Delta_R}, \Delta_C \in \mathbf{\Delta_C}\} \subseteq \mathbf{C}^{n \times n}$$

Partition $M \in \mathbf{C}^{n \times m}$ accordingly. If $\mu_{\mathbf{\Delta_R}}(M_{11}) < \mu_{\mathbf{\Delta}}(M)$, then $\mu_{\mathbf{\Delta}} : \mathbf{C}^{n \times n} \to \mathbf{R}$ is continuous at M.

Example of $\mu_{\mathbf{\Delta_R}}(M_{11}) < \mu_{\mathbf{\Delta}}(M)$

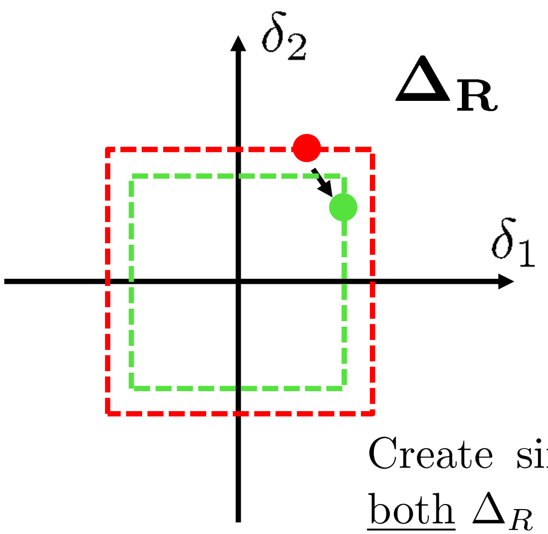


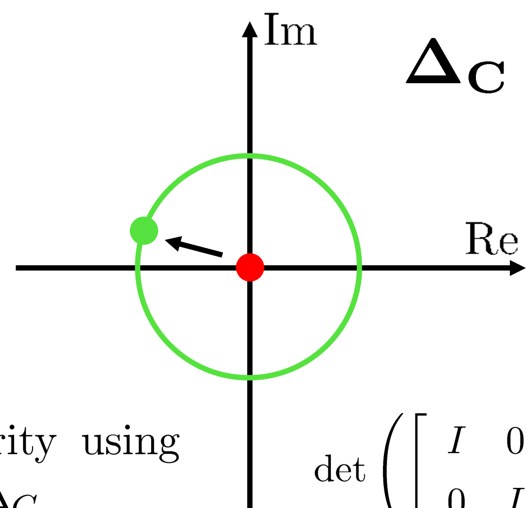
Create singularity using <u>only</u> Δ_R (with $\Delta_C := 0$)



$$\det\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \Delta_R & 0 \\ 0 & 0 \end{bmatrix}\right) = \det\left(I - M_{11}\Delta_R\right)$$

There are complex blocks, and they "matter"





Remark: Likewise, if $\mu_{\mathbf{\Delta}}$ is not continuous at M, then $\mu_{\mathbf{\Delta_R}}(M_{11}) = \mu_{\mathbf{\Delta}}(M)$. This means "the complex blocks do not matter" and hence can be set to 0.

Create singularity using <u>both</u> Δ_R and Δ_C

$$\det\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \Delta_R & 0 \\ 0 & \Delta_C \end{bmatrix}\right)$$

Canonical Block Structure

All important mathematical features are captured in $\mathbf{\Delta}$ of the form

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

The individual blocks on the diagonal are referred to as

- a *repeated real* block
- a *repeated complex* block, and
- a *full complex* block.

An actual robustness analysis might contain several of such blocks.

Canonical Block Structure

Consider an illustrative $\mathbf{\Delta}$, and associated set $\mathcal{D}$ of invertible matrices

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 \in \mathbf{C}^{t_1 \times t_1}, D_2 \in \mathbf{C}^{r_2 \times r_2}, d_3 \in \mathbf{C} \right\}$$

For every $\Delta \in \mathbf{\Delta}$ and $D \in \mathcal{D}$, $D\Delta = \Delta D$. Hence $\Delta = D^{-1} \Delta D$, and

$$\det(I - M\Delta) = \det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta).$$

Since $\mu_{\mathbf{\Delta}}(M)$ is defined entirely in terms of $\det(I - M\Delta)$,

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}\left(DMD^{-1}\right)$$

But $\mu_{\mathbf{\Delta}}(\cdot) \leq \bar{\sigma}(\cdot)$, always, hence

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}\left(DMD^{-1}\right) \leq \bar{\sigma}\left(DMD^{-1}\right)$$

Canonical Block Structure

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 \in \mathbf{C}^{t_1 \times t_1}, D_2 \in \mathbf{C}^{r_2 \times r_2}, d_3 \in \mathbf{C} \right\}$$

For any $D \in \mathcal{D}$, $\quad \mu_{\mathbf{\Delta}}(M) \leq \bar{\sigma}\left(DMD^{-1}\right)$

This bound only uses the block-diagonal structure information regarding $\mathbf{\Delta}$, namely for $D \in \mathcal{D}, \Delta \in \mathbf{\Delta}$, $D\Delta = \Delta D$.

The "best" upper bound obtained with this technique would come from optimizing the choice of $D \in \mathcal{D}$, namely

$$\mu_{\mathbf{\Delta}}(M) \leq \min_{D \in \mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right),$$

commonly referred to as the "d-m-d-inverse upper bound of μ."

Real versus Complex uncertain elements

The upper bound derived thusfar,

$$\mu_{\boldsymbol{\Delta}}(M) \leq \min_{D \in \mathcal{D}} \bar{\sigma}\left(DMD^{-1}\right),$$

does not use/exploit known information that an uncertain element is real (as opposed to complex).

For example, in the block structure below, δ_1 and δ_2 are handled the same way (using D_1 and D_2) even though more partial information is known for δ_1.

$$\boldsymbol{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D} := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 \in \mathbf{C}^{t_1 \times t_1}, D_2 \in \mathbf{C}^{r_2 \times r_2}, d_3 \in \mathbf{C} \right\}$$

Improved Upper Bound

Define sets associated with $\mathbf{\Delta}$

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 I_{t_1} & 0 & 0 \\ 0 & \delta_2 I_{r_2} & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbf{R}, \delta_2 \in \mathbf{C}, \Delta_3 \in \mathbf{C}^{m_3 \times m_3} \right\}$$

$$\mathcal{D}_+ := \left\{ \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & d_3 I_{m_3} \end{bmatrix} : D_1 = D_1^* \succ 0, D_2 = D_2^* \succ 0, d_3 > 0 \right\}$$

$$\mathcal{G} := \left\{ \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : G_1 = G_1^* \right\}$$

Due the structure of the various matrices

- if $D \in \mathcal{D}_+$, then $D\Delta = \Delta D$ for all $\Delta \in \mathbf{\Delta}$ and $D^{\frac{1}{2}} \in \mathcal{D}_+$
- if $G \in \mathcal{G}$, then $G\Delta = \Delta^* G$ for all $\Delta \in \mathbf{\Delta}$

Improved Upper Bound

Suppose $\beta > 0$, and $G \in \mathcal{G}, D \in \mathcal{D}_+$ satisfy

$$M^* D M - \beta^2 D + j(GM - M^* G) \preceq 0.$$

Then $\mu_{\mathbf{\Delta}}(M) \leq \beta$.

These type of formulae are due to Doyle (conference papers in mid 80s) and:

M. Fan, A. Tits and J. Doyle, "Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics," *IEEE Transaction on Automatic Control*, vol. 36, no. 1, pp. 25-38, January 1991.

The short proof (next slide) is taken from 1995 PhD Thesis, Anders Helmersson, "Methods for robust gain scheduling," Linköping

Improved Upper Bound

Suppose $\beta > 0$, and $G \in \mathcal{G}, D \in \mathcal{D}_+$ satisfy

$$M^* D M - \beta^2 D + j(GM - M^* G) \preceq 0.$$

Then $\mu_{\boldsymbol{\Delta}}(M) \leq \beta$.

Proof: If $\Delta \in \boldsymbol{\Delta}$ has $\det(I - M\Delta) = 0$, there exist nonzero $w, z \in \mathbf{C}^n$ with $w = Mz, z = \Delta w$. Use $D^{\frac{1}{2}}\Delta = \Delta D^{\frac{1}{2}}$ and $\Delta^* G = G\Delta$,

$$\begin{aligned} 0 &\geq z^*(M^* D M - \beta^2 D + j(GM - M^* G))z \\ &= w^* D w - \beta^2 w^* \Delta^* D \Delta w + j w^* \Delta^* G w - j w^* G \Delta w \\ &= w^* D^{\frac{1}{2}} D^{\frac{1}{2}} w - \beta^2 w^* \Delta^* D^{\frac{1}{2}} D^{\frac{1}{2}} \Delta w + j w^* G \Delta w - j w^* G \Delta w \\ &= w^* D^{\frac{1}{2}} D^{\frac{1}{2}} w - \beta^2 w^* D^{\frac{1}{2}} \Delta^* \Delta D^{\frac{1}{2}} w \\ &= w^* D^{\frac{1}{2}} \left(I - \beta^2 \Delta^* \Delta\right) D^{\frac{1}{2}} w. \end{aligned}$$

Since D is invertible and $w \neq 0_n$, it must be that $\bar{\sigma}(\Delta) \geq \beta^{-1}$. Hence the minimum (in definition of $\mu_{\boldsymbol{\Delta}}(M)$) is $\geq \frac{1}{\beta}$, making $\mu_{\boldsymbol{\Delta}}(M) \leq \beta$.

Robustness test with μ

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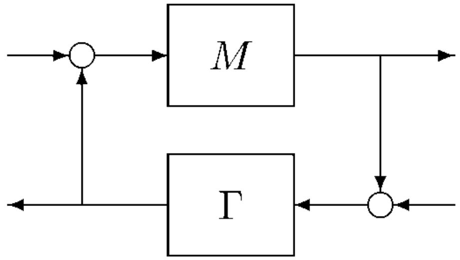
Dynamic, Structured, Linear Uncertain Elements



- $M \in \mathcal{S}^{n \times m}$ is given
- $\mathbf{\Delta} \subseteq \mathbf{C}^{m \times n}$, and associated $\mathbf{\Gamma}$

$$\mathbf{\Delta} = \{\text{diag}\,[\delta_1^r I_{t_1}, \ldots, \delta_V^r I_{t_V}, \delta_1^c I_{r_1}, \ldots, \delta_S^c I_{r_S}, \Delta_1, \ldots, \Delta_F] : \\ \delta_k^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_j \in \mathbf{C}^{m_j \times n_j}\}$$

$$\mathbf{\Gamma} := \{\text{diag}\,[\gamma_1^r I_{t_1}, \ldots, \gamma_V^r I_{t_V}, \gamma_1(s) I_{r_1}, \ldots, \gamma_S(s) I_{r_S}, \Gamma_1(s), \ldots, \Gamma_F(s)] : \\ \gamma_k^r \in \mathbf{R}, \gamma_i \in \mathcal{S}, \Gamma_j \in \mathcal{S}^{m_j \times n_j}\}$$

- Partial knowledge is $\Gamma \in \mathbf{\Gamma}$ and $\|\Gamma\|_\infty < 1$

Determine if $(I_n - M(s)\Gamma(s))^{-1} \in \mathcal{S}^{n \times n}$ for all such Γ

Equivalently: is $\det(I - M(s_0)\Gamma(s_0)) \neq 0$ for all $\text{Re}(s_0) \geq 0$ and all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < 1$.

Robust Stability of Interconnection as μ-test

Theorem: (M, Γ) interconnection is stable for all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < \beta$ if and only if $M \in \mathcal{S}^{n\times n}$ and

$$\max_{\omega \in \mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) := \max \left\{ \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \; , \; \mu_{\mathbf{\Delta}}(M_\infty) \right\} \leq \frac{1}{\beta}$$

Proof: $(\Rightarrow)$ If $M \notin \mathcal{S}^{n\times n}$ then the interconnection is unstable at $\Gamma = 0$.

If $\mu_{\mathbf{\Delta}}(M(j\bar{\omega})) > \beta^{-1}$ at some nonzero, finite frequency $\bar{\omega}$, there is a $\Delta \in \mathbf{\Delta}$ with $\bar{\sigma}(\Delta) < \beta$ such that $I - M(j\bar{\omega})\Delta$ is singular. Now proceed block-by-block:

- replace each complex block Δ_i in Δ with stable, real-rational Γ_i that has $\|\Gamma_i\|_\infty = \bar{\sigma}(\Delta_i) < \beta$ and $\Gamma_i(j\bar{\omega}_i) = \Delta_i$
- real-valued blocks in Δ are copied into Γ

The constructed Γ satisfies: $\Gamma \in \mathbf{\Gamma}, \|\Gamma\|_\infty < \beta$ and the (M, Γ) interconnection is unstable, with a pole at $s = j\bar{\omega}$

Robust Stability of Interconnection as μ-test

Theorem: (M,Γ) interconnection is stable for all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < \beta$ if and only if $M \in \mathcal{S}^{n\times n}$ and

$$\max_{\omega\in\mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) := \max\left\{\sup_{\omega\in\mathbf{R}} \mu_{\mathbf{\Delta}}(M\,(j\omega))\ \ ,\ \ \mu_{\mathbf{\Delta}}(M_\infty)\right\} \leq \frac{1}{\beta}$$

Proof: $(\Leftarrow)$ Since M is proper, stable and μ satisfies a maximum-modulus property,

$$\max\left\{\sup_{\mathrm{Re}(s)\geq 0} \mu_{\mathbf{\Delta}}(M(s))\ \ ,\ \ \mu_{\mathbf{\Delta}}(M_\infty)\right\} \leq \frac{1}{\beta}$$

Therefore, at all $s_0 \in \mathbf{C}$, with $\mathrm{Re}(s_0) \geq 0$ (including $s = \infty$)

$$\mu_{\mathbf{\Delta}}(M(s_0)) \leq \frac{1}{\beta}.$$

Take any $\Gamma \in \mathbf{\Gamma}$ (also stable) with $\|\Gamma\|_\infty < \beta$. Γ satisfies the (usual) maximum-modulus property, so $\bar{\sigma}\,(\Gamma(s_0)) \leq \|\Gamma\|_\infty < \beta$ and $\Gamma(s_0) \in \mathbf{\Delta}$. Hence $I - M(s_0)\Gamma(s_0)$ is nonsingular. This holds at all $\mathrm{Re}(s_0) \geq 0$, so $(I - M\Gamma)^{-1}$ is proper and stable.

Robust Stability of Interconnection as μ-test

Theorem: (M, Γ) interconnection is stable for all $\Gamma \in \mathbf{\Gamma}$ with $\|\Gamma\|_\infty < \beta$ if and only if $M \in \mathcal{S}^{n\times n}$ and

$$\max_{\omega \in \mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) := \max \left\{ \sup_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \ , \ \mu_{\mathbf{\Delta}}(M_\infty) \right\} \leq \frac{1}{\beta}$$

There are a few technical details to pay attention to get it all correct, constructing the destabilizing $\Gamma \in \mathbf{\Gamma}$ when $\max_{\omega \in \mathbf{R}^e} \mu_{\mathbf{\Delta}}(M(j\omega)) > \beta^{-1}$.

- the peak may occur at $\omega_P = \infty$ or 0. The frequency-response at $\omega = \infty$ or $\omega = 0$ of an element $\Gamma \in \mathbf{\Gamma}$ is real-valued, not complex-valued. So, the "equivalence" between constant-complex-valued and real-dynamic uncertainty is more delicate.
- If $\mu_{\mathbf{\Delta}}$ is continuous at $M(j\omega_P)$, then $\mu_{\mathbf{\Delta}}(M(j\omega)) > \beta^{-1}$ at some finite, nonzero ω and the previous construction works.
- If $\mu_{\mathbf{\Delta}}$ is not continuous at $M(j\omega_P)$, then without loss in generality, all complex-blocks in Δ can be chosen as 0, and the destabilizing Γ is constant (and real-valued).

Performance characterized as Robustness

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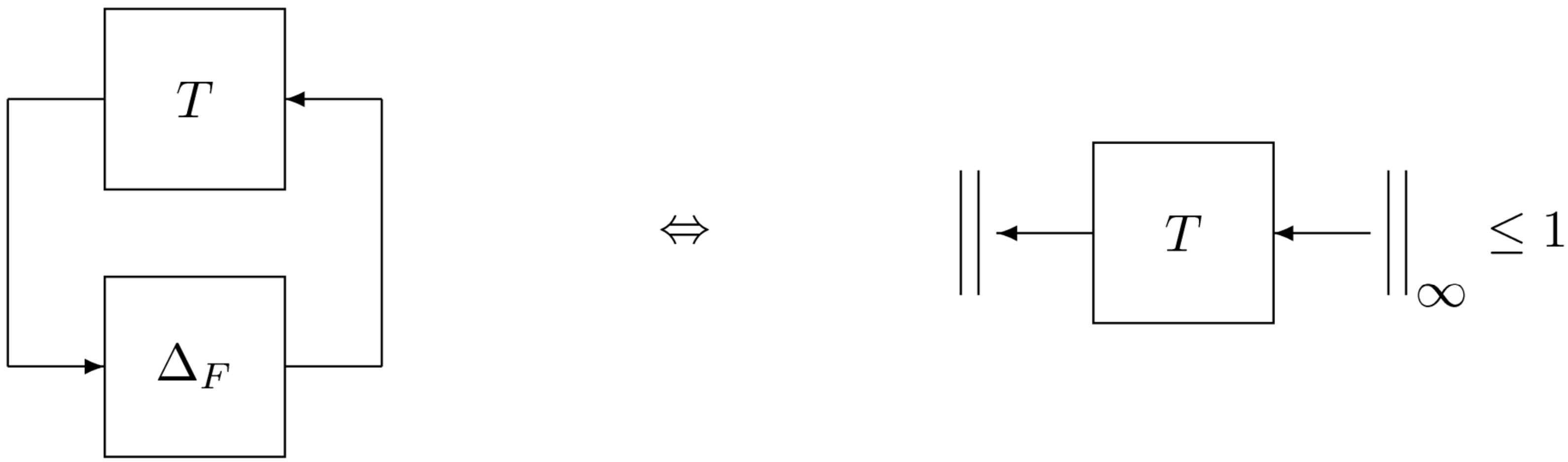
Performance as Robustness-to-Uncertainty



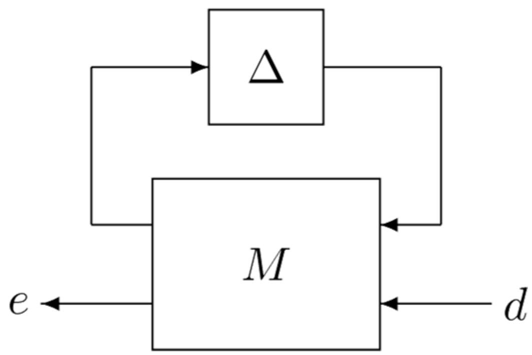
Stable for all $\|\Delta_F\|_\infty < 1$

The norm of a transfer function can be determined using a robust stability test

Pose **robust performance** questions as robust stability questions.

Robust Performance as Robust Stability

$$T = F_U(M, \Delta)$$

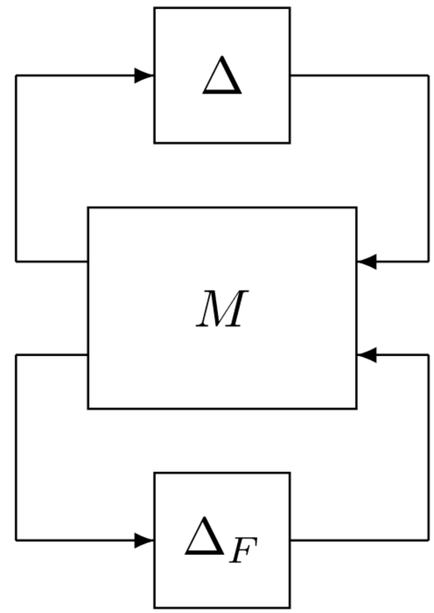


Exactly a *Robust Stability* problem for M, subjected to perturbation matrices with block diagonal structure,

$$\Delta_P = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix}$$

Hence, robust *stability* techniques – on a larger problem, to test/guarantee robust *performance* of original problem

$\|F_U(M, \Delta)\|_\infty \leq 1$ for all $\Delta \in \boldsymbol{\Delta}$ if and only



is stable for all $\Delta \in \boldsymbol{\Delta}$ **and** all $\Delta_F \in \mathbf{C}^{m \times n}$ satisfying

$$\bar{\sigma}(\Delta_F) < 1.$$

$\mathcal{H}_\infty$ control and $\mathcal{H}_\infty$-loop Shaping

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(For the ACC 2014Workshop
40 years of robust control)
ACC 2014

Contents

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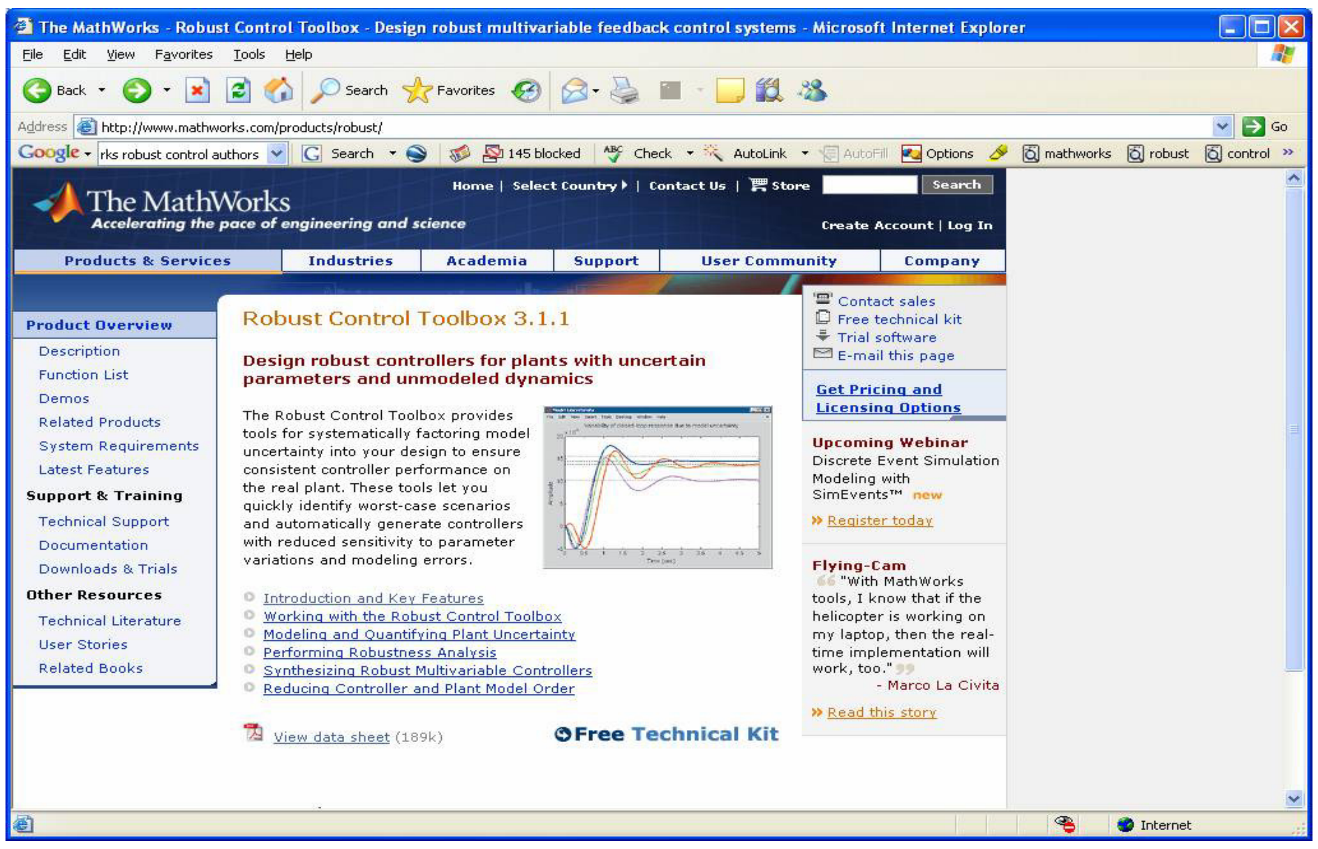
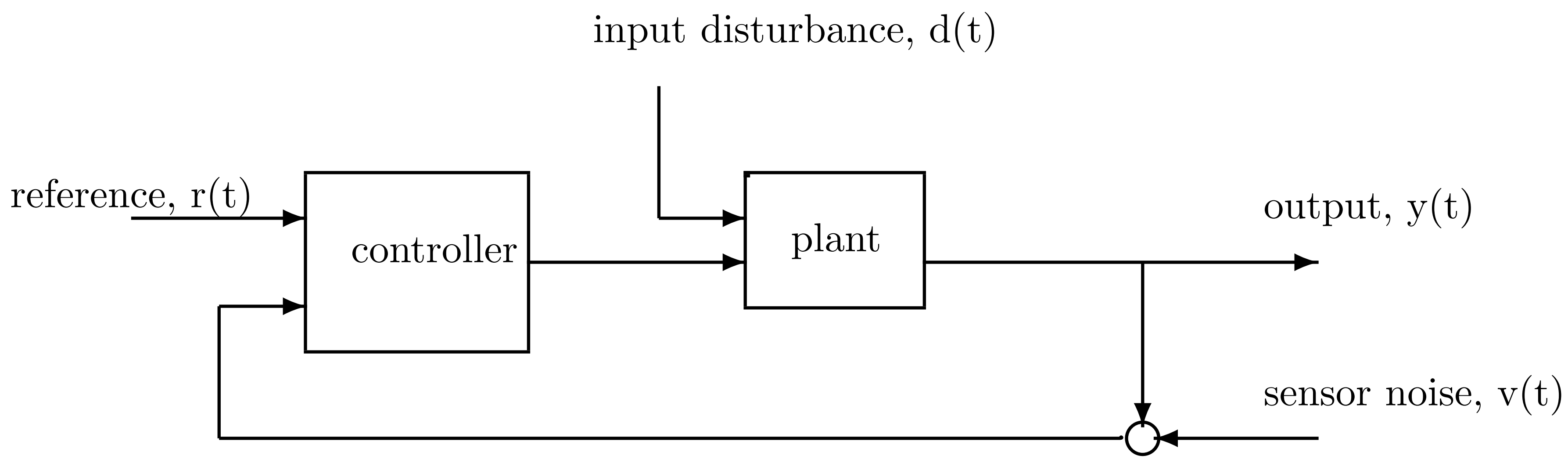


Figure 1: Robust Control Toolbox

2 Introduction

2.1 General Problem

A typical feedback system is given below:



Given a **plant** whose dynamics are only known approximately our **objective** is to design a controller so that the output "follows" the reference despite the "uncertainty" in the plant and the "unknown" disturbances.

To pose a formal problem and use analytic techniques we need to be more precise:

- measure the size of the error between the output, $y(t)$, and the reference, $r(t)$. e.g. $\|y - r\|_2 = \sqrt{\int_0^\infty (y(t) - r(t))^2 \; dt}$.
- characterize the uncertainty in the plant, e.g. $|G(j\omega) - G_o(j\omega)| < \epsilon$ for all ω.
- characterize the unknown disturbances, e.g. $\int_0^\infty d(t)^2 \; dt < 1$, or $d(t)$ is white noise.

Analysis - given such a set-up with a given controller is the size of the error suitably small for all disturbances in its class and all plants in its class?

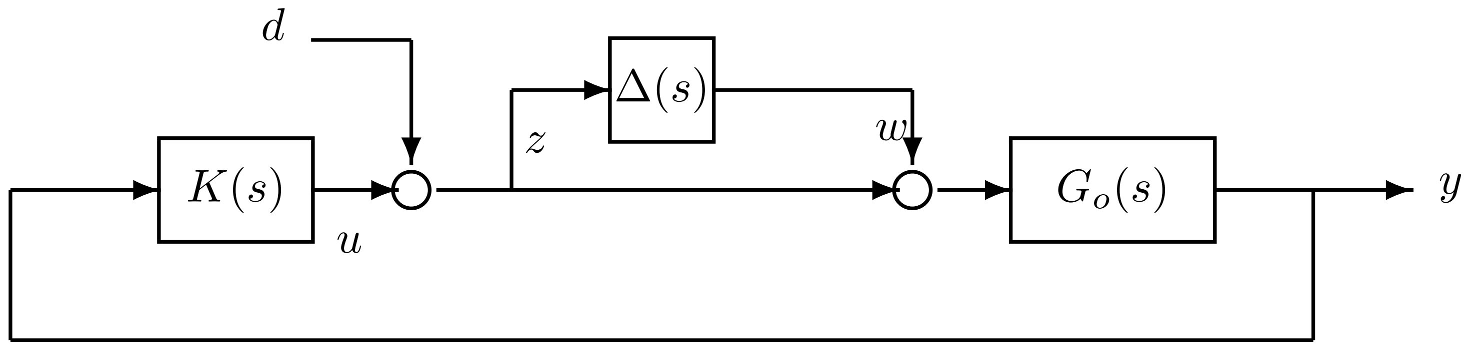
Synthesis - find a controller to meet such a specification.

Example

Consider the example where the plant's transfer function,

$$G(s) = G_o(s)(1 + \Delta(s)) \quad \text{where} \quad |\Delta(j\omega)| < \epsilon \ \text{ for all } \omega,$$

and a disturbance, $d(t)$, enters the system and the plant input as follows:



Suppose that we want the output due to the disturbance to be limited in the sense that the transfer function from d to y satisfies:

$$|T_{d\to y}(j\omega)| < \alpha(\omega) \ \text{ for all } \omega,$$

and for all plants perturbed as above.

$$
\begin{aligned}
y &= G_o(1+\Delta)\,[d + Ky] \\
(1 - G_o(1+\Delta)K)y &= G_o(1+\Delta)d \\
\Rightarrow T_{d\to y} &= \frac{G_o(1+\Delta)}{1 - G_o(1+\Delta)K}
\end{aligned}
$$

Therefore we require,

$$
\begin{aligned}
\left|\frac{G_o(1+\Delta)}{1 - G_o(1+\Delta)K}\right| &< \alpha \quad \text{for all } \omega \text{ and for all } |\Delta| < \epsilon. \\
\Leftrightarrow \left|\frac{1}{G_o(1+\Delta)} - K\right| &> \frac{1}{\alpha} \quad \text{for all } \omega \text{ and for all } |\Delta| < \epsilon. \\
\Leftrightarrow \left|\frac{1}{(1+\Delta)} - G_oK\right| &> \frac{|G_o|}{\alpha} \quad \text{for all } \omega \text{ and for all } |\Delta| < \epsilon.
\end{aligned}
$$

Given α and ϵ this gives a condition on $G_o(j\omega)$ and $K(j\omega)$ for each ω and to make it easily computed we need to eliminate the term Δ. Consider the term $1/(1+\Delta)$ for all Δ with $|\Delta| < \epsilon$; we will show that this set of points in the complex plane gives the inside of a disk with centre $1/(1-\epsilon^2)$ and radius $\epsilon/(1-\epsilon^2)$, as Δ varies with $|\Delta| < \epsilon$.

Firstly note that given complex numbers β and z with $|\beta| < 1$ then,

$$\begin{aligned} \left|\frac{\beta + z}{1 + \beta^* z}\right|^2 &= \frac{|\beta|^2 + \beta z^* + \beta^* z + |z|^2}{1 + \beta^* z + \beta z^* + |\beta|^2 |z|^2} \\ &= 1 - \frac{(1 - |\beta|^2)(1 - |z|^2)}{|1 + \beta^* z|^2} \quad \begin{cases} < 1 & \text{if } |z| < 1 \\ = 1 & \text{if } |z| = 1 \end{cases} \end{aligned}$$

So that the set of points $w = \frac{\beta + z}{1 + \beta^* z}$ map out a disk in the complex plane centred at the origin with unit radius as z varies inside the circle of unit radius. Note also that $z = \frac{(-\beta) + w}{1 + (-\beta)^* w}$ so there is a unique correspondence between the points z and w inside the unit disk.

Now we note that,

$$\frac{1}{1 + \Delta} = \frac{1}{1 - \epsilon^2} + \hat{\Delta}, \quad \text{where } \hat{\Delta} = -\frac{\epsilon}{1 - \epsilon^2} \cdot \left(\frac{\epsilon + \Delta/\epsilon}{1 + \epsilon\Delta/\epsilon}\right) \Rightarrow |\hat{\Delta}| < \frac{\epsilon}{1 - \epsilon^2}.$$

Now substituting into the condition $\left|\frac{1}{(1+\Delta)} - G_o K\right| > \frac{|G_o|}{\alpha}$, we obtain,

$$\begin{aligned} \left|\frac{1}{(1 + \Delta)} - G_o K\right| &> \frac{|G_o|}{\alpha} \text{ for all } |\Delta| < \epsilon \\ \Leftrightarrow \left|\frac{1}{1 - \epsilon^2} + \hat{\Delta} - G_o K\right| &> \frac{|G_o|}{\alpha} \text{ for all } |\hat{\Delta}| < \epsilon/(1 - \epsilon^2) \end{aligned}$$

$$\Leftrightarrow \boxed{\left|\frac{1}{1-\epsilon^2} - G_o K\right| > \frac{|G_o|}{\alpha} + \frac{\epsilon}{1-\epsilon^2}}$$

This final condition gives the exact condition for the so-called **Robust Performance** of the uncertain system.

If $\epsilon = 0$ then this reduces to the **Nominal Performace condition**,

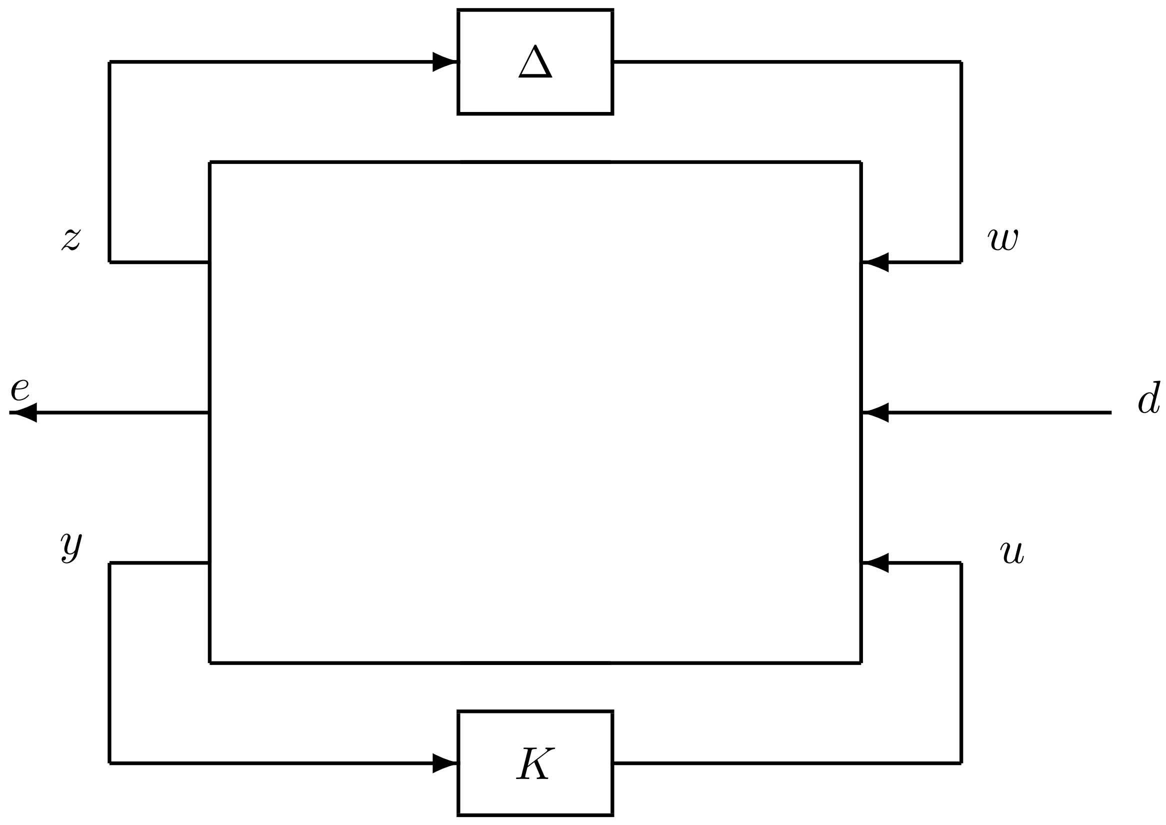
$$|1 - G_o K| > \frac{|G_o|}{\alpha} \quad \text{i.e.} \quad \boxed{\left|\frac{G_o}{1 - G_o K}\right| < \alpha}$$

Alternatively if we remove the performance condition by letting $\alpha \to \infty$ then defining $T_o = G_o K/(1 - G_o K)$, $(\Rightarrow G_o K = T_o/(1 + T_o))$, the **Robust Stability** condition becomes:

$$\begin{aligned}
\left|\frac{1}{1-\epsilon^2} - \frac{T_o}{1+T_o}\right| &> \frac{\epsilon}{1-\epsilon^2} \\
\Leftrightarrow \left|1 + \epsilon^2 T_o\right|^2 &> \epsilon^2 \left|1 + T_o\right|^2 \\
\Leftrightarrow \ 1 + \epsilon^2 (T_o + T_o^*) + \epsilon^4 \left|T_o\right|^2 &> \epsilon^2 \left(1 + T_o + T_o^* + \left|T_o\right|^2\right) \\
\Leftrightarrow \ (1 - \epsilon^2) &> \epsilon^2 (1 - \epsilon^2) \left|T_o\right|^2
\end{aligned}$$

$$\Leftrightarrow \boxed{|T_o| < \frac{1}{\epsilon}}$$

This example illustrates a general **robust performance problem** which can be put in the following general framework:



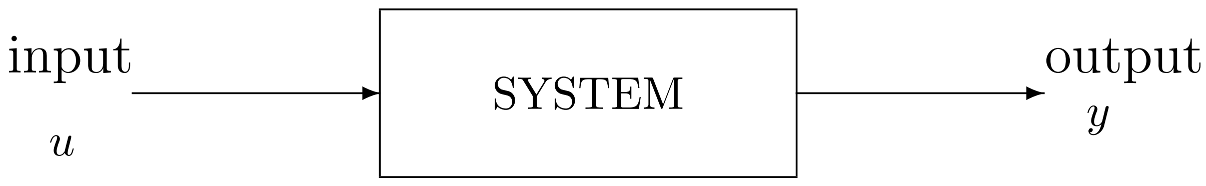
We will consider:

- Stability
- Robust Stability
- Performance
- Robust Performance

3 Systems and Signals

3.1 Scalar Case

A system can be thought of as a mapping from its inputs to outputs:



For a quantitative theory we need a measure of the **size** of the signals and this induces the gain of the system as the maximum ratio of the size of the output to the size of the input.

There are a number of different choices that can be used but the choices we give below have been found to be both physically sensible and able to exploit an elegant underlying mathematical theory.

Definition 3.1

$$\|u\|_2 = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2 dt}$$

is called the $\mathcal{L}_2$-norm of the signal u. This is a measure of the size of the signal with $\|u\|_2^2$ the energy of the signal. ($\mathcal{L}$ stands for Lebesgue space)

Definition 3.2 *If $\|u\|_2 < \infty$ then the Fourier transform of the signal u is given by*

$$\hat{u}(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt$$

and we can define the $\mathcal{L}_2$-norm of $\hat{u}(j\omega)$ as

$$\|\hat{u}\|_2 = \sqrt{\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 \, d\omega}$$

The following is a remarkable result connects the norms of functions and their transforms.

Theorem 3.3 (Parseval's Theorem)

$$\int_{-\infty}^{\infty} u(t)^* y(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{y}(j\omega)d\omega$$

and this immediately implies that $\|u\|_2 = \frac{1}{\sqrt{2\pi}}\|\hat{u}\|_2$.

Definition 3.4 *A transfer function is said to be in the space* $\mathcal{H}_\infty$ *(where the* $\mathcal{H}$ *stands for Hardy space), if*

$$\sup_{\Re e(s)>0} |G(s)| < \infty.$$

when the $\mathcal{H}_\infty$*-norm is defined as*

$$\|G(s)\|_\infty = \sup_{\Re e(s)>0} |G(s)|.$$

[sup is like max except need not be achieved]

Note that if $G(s)$ is in $\mathcal{H}_\infty$ then all its poles must be in the left half plane and hence this will be a stable transfer function.

Theorem 3.5 (Maximum Modulus Theorem) *If* $G(s)$ *is in* $\mathcal{H}_\infty$ *then*

$$\|G(s)\|_\infty = \sup_{\Re e(s)>0} |G(s)| = \sup_{-\infty<\omega<\infty} |G(j\omega)|.$$

This result shows that the $\mathcal{H}_\infty$-norm can be calculated by just examining $G(s)$ for s on the imaginary axis and it is not required to consider s in the whole of the right half plane.

The **gain** of a system with input u and output y will be defined as,

$$\sup_{u\neq 0} \frac{\|y\|_2}{\|u\|_2} = \sup_{\hat{u}\neq 0} \frac{\|\hat{y}\|_2}{\|\hat{u}\|_2}$$

Theorem 3.6 *For a stable Linear Time Invariant system with transfer function* $G(s)$ *its gain* *is given by,*

$$\sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2} = \|G(s)\|_\infty$$

Proof:

$$\begin{aligned}
\|\hat{y}\|_2^2 &= \|G\,\hat{u}\|_2^2 = \int_{-\infty}^{\infty} |G(j\omega)|^2 \, |\hat{u}(j\omega)|^2 \; d\omega \\
&\leq \|G(j\omega)\|_\infty^2 \int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 \; d\omega = \|G\|_\infty^2 \, \|\hat{u}\|_2^2 \\
\Rightarrow \frac{\|\hat{y}\|_2}{\|\hat{u}\|_2} &\leq \|G\|_\infty \;\text{ for any } u \neq 0
\end{aligned}$$

To show that maximising LHS gives equality requires a judicious choice of u. Idea: find ω_o where $|G(j\omega)|$ achieves maximum then choose

$$\begin{aligned}
u(t) &= \sin \omega_o t \\
\Rightarrow y(t) &\to |G(j\omega_o)| \,.\, \sin\left(\omega_o t + \angle G(j\omega_o)\right) \\
\Rightarrow \sqrt{\text{energy ratio}} &\to |G(j\omega_o)|
\end{aligned}$$

(Technical point: the integral of $u^2(t)$ will $\to \infty$, so we need to take a sinusoid of finite but very long duration). $\square$

Ex. (i) $\left\|\frac{1}{1+s}\right\|_\infty = 1$ and the max is achieved at $s = 0$ and $\left|\frac{1}{1+\sigma+j\omega}\right| \leq 1$ for all $\sigma > 0$, and all ω.

(ii) e^s in analytic in whole complex plane but $\sup_{\Re e(s)>0} |e^s| = \infty$
$\Rightarrow e^s$ is not in $\mathcal{H}_\infty$
whereas e^{-s} is in $\mathcal{H}_\infty$.

3.2 Vector/matrix generalisations

(i) Vector version of $\mathcal{L}_2$

Take (column) vector functions $\underline{u}(t)$ of length r and define

$$\begin{aligned}\|\underline{u}\|_2^2 &= \int_{-\infty}^{\infty} \underline{u}(t)^* \underline{u}(t)\, dt \quad (\text{* denotes complex conjugate transpose.}) \\ &= \int_{-\infty}^{\infty} \sum_{i=1}^{r} |u_i(t)|^2 \; dt\end{aligned}$$

Parseval's Theorem then becomes,

$$\int_{-\infty}^{\infty} \underline{u}(t)^* \underline{y}(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\underline{u}}(j\omega)^* \hat{\underline{y}}(j\omega)\, d\omega$$

(ii) Matrix version of $\mathcal{H}_\infty$ space:

Let A be any complex matrix then $\lambda_i(A^*A)$ are real and ≥ 0. **Proof:** Let $A^*A\underline{w} = \lambda\underline{w}$ then $\underline{w}^*A^*A\underline{w} = \lambda\underline{w}^*\underline{w}$ so that $\lambda = \frac{\|A\underline{w}\|_2^2}{\|\underline{w}\|_2^2} \geq 0$.

Let $\lambda_i(A^*A) = \sigma_i^2$ then $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_n \geq 0$ are called the **singular values** of A. Indeed just as in the real case A will have a singular value decomposition,

$$A = U\Sigma V^*, \quad \text{where } U^*U = I,\ V^*V = I,\ \Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{bmatrix},\ \Sigma_{11} = \text{diag}(\sigma_1, \cdots, \sigma_r)$$

with U and V complex matrices.

Denote max. sing. value of A by $\overline{\sigma}(A)$. If $G(s)$ is a $p \times m$ matrix function of s, whose elements are analytic in RHP (i.e. no poles in $\Re e(s) \geq 0$) and such that

$\sup_{\Re e(s)>0} \overline{\sigma}(G(s))$ is finite then define

$$\|G(s)\|_\infty = \sup_{\Re e(s)>0} \overline{\sigma}(G(s)) = \sup_\omega \overline{\sigma}(G(j\omega))$$

With these defns. then Theorem 3.6 still holds, namely:

$$\|\underline{y}\|_2 \leq \|G(s)\|_\infty \cdot \|\underline{u}\|_2$$

Recap:

$$\|\underline{u}\|_2^2 = \int_{-\infty}^{\infty} \underline{u}(t)^* \underline{u}(t) dt.$$

$$\text{maximum system gain} = \sup_{\|\underline{u}\| \neq 0} \left\|\underline{y}\right\|_2 / \|\underline{u}\|_2$$

$$= \|G(s)\|_\infty \qquad \mathcal{H}_\infty - \text{norm}$$

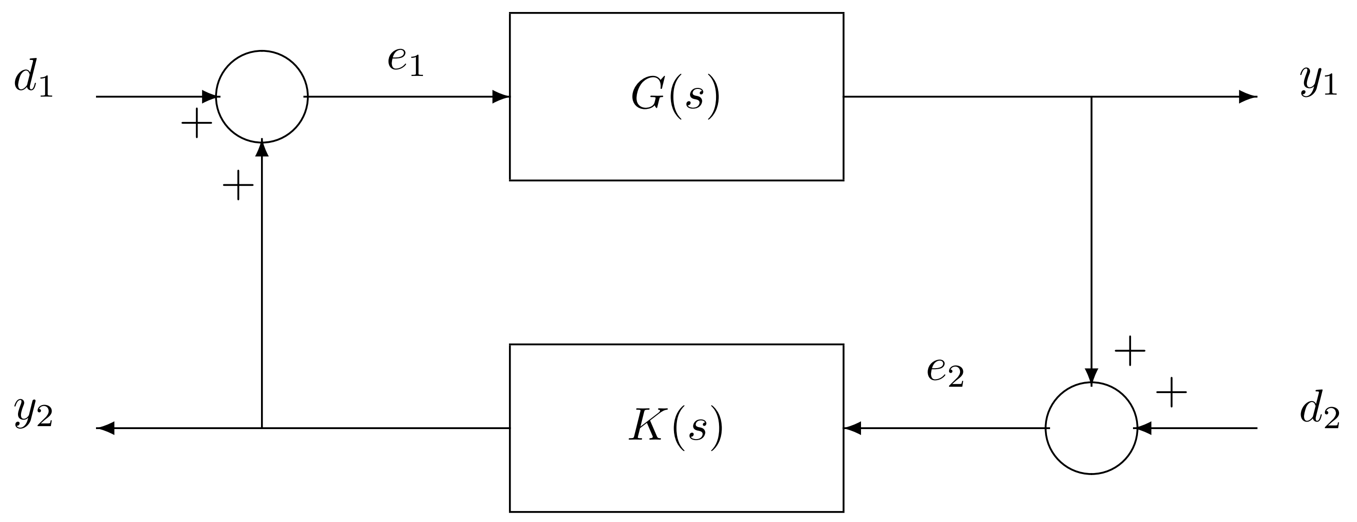
$$= \sup_\omega \overline{\sigma}\left(G(j\omega)\right)$$

We can write a number of frequency domain specifications as H_∞ norms of closed-loop transfer functions.

e.g. the requirement that $|G(j\omega)| < |\alpha(j\omega)|$ for all ω is equivalent to $\left\|\frac{G(j\omega)}{\alpha(j\omega)}\right\|_\infty < 1$. (assuming that all the zeros of $\alpha(s)$ are in the left half plane.

4 Robust stability

4.1 Internal Stability



In state space:

$$
\begin{aligned}
\dot{\underline{x}} &= A\underline{x} + B\underline{e}_1 \\
\underline{e}_2 &= C\underline{x} + D\underline{e}_1 + \underline{d}_2 \\
\dot{\hat{\underline{x}}} &= \hat{A}\hat{\underline{x}} + \hat{B}\underline{e}_2 \\
\underline{e}_1 &= \hat{C}\hat{\underline{x}} + \hat{D}\underline{e}_2 + \underline{d}_1
\end{aligned}
$$

In matrix form we have:

$$\begin{aligned}
\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} \\
\frac{d}{dt} \begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} &= A_{CL} \begin{bmatrix} \underline{x} \\ \underline{\hat{x}} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} \\
\text{where } A_{CL} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}
\end{aligned}$$

We will call this state-space feedback system *stable* if A_{CL} is a stable matrix.

Stability of the feedback system can also be considered using transfer functions when we call the feedback system is called *internally stable* if all transfer functions from d_1 and d_2 to e_1, e_2, y_1 and y_2 are in $\mathcal{H}_\infty$.

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix}$$

$$\begin{bmatrix} \underline{y}_2 \\ \underline{y}_1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} - \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} = \left\{ \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix}$$

Hence internally stable if and only if $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1}$ in H_∞. This is equivalent to A_{CL} being a stable matrix if the realizations of G and K are controllable and observable.

Note this includes all the closed loop transfer functions.

4.2 Singular Value Inequalities

For a general rectangular complex matrix, A in $\mathbb{C}^{m\times n}$, recall that A will have a *Singular Value Decomposition*, $A = U\Sigma V^*$ where $U^*U = UU^* = I$, $V^*V = VV^* = I$, and $\Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{bmatrix}$ with $\Sigma_{11} = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(1) Now denote,

$$\begin{aligned} \overline{\sigma}(A) &= \text{max singular value} = \sup_{\underline{x}\neq\underline{0}} \|A\underline{x}\|_2/\|\underline{x}\|_2 \\ (\text{ assuming } n=m) \quad \underline{\sigma}(A) &= \text{min singular value} = \min_{\underline{x}\neq\underline{0}} \|A\underline{x}\|_2/\|\underline{x}\|_2. \end{aligned}$$

Proof: Suppose $\underline{x}^*\underline{x} = 1$,

$$\begin{aligned} \|A\underline{x}\|_2^2 &= \|U\Sigma V^*\underline{x}\|_2^2 = \|\Sigma\underline{z}\|_2^2 \quad \text{where } \underline{z} = V^*\underline{x}, \quad \text{and } \|\underline{z}\|_2 = \|\underline{x}\|_2 \\ &= \sum_{i=1}^{r} \sigma_i^2\,|z_i|^2 \\ &= \sigma_1^2(1 - |z_2|^2 - |z_3|^2 \cdots - |z_r|^2) + \sigma_2^2|z_2|^2 + \cdots + \sigma_r^2|z_r|^2 \\ &= \sigma_1^2 - (\sigma_1^2 - \sigma_2^2)|z_2|^2 - \cdots - (\sigma_1^2 - \sigma_r^2)|z_r|^2 \\ &\leq \sigma_1^2 \end{aligned}$$

A similar argument gives the minimum gain. □

(2) $\overline{\sigma}(A) - \overline{\sigma}(B) \leq \overline{\sigma}(A+B) \leq \overline{\sigma}(A) + \overline{\sigma}(B)$.

Proof:

$$
\begin{aligned}
\|(A+B)\underline{x}\|_2 &= \|A\underline{x} + B\underline{x}\|_2 \\
&\leq \|A\underline{x}\|_2 + \|B\underline{x}\|_2 \quad \text{by the triangle inequality} \\
&\leq \overline{\sigma}(A)\,\|\underline{x}\|_2 + \overline{\sigma}(B)\,\|\underline{x}\|_2 = (\overline{\sigma}(A) + \overline{\sigma}(B))\,\|\underline{x}\|_2 \\
\text{hence } \overline{\sigma}(A+B) &\leq \overline{\sigma}(A) + \overline{\sigma}(B).
\end{aligned}
$$

The left hand inequality comes from
$\overline{\sigma}((A+B) + (-B)) \leq \overline{\sigma}(A+B) + \overline{\sigma}(-B) = \overline{\sigma}(A+B) + \overline{\sigma}(B)$.

(3) $\overline{\sigma}\left(A^{-1}\right) = 1/\underline{\sigma}(A)$

(4) $\underline{\sigma}(A) - \overline{\sigma}(B) \leq \underline{\sigma}(A+B) \leq \underline{\sigma}(A) + \overline{\sigma}(B)$.

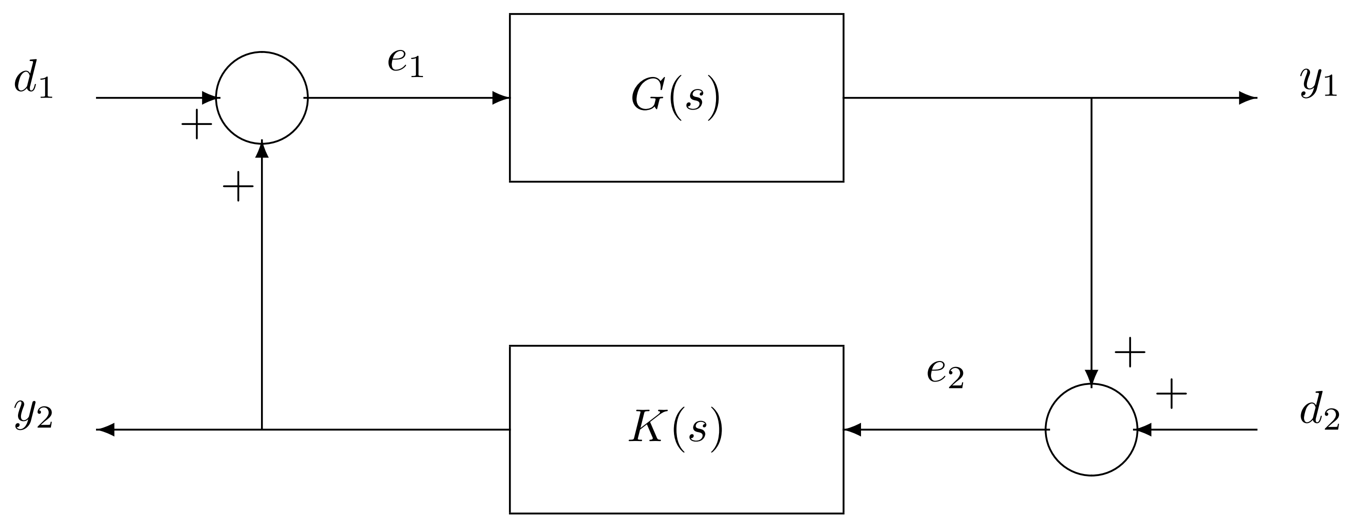
(5) $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$

e.g.

$$\overline{\sigma}\left((I - GK)^{-1}\right) = \frac{1}{\underline{\sigma}\,(I - GK)} \leq \begin{cases} \dfrac{1}{1 - \overline{\sigma}(GK)} & \text{if} \quad \overline{\sigma}(GK) < 1 \\ \dfrac{1}{\underline{\sigma}(GK) - 1} & \text{if} \quad \underline{\sigma}(GK) > 1. \end{cases}$$

Hence notions of high and low loop gain and bandwidth carry over to multivariable systems but with more 'slack' in results.

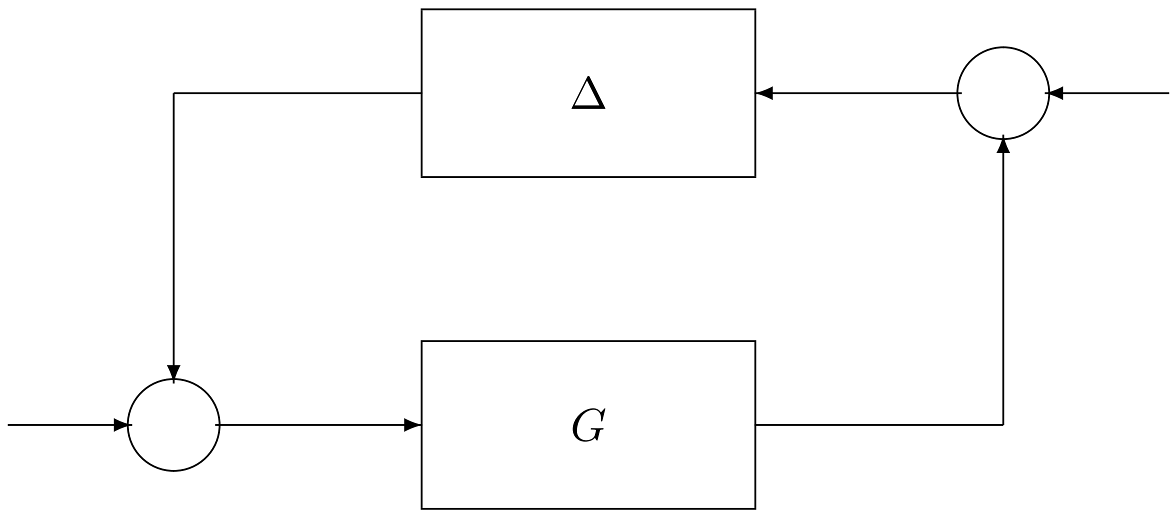
4.3 Small Gain Theorems



Theorem 4.1 *If G and K are both stable then the closed loop is stable if*

$$\|GK\|_\infty < 1 \text{ or if } \|KG\|_\infty < 1.$$

Now consider uncertain systems with G and Δ stable



Theorem 4.2 *Suppose Δ in H_∞ is unknown but $\|\Delta\|_\infty < \epsilon$ and G in $\mathcal{H}_\infty$ is known. Then the feedback system is closed loop stable for all such Δ* if and only if

$$\|G\|_\infty \leq 1/\epsilon$$

Proof: If $\|G\|_\infty \leq 1/\epsilon$ and $\|\Delta\|_\infty < \epsilon$ then,

$$\begin{aligned}
\|G\Delta\|_\infty &= \sup_\omega \overline{\sigma}(G\Delta) \\
&\leq \sup_\omega \overline{\sigma}(G).\overline{\sigma}(\Delta) \\
&\leq \sup_\omega \overline{\sigma}(G).\sup_\omega \overline{\sigma}(\Delta) \\
&= \|G\|_\infty . \|\Delta\|_\infty \\
&< \frac{1}{\epsilon}\epsilon = 1 \\
&\Rightarrow \text{ stable by small gain theorem.}
\end{aligned}$$

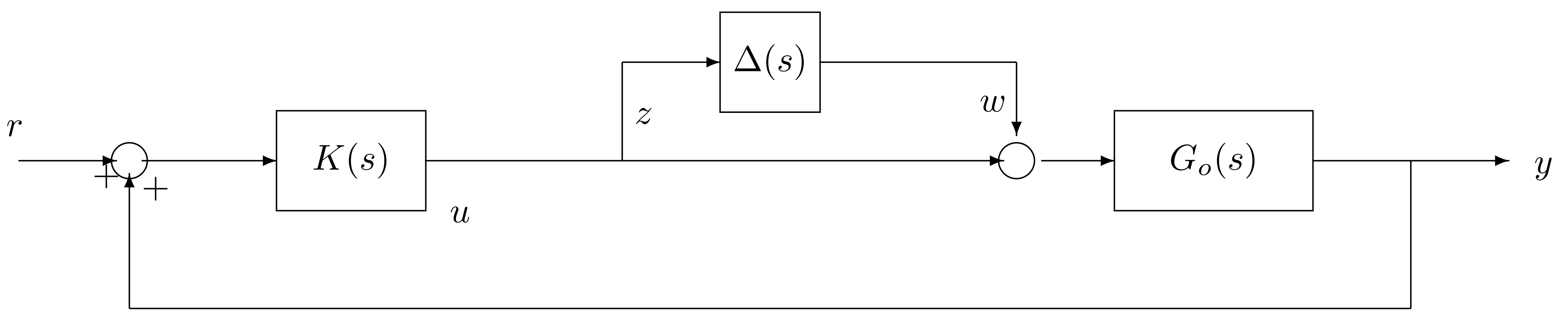
Now suppose that $\overline{\sigma}\left(G(j\omega_o)\right) > \frac{1}{\epsilon}$ for some ω_o, then we can construct (with some effort) a Δ s.t. $\|\Delta\|_\infty < \epsilon$ and $\det\left(I - G(j\omega_o)\Delta(j\omega_o)\right) = 0$

$\Rightarrow j\omega_o$ is a closed loop pole $\Rightarrow$ not stable. That is we have constructed a destabilizing perturbation in the set of Δ and hence for *robust stability* we need $\|G\|_\infty \leq 1/\epsilon$.

4.4 Robust Stability Tests

Multiplicative Uncertainty

Let an uncertain system have transfer function $G(s) = (I + \Delta(s))G_o(s)$, where $\|\Delta(s)\|_\infty < \epsilon$,



rewrite as $z = K(r + G_o(w + z)) \Rightarrow z = (I - KG_o)^{-1} K (r + G_o w)$
and with $r = 0$ we get

Closed loop internally stable for all $\|\Delta\|_\infty < \epsilon \Leftrightarrow (G_o, K)$ is internally stable and

$$\left\| (I - KG_o)^{-1} KG_o \right\|_\infty \leq \frac{1}{\epsilon}.$$

Additive Uncertainty

So $G = G_o + W_2 \Delta W_1$, with $\|\Delta(s)\|_\infty < \epsilon$, or

$$\left\| W_2^{-1} \left(G - G_o \right) W_1^{-1} \right\|_\infty < \epsilon.$$

In the SISO case this is the same as $|G - G_o| < |W_1 . W_2|$ for all ω

As in the case of multiplicative uncertainty we now obtain internal stability of the perturbed closed loop if (G_o, K) is internally stable and

$$\left\| W_1 \left(I - K G_o \right)^{-1} K W_2 \right\|_\infty < 1/\epsilon$$

5 Perturbations to Coprime Factors

5.1 Coprime Factorization of Transfer Functions

Given any $p \times m$ transfer function $G_o(s) = C(sI - A)^{-1}B$ (with A not necessarily stable, (A, C) observable and (A, B) controllable), we can write,

$$G_o = \tilde{M}^{-1}\tilde{N} = NM^{-1}$$

with $\tilde{M}, \tilde{N}, M, N$ all in H_∞. These factorizations are called respectively **left (and right) coprime factorizations of $G_o(s)$ over H_∞** if in addition

$$\begin{aligned} \text{rank}\begin{bmatrix}\tilde{N}(s) & \tilde{M}(s)\end{bmatrix} &= p \text{ for all } Re(s) \geq 0 \\ \text{rank}\begin{bmatrix}N(s) \\ M(s)\end{bmatrix} &= m \text{ for all } Re(s) \geq 0 \end{aligned}$$

i.e. there are no "common zeros" in $N(s)$ and $M(s)$ in the right half plane.

A *state-space procedure* for this is as follows:

The system equations with input, $\underline{u}$, state, $\underline{x}$ and output, $\underline{y}$ will be,

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad \underline{y} = C\underline{x}$$

and these can be rewritten as,

$$\dot{\underline{x}} = (A + LC)\underline{x} + B\underline{u} - Ly, \quad \underline{y} = C\underline{x}$$

where L is chosen so that $(A + LC)$ is stable (c.f. observer design). Hence,

$$\begin{aligned} \underline{y} &= C(sI - A - LC)^{-1} \begin{bmatrix} B & -L \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{y} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{N}(s) & I - \tilde{M}(s) \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{y} \end{bmatrix} \\ \Rightarrow \tilde{M}(s)\underline{y}(s) &= \tilde{N}(s)\underline{u}(s) \end{aligned}$$

The other factorization is derived by finding a F such that $(A+BF)$ is stable (c.f. state feedback pole placement) and writing the state equation as,

$$\underline{\dot{x}} = (A+BF)\underline{x} + B\underline{e}, \quad \underline{e} = \underline{u} + \underline{z}, \quad \underline{z} = -F\underline{x}, \quad \underline{y} = C\underline{x}$$

and hence

$$\begin{aligned} \begin{bmatrix} \underline{y} \\ \underline{z} \end{bmatrix} &= \begin{bmatrix} C \\ -F \end{bmatrix} (sI - A - BF)^{-1} B\underline{e} = \begin{bmatrix} N(s) \\ I - M(s) \end{bmatrix} \underline{e} \\ \Rightarrow \underline{e} &= \underline{u} + (I - M(s))\underline{e} \quad \Rightarrow \underline{e} = M(s)^{-1}\underline{u}, \quad \underline{y} = N(s)\underline{e} = N(s)M(s)^{-1}\underline{u} \end{aligned}$$

It is also possible to demonstrate that these two factorizations are coprime.

Normalized Coprime Factorizations

A left coprime factorization will be called a **normalized left coprime factorization** of $G_o(s)$ if

$$\tilde{M}(j\omega)\tilde{M}(j\omega)^* + \tilde{N}(j\omega)\tilde{N}(j\omega)^* = I \text{ for all } \omega$$

Note that given any coprime factorization of $G_o = \tilde{M}^{-1}\tilde{N}$ then

$$\begin{aligned} G_o &= (R\tilde{M})^{-1}(R\tilde{N}) \\ \text{and} \quad (R\tilde{M})(\tilde{M}^*R^*) + (R\tilde{N})(\tilde{N}^*R^*) &= R(\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^*)R^* \end{aligned}$$

so normalisation is possible by choice of R (need the poles and zeros of R to be in the LHP).

e.g.

$$\begin{aligned} G_o(s) = \frac{1}{s} &= \left(\frac{1}{s+1}\right) \Big/ \left(\frac{s}{s+1}\right) = N/M \\ MM^* + NN^* &= \frac{j\omega}{j\omega+1} \cdot \frac{(-j\omega)}{(-j\omega+1)} + \frac{1}{j\omega+1} \cdot \frac{1}{-j\omega+1} \\ &= \frac{\omega^2}{1+\omega^2} + \frac{1}{1+\omega^2} = 1 \end{aligned}$$

e.g.

$$\begin{aligned}
\frac{s-1}{s+1} &= \frac{1}{\sqrt{2}}\left(\frac{s-1}{s+1}\right)\bigg/\frac{1}{\sqrt{2}}\cdot 1.\\
\frac{s+1}{s-1} &= \frac{1}{\sqrt{2}}\cdot 1\bigg/\frac{1}{\sqrt{2}}\cdot\left(\frac{s-1}{s+1}\right)
\end{aligned}$$

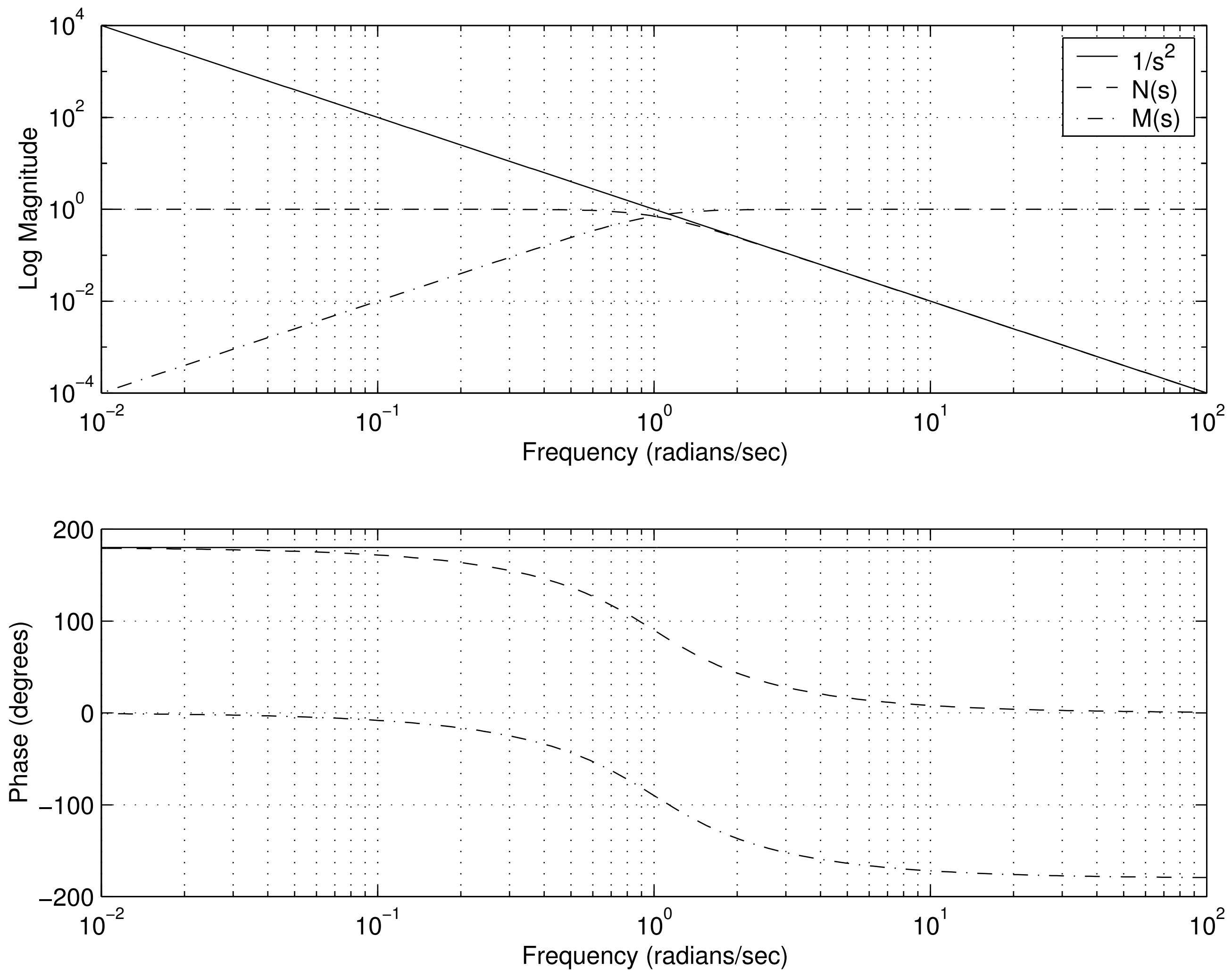


Figure 2: Bode Diagrams for $1/s^2$ and its normalised coprime factors

5.2 Uncertainty in Coprime Factorisations

Suppose

$$G_\Delta = \left(\tilde{M} + \Delta_M\right)^{-1} \left(\tilde{N} + \Delta_N\right)$$

with

$$\|[\Delta_M, \Delta_N]\|_\infty < \epsilon, \quad \Delta_M, \Delta_N \text{ in } H_\infty.$$

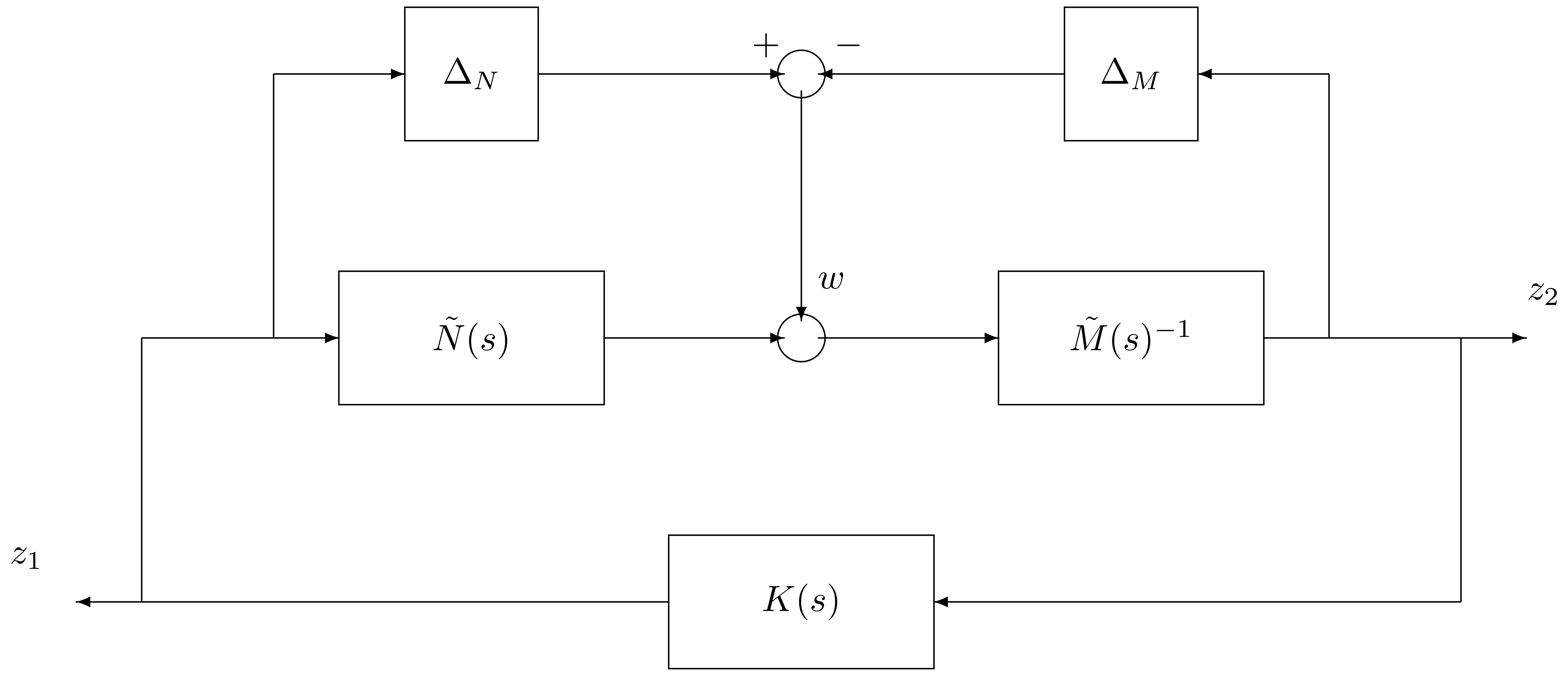
e.g.

$$\begin{aligned} G_o &= 1/s = \frac{1}{s+1} \Big/ \frac{s}{s+1} \\ G_\Delta &= \frac{\frac{1}{s+1} + \Delta_N}{\frac{s}{s+1} + \Delta_M} = \frac{1 + \Delta_N(s+1)}{s + \Delta_M(s+1)} \text{ with } |\Delta_N|^2 + |\Delta_M|^2 < \epsilon^2 \end{aligned}$$

If Δ_M real constant then pole is moved to $-\frac{\Delta_M}{1+\Delta_M}$

Hence poles move across $s = j\omega$ with small $|\Delta_M|$ but very large $|G_\Delta - G_o|$ changes.

Including a controller, $K(s)$, the block diagram now becomes:



And we obtain:

$$
\begin{aligned}
z_2 &= \tilde{M}^{-1}\left(-\Delta_M z_2 + \left(\Delta_N + \tilde{N}\right) z_1\right) \\
\Rightarrow \left(\tilde{M} + \Delta_M\right) z_2 &= \left(\tilde{N} + \Delta_N\right) z_1 \\
\Rightarrow z_2 &= \underbrace{\left(\tilde{M} + \Delta_M\right)^{-1}\left(\tilde{N} + \Delta_N\right)}_{G_\Delta} z_1 \quad \text{as desired.}
\end{aligned}
$$

Also considering the controlled system gives,

$$\begin{aligned}
z_2 &= \tilde{M}^{-1}\left\{w + \tilde{N}Kz_2\right\} \\
\left(I - \tilde{M}^{-1}\tilde{N}K\right) z_2 &= \tilde{M}^{-1}w \\
z_2 &= (I - GK)^{-1}\,\tilde{M}^{-1}w \\
z_1 &= Kz_2 \\
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\,\tilde{M}^{-1}w \\
w &= [\Delta_N, -\Delta_M] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\end{aligned}$$

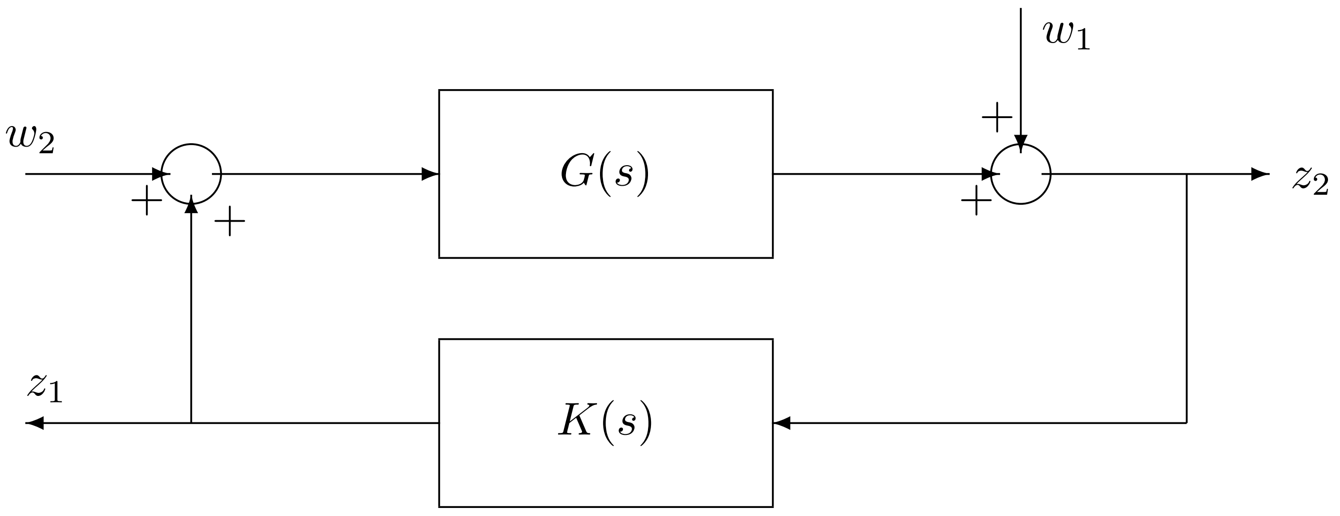
Theorem 5.1 *The above closed loop is internally stable for all* $\|[\Delta_N, \Delta_M]\|_\infty < \epsilon$

$$\Leftrightarrow \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\,\tilde{M}^{-1} \right\|_\infty \leq 1/\epsilon, \quad \textit{(by the Small Gain Theorem).}$$

Note that since $\left[\tilde{M}\ \tilde{N}\right]\left[\begin{array}{c}\tilde{M}^*\\ \tilde{N}^*\end{array}\right]=I$, we have $\lambda_i\left(XX^*\right)=\lambda_i\left(X\left[\tilde{M}\ \tilde{N}\right]\left[\begin{array}{c}\tilde{M}^*\\ \tilde{N}^*\end{array}\right]X^*\right)$, and hence $\overline{\sigma}(X)=\overline{\sigma}\left(X\left[\tilde{M}\ \tilde{N}\right]\right)$.

Hence the closed loop will be internally stable for all $\|[\Delta_N,\Delta_M]\|_\infty<\epsilon$ if and only if

$$\begin{aligned}\left\|\left[\begin{array}{c}K\\ I\end{array}\right](I-GK)^{-1}\tilde{M}^{-1}\left[\tilde{M},\ \tilde{N}\right]\right\|_\infty &\leq 1/\epsilon\\ \Leftrightarrow\left\|\left[\begin{array}{c}K\\ I\end{array}\right](I-GK)^{-1}[I,\ G]\right\|_\infty &\leq 1/\epsilon\end{aligned}$$



i.e.

$$\left\| T_{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \to \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}} \right\|_\infty \leq 1/\epsilon.$$

This closed-loop therefore includes all the standard transfer functions for stability and *performance*.

We will now define the "stability margin" for coprime factor perturbations to be:

$$b\left(G,K\right) \stackrel{def}{=} \left\| \begin{bmatrix} K \\ I \end{bmatrix} \left(I - GK\right)^{-1} \left[I \; G\right] \right\|_\infty^{-1}$$

It can be thought of as a generalisation of gain and phase margins.

We have that the closed loop will be stable for all $\|[\Delta_N, \Delta_M]\|_\infty < \epsilon \quad \Leftrightarrow \quad b\left(G,K\right) \geq \epsilon$.

Experience indicates that $b(G,K) > 0.2 - 0.3$ is satisfactory for good robustness.

It can in fact also be shown that for single-input/single-output systems:

Theorem 5.2

$$\begin{aligned} \mathit{GAIN\ MARGIN} &\geq \frac{1+b(G,K)}{1-b(G,K)} \\ \mathit{PHASE\ MARGIN} &\geq 2\arcsin(b(G,K)) \end{aligned}$$

Proof: The proof of the gain margin result is as follows:

Let $\beta = b(G,K)$ and note that when G and K are both scalar, Lemma 5.3(b) gives that

$$\overline{\sigma}^2 \left\{ \begin{bmatrix} K \\ 1 \end{bmatrix} (1-GK)^{-1} \begin{bmatrix} 1 & G \end{bmatrix} \right\} = (1+|K|^2)|1-GK|^{-2}(1+|G|^2)$$

Hence $b(G,K) = \beta$ implies that

$$(1+|K|^2)|1-GK|^{-2}(1+|G|^2) \leq \beta^{-2} \quad \text{for all } \omega$$

Now to calculate gain margin we need to consider the case when the loop gain $GK = \alpha$ and α is positive and real (positive feedback convention). Hence

$$
\begin{aligned}
\beta^2(1+|K|^2)(1+\frac{\alpha^2}{|K|^2}) &\leq |1-\alpha|^2 \\
\Rightarrow \beta^2\left((1+\alpha)^2 + \left(|K| - \frac{\alpha}{|K|}\right)^2\right) &\leq (1-\alpha)^2 \\
\Rightarrow \beta^2(1+\alpha)^2 &\leq (1-\alpha)^2 \\
\text{for } 0 \leq \alpha \leq 1 \Rightarrow \alpha \leq \frac{1-\beta}{1+\beta} &
\end{aligned}
$$

□

The following linear algebra result was needed above.

Lemma 5.3 *(a) For any $n \times m$ matrix A and $m \times n$ matrix B, the non-zero eigen values of AB equal those of BA.*

(b) $\sigma_i^2(XYZ) = \lambda_i(XYZZ^*Y^*X^*) = \lambda_i(YZZ^*Y^*X^*X)$.

Proof: (a) The general idea is that if $\lambda \neq 0$ is such that $AB\underline{x} = \lambda\underline{x}$, then $BA \underbrace{B\underline{x}}_{=\underline{y}\neq\underline{0}} = \lambda \underbrace{B\underline{x}}_{\underline{y}}$. If eigen values are repeated this argument is not quite complete, and this case is handled by the following identity:

$$\begin{aligned} \begin{bmatrix} \lambda I_n & \lambda A \\ B & \lambda I_m \end{bmatrix} &= \begin{bmatrix} \lambda I_n & 0 \\ B & I_m \end{bmatrix} \begin{bmatrix} I_n & A \\ 0 & \lambda I_m - BA \end{bmatrix} \\ &= \begin{bmatrix} I_n & A \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I_n - AB & 0 \\ B & \lambda I_m \end{bmatrix} \\ \Rightarrow \lambda^n \det(\lambda I_m - BA) &= \lambda^m \det(\lambda I_n - AB) \end{aligned}$$

(b) is immediate from (a). $\square$

5.3 Gap Metric

Coprime Factor perturbations are not unique. The smallest value of $\|[\Delta_N(j\omega),\ \Delta_M(j\omega)]\|_\infty$ that perturbs G_o into G_1 is called the *gap* between G_o and G_1 and is denoted $\delta_g\,(G_o, G_1)$.

Hence if $\delta_g\,(G_o, G_1) < b(G_0, K)$ then the closed loop system with G_1 and K will also be stable.

The $\nu - gap$ (δ_ν) between G_0 and G_1 is an important development of the gap whose details are beyond our present scope. However we note that both δ_g and δ_ν are metrics (i.e. distance measures) and hence satisfy e.g.

(1) $0 \leq \delta_\nu\,(G_0, G_1) \leq 1$

(2) $\delta_\nu\,(G_0, G_1) = 0 \Rightarrow G_0 = G_1$

(3) $\delta_\nu\,(G_0, G_1) = \delta_\nu\,(G_1, G_0)$

(4) $\delta_\nu\,(G_0, G_2) \leq \delta_\nu\,(G_0, G_1) + \delta_\nu\,(G_1, G_2)$ (Triangle inequality).

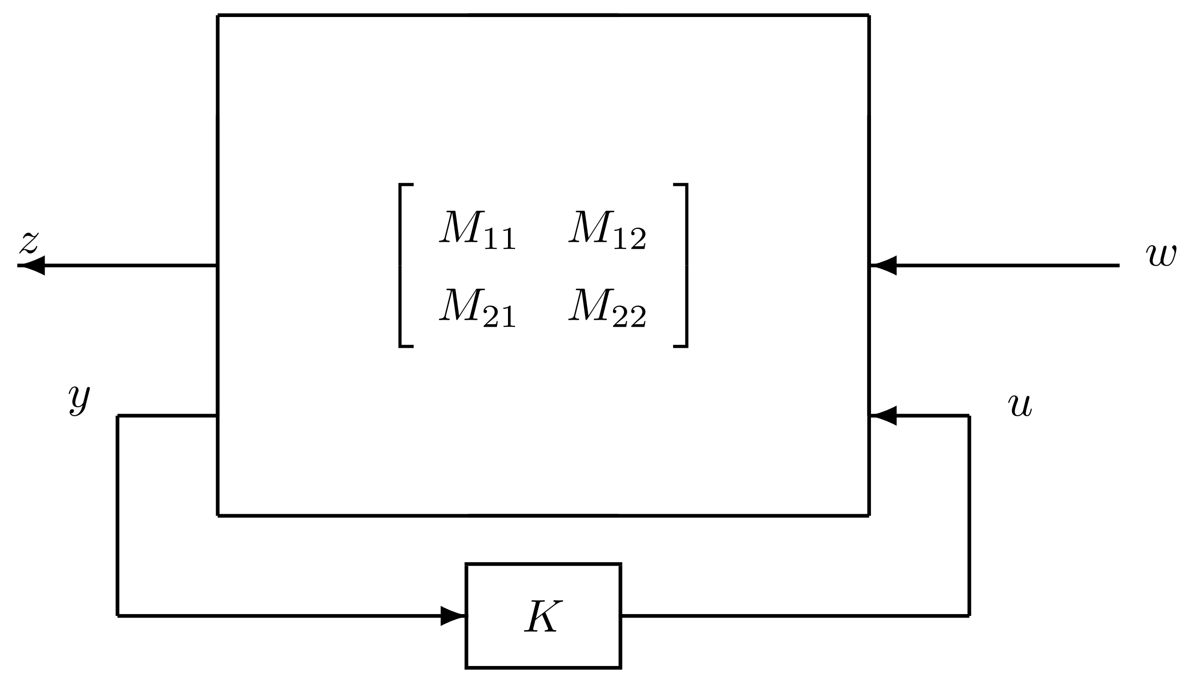
In addition, it can be shown that if $\delta_\nu\,(G_o, G_1) < b\,(G_0, K)$ then we have closed-loop stability of G_1 and K.

Thus: $b\,(G_0, K)$ gives the radius (in terms of the distance in the ν-gap metric) of the largest "ball" of plants stabilised by K.

6 H_∞ Control Synthesis

6.1 The Youla Parameterization of All Stabilizing Controllers

Now consider the problem of synthesizing a controller, $K(s)$, that minimises the $\mathcal{H}_\infty$-norm of the closed-loop system:



$$\begin{aligned} z &= Hw \\ H &= M_{11} + M_{12}K\left(I - M_{22}K\right)^{-1} M_{21} \end{aligned}$$

The $\mathcal{H}_\infty$ control synthesis problem is then to find K that internally stabilises this feedback system and minimises $\|H\|_\infty$.

We can for example compute

$$b_{opt}(G) = \max_K b(G, K)$$

Consider the coprime factorisations $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ of the plant.

It is possible to solve the Double Bezout Equation:

$$\begin{bmatrix} \tilde{V}_o & -\tilde{U}_o \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_o \\ N & V_o \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The Youla parameterisation of *all* stabilising controllers is then given by:

$$K = (U_o + MQ)(V_o + NQ)^{-1} = (\tilde{V}_o + Q\tilde{N})^{-1}(\tilde{U}_o + Q\tilde{M}), \text{ for } Q \text{ in } \mathcal{H}_\infty$$

Now consider the closed loop transfer function for $b(G,K)$:

$$\begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\,[I,\ G]$$

$$= \begin{bmatrix} (U_o+MQ) \\ (V_o+NQ) \end{bmatrix} (V_o+NQ)^{-1}\left(I-\tilde{M}^{-1}\tilde{N}(U_o+MQ)(V_o+NQ)^{-1}\right)^{-1}\tilde{M}^{-1}\left[\tilde{M},\ \tilde{N}\right]$$

$$= \begin{bmatrix} (U_o+MQ) \\ (V_o+NQ) \end{bmatrix} \underbrace{\left(\tilde{M}(V_o+NQ)-\tilde{N}(U_o+MQ)\right)^{-1}}_{=I}\left[\tilde{M},\ \tilde{N}\right]$$

$$= \begin{bmatrix} U_o \\ V_o \end{bmatrix}\left[\tilde{M},\ \tilde{N}\right] + \begin{bmatrix} M \\ N \end{bmatrix} Q\left[\tilde{M},\ \tilde{N}\right]$$

Hence $\min_K b^{-1}(G,K) = \min_Q \|$ a linear function of $Q\ \|_\infty$ which is a CONVEX PROBLEM!

For a more general problem all stable closed-loop transfer funtions can be written as:

$$T_{11}+T_{12}QT_{21} = \mathcal{F}_\ell(T,Q) \quad \text{for } Q \text{ in } \mathcal{H}_\infty$$

The first solutions to the $\mathcal{H}_\infty$ control problem used this as the first step with solutions from interpolation theory, and state-space representations of these transfer functions.

6.2 State-space solution to the $\mathcal{H}_\infty$ control problem

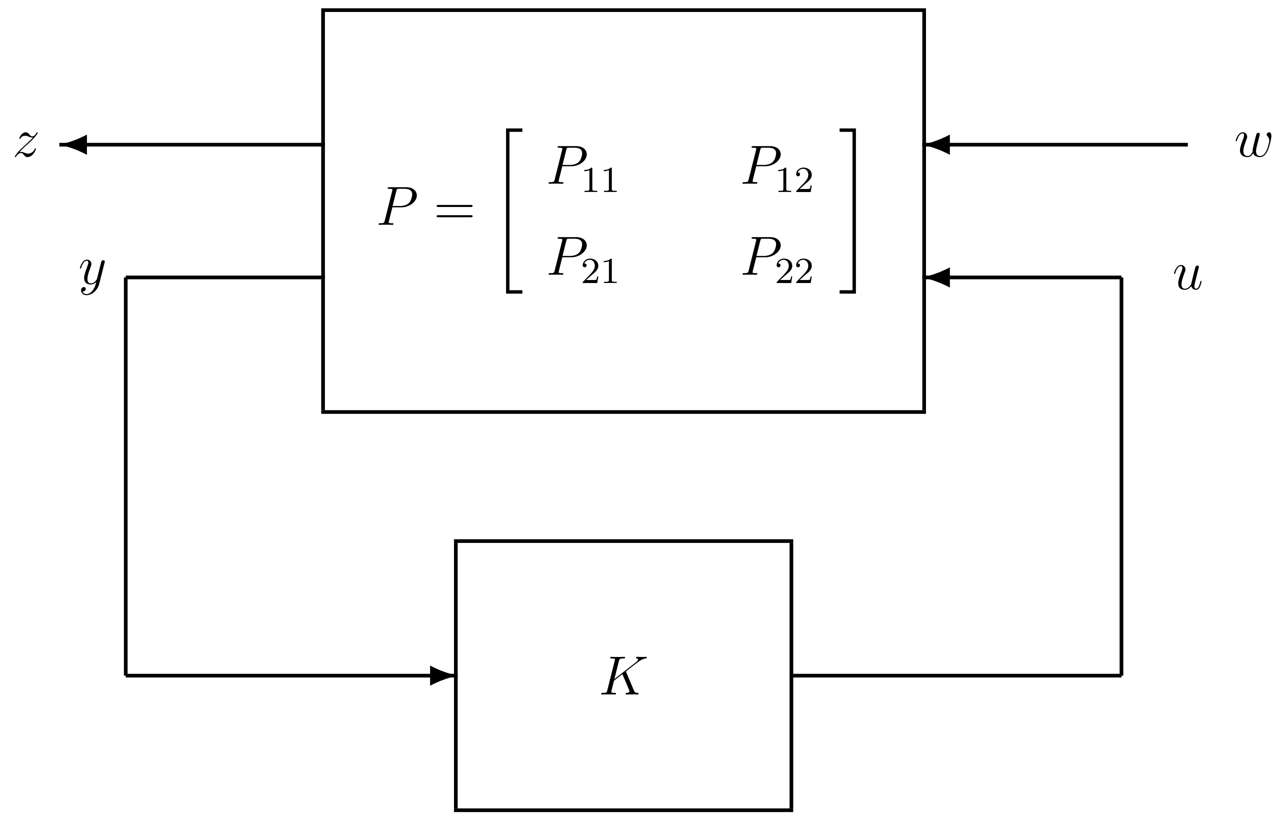


Figure 3: (lower) Linear Fractional Transformation - Feedback System

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) && (6.1) \\ z(t) &= C_1 x(t) + D_{12} u(t) && (6.2) \\ y(t) &= C_2 x(t) + D_{21} w(t) && (6.3) \end{aligned}$$

i.e. in Fig. 3

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$$

where we also assume, with little loss of generality, that $D_{12}^* D_{12} = I$, $D_{21} D_{21}^* = I$, $D_{12}^* C_1 = 0$ and $B_1 D_{21}^* = 0$. Since we wish to have $\|T_{z \leftarrow w}\|_\infty < \gamma$ we need to find u such that

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \quad \text{for all} \quad w \neq 0 \text{ in } \mathcal{L}_2(0, \infty).$$

Suppose that there exists a solution, X_∞, to the Algebraic Riccati Equation (ARE),

$$A^* X_\infty + X_\infty A + C_1^* C_1 + X_\infty (\gamma^{-2} B_1 B_1^* - B_2 B_2^*) X_\infty = 0 \qquad (6.4)$$

with $X_\infty \geq 0$ and $A + (\gamma^{-2} B_1 B_1^* - B_2 B_2^*) X_\infty$ a stable 'A-matrix'. A simple substitution then gives that

$$\frac{d}{dt}(x(t)^* X_\infty x(t)) = -z^* z + \gamma^2 w^* w + v^* v - \gamma^2 r^* r$$

where,

$$v := u + B_2^* X_\infty x, \quad r := w - \gamma^{-2} B_1^* X_\infty x.$$

Now let $x(0) = 0$ and assuming stability so that $x(\infty) = 0$, then integrating from 0 to ∞ gives,

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \|v\|_2^2 - \gamma^2 \|r\|_2^2 \tag{6.5}$$

If the state is available to u then the control law $u = -B_2^* X_\infty x$ gives $v = 0$ and $\|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$ for all $w \neq 0$. It can be shown that (6.4) has a solution if there exists a controller such that $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$. In addition since transposing a system does not change its $\mathcal{H}_\infty$-norm the following dual ARE will also have a solution, $Y_\infty \geq 0$,

$$AY_\infty + Y_\infty A^* + B_1 B_1^* + Y_\infty(\gamma^{-2} C_1^* C_1 - C_2^* C_2) Y_\infty = 0 \tag{6.6}$$

To obtain a solution to the output feedback case note that (6.5) implies that $\|z\|_2^2 < \gamma^2 \|w\|_2^2$ if and only if $\|v\|_2^2 < \gamma^2 \|r\|_2^2$ and $\bar{v} = \mathcal{F}_l(P_{\text{tmp}}, K)\bar{r}$ where,

$$\begin{bmatrix} \bar{v} \\ \bar{y} \end{bmatrix} = P_{\text{tmp}} \begin{bmatrix} \bar{r} \\ \bar{u} \end{bmatrix}, \quad \text{where} \quad P_{\text{tmp}} = \left[\begin{array}{c|cc} A + \gamma^{-2} B_1 B_1^* X_\infty & B_1 & B_2 \\ \hline B_2^* X_\infty & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

The special structure of this problem enables a solution to be derived in much the same way as the dual of the state feedback problem. The corresponding ARE will have a solution $Y_{\text{tmp}} = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty \geq 0$ if and only if the spectral radius, $\rho(Y_\infty X_\infty) < \gamma^2$.

The above outline, supported by significant technical detail and assumptions, will therefore demonstrate that there exists a stabilising controller, $K(s)$, such that the system described by (6.1-6.3) satisfies $\|T_{z\leftarrow w}\|_\infty < \gamma$ if and only if there exist stabilising solutions to the ARE's in (6.4) and (6.6) such that,

$$X_\infty \geq 0, \quad Y_\infty \geq 0, \quad \rho(Y_\infty X_\infty) < \gamma^2 \tag{6.7}$$

The state equations for the resulting controller can be writen as,

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_1\hat{w}_{\text{worst}} + B_2u + Z_\infty L_\infty(C_2\hat{x} - y) \\
u &= F_\infty\hat{x}, \quad \hat{w}_{\text{worst}} = \gamma^{-2}B_1^*X_\infty\hat{x} \\
F_\infty &:= -B_2^*X_\infty, \quad L_\infty := -Y_\infty C_2^*, \\
Z_\infty &:= (I - \gamma^{-2}Y_\infty X_\infty)^{-1}
\end{aligned}$$

giving feedback from a state estimator in the presence of an estimate of the worst-case disturbance.

7 H_∞ loop shaping design procedure

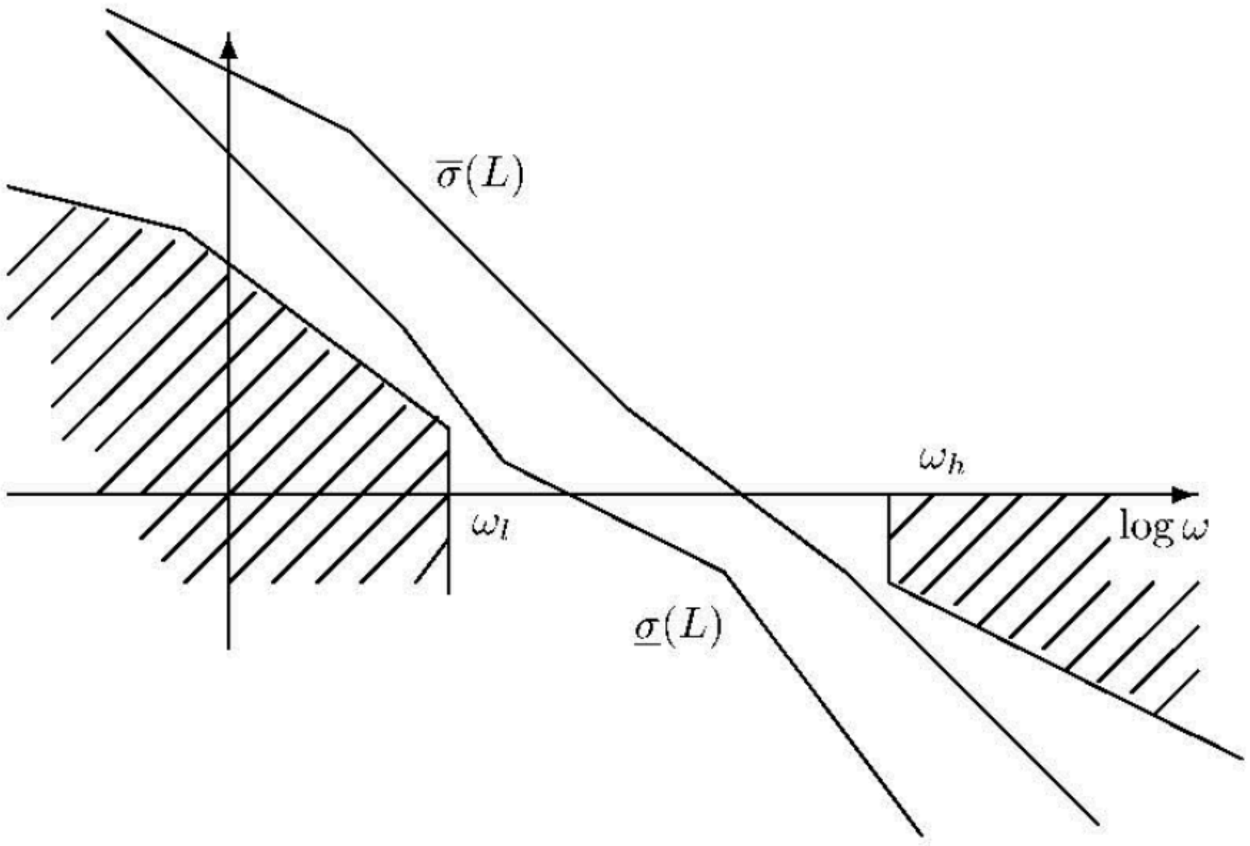


Figure 4: Desirable Loop Shapes

This method of control system design chooses a pre-compensator, $W(s)$, and then uses a controller that maximises $b\,(G_0W, K)$ over all stabilizing K.

Steps

(1) Scale inputs and outputs so that a unit change on each input are similarly important, also for outputs.

(2) Plot singular values of $G_0(j\omega)$ (after scaling).

(3) Insert a pre-compensator $W(j\omega)$ (with poles and zeros in lhp) to shape the singular values as desired. (e.g. proportional plus integral action diagonal precompensator).

(4) Design a K to maximise $b\,(G_0W, K)$ (say K_∞). If $b\,(G_0W, K_\infty)$ is $\lesssim 0.2$ change W and return to (3).

(5) Implement controller WK_∞.

It can be shown that, as long as $b\,(G_0W, K_\infty)$ is large (ie $\gtrsim 0.3$) then $\sigma_i(G_0W) \approx \sigma_i(G_0WK)$. In this case, K doesn't change the desired "loop shape" too much. However, it *does* modify the phase of the individual frequency responses in order to get good multivariable stability margins.

7.1 Example of the ν-gap metric and loop shaping

.

Calculate the ν-gap between two transfer functions:

$$G(s) = 1/(s^2 + 1) \quad \text{and} \quad G_2(s) = (-0.5s + 1)/(s^2 + 1.5)$$

then the gap can be calculated as:

$$\delta_\nu(G, G_2) = 0.4632$$

Now the maximum stability margin to coprime factor perturbations is given by:

$$b_{opt}(G) = 0.5556$$

which is more than the gap so both systems will be stabilised with $K_\infty(s)$ achieving this margin. Look at the resulting closed-loop poles for (G, K_∞) are:

```
-0.4551 + 1.0987i
-0.4551 - 1.0987i
-1.1892
```

and for (G_2, K_∞) are

```
-0.1934 + 1.6718i
-0.1934 - 1.6718i
-0.9644
```

Loop shaping:

Now let's consider the robust stabilization in the gap metric of the systems:

$$G(s) = f/(s^2 + 1)$$

for $f = 0.1, 1, 10, 100$.

f	0.1	1	10	100
b_{opt}	0.6893	0.5556	0.4056	0.3850
closed-loop poles	-0.0499 ± 1.0012i -1.0025	-0.4551 ± 1.0987i -1.1892	-2.1272 ± 2.3505i -3.1702	-7.0358 ± 7.1065i -10.0002

Figure 5: Loop shaping for $f/(s^2 + 1)$

Analysis of the Bode diagrams shows that the stability margins are always satisfactory. The loop gains are given in Fig. 6.

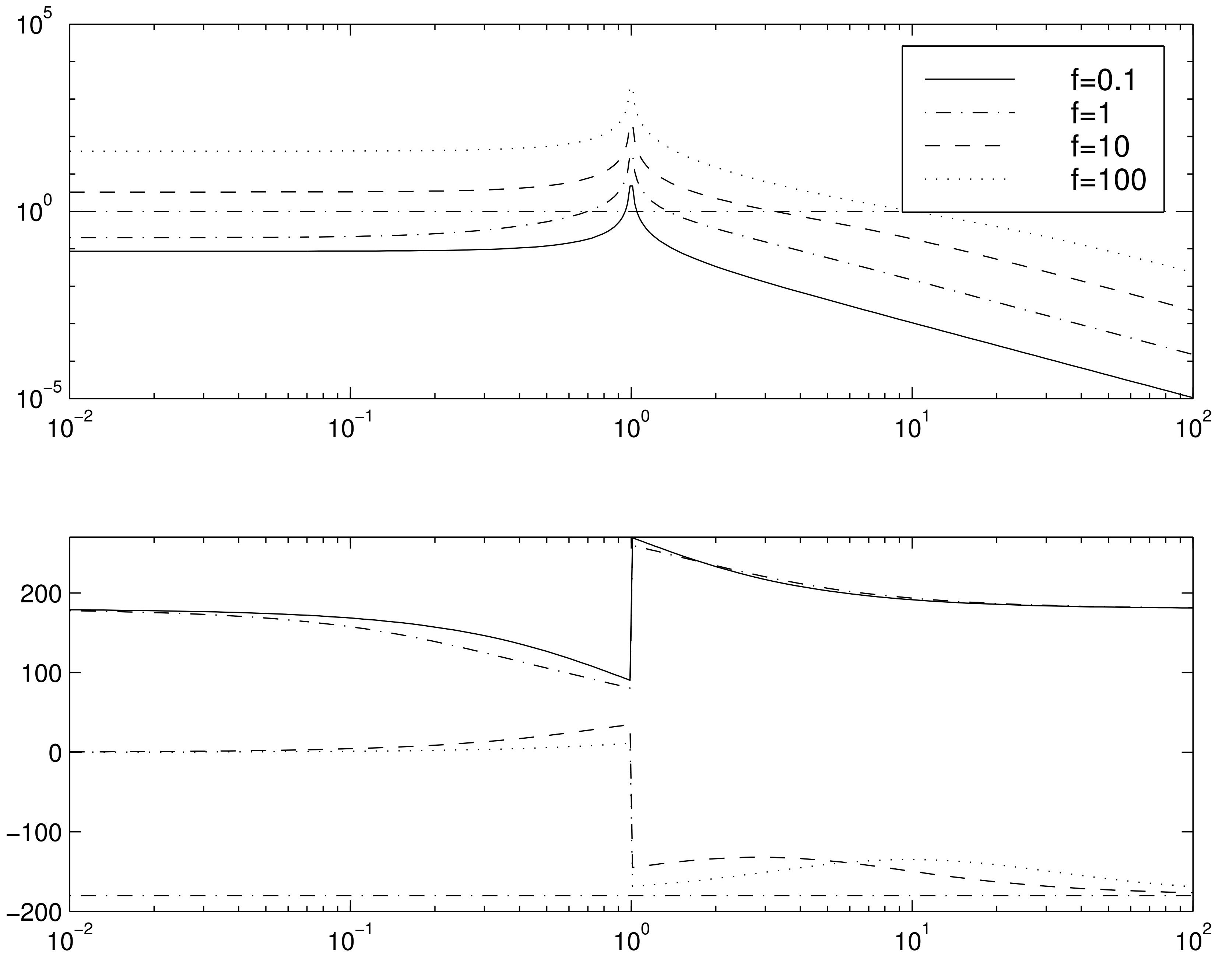


Figure 6: Bode Diagrams for Loop Gains

Finally let's look at

$$G_3(s) = 10(-s+1)/(s+1)(s^2+1)$$

when $b_{opt}(G_3) = 0.0975$

Here the maximum stability margin is less than 0.1 which is unsatisfactory and the desired loop shape will have to be changed (e.g. by reducing the gain and hence the desired closed loop bandwidth).

7.2 Robust Performance in the ν-Gap Metric

The ν-Gap Metric between two systems was briefly mentioned in section 5.3 where it was asserted that if there exist Δ_N, Δ_M in H_∞ satisfying $\|[\Delta_N,\ \Delta_M]\|_\infty < \beta$ and $G_1 = \left(\tilde{M} + \Delta_M\right)^{-1}\left(\tilde{N} + \Delta_N\right)$ then it will necessarily be the case that $\delta_\nu(G_0, G_1) < \beta$. Furthermore, if K stabilizes G_0 with $b(G_0, K) \geq \beta$ then K will also stabilize G_1.

So, $b(G, K)$ gives both a measure of the stability margins as well as the (nominal) performance to input and output disturbances. A bound on the robust performance can also be stated in this framework when both the plant and controller are perturbed:

$$\arcsin(b(G_1, K_1)) \geq \arcsin(b(G_o, K_o)) - \arcsin(\delta_\nu(G_1, G_o)) - \arcsin(\delta_\nu(K_1, K_o))$$

(The derivation of this is due to Vinnicombe and is non-trivial and omitted.)

Note that (since $\sin(A - B - C) \geq \sin(A) - \sin(B + C) \geq \sin(A) - \sin(B) - \sin(C)$ and by taking the sine of the above inequality) this inequality is a slightly stronger inequality than

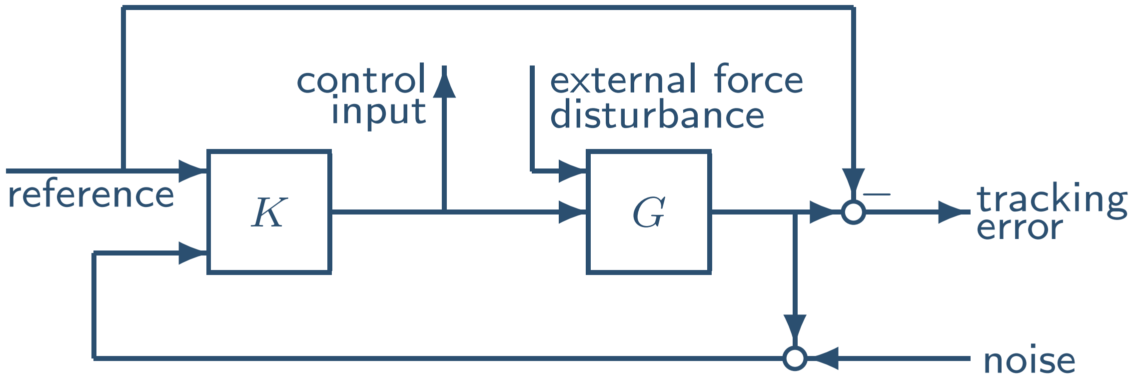
$$\underbrace{b(G_1, K_1)}_{\text{perturbed performance}} \geq \underbrace{b(G_o, K_o)}_{\text{nominal performance}} - \underbrace{\delta_\nu(G_1, G_o)}_{\text{plant perturbation}} - \underbrace{\delta_\nu(K_1, K_o)}_{\text{controller perturbation}}$$

which is also true and shows clearly how the performance can be degraded by perturbations to the plant and controller.

Generalized Disturbance Rejection

Generalized Disturbance Rejection

Consider a problem with many exogenous inputs/errors:



Objective:

> "Design K to keep tracking errors *and* control input signal *small* for all reasonable reference commands, sensor noises, *and* external force disturbances"

Assess 'performance" by measuring the "gain" from **outside influences** to **regulated variables**

$$\underbrace{\begin{bmatrix} \text{tracking error} \\ \text{control input} \end{bmatrix}}_{regulated\ variables} = T \underbrace{\begin{bmatrix} \text{reference} \\ \text{external force} \\ \text{noise} \end{bmatrix}}_{outside\ influences}$$

Definition: Good Performance $\Leftrightarrow$ T is "small"

Generalized Disturbance Rejection (cont'd)

Since the closed-loop system T is a MIMO dynamical system, two aspects to the gain:

- Spatial (*vector* disturbances and *vector* errors)
- Temporal (dynamical relationship between input/output signals)

In any performance criterion, we must account for the *relative*

- magnitude of outside influences;
- importance of the magnitudes of regulated variables.

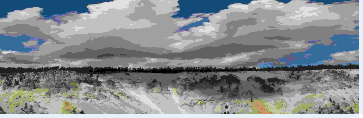
Recall from the SISO sensitivity discussion

- Closed-loop maps can't necessarily be small at all frequencies.
- Tradeoffs between the different objectives.

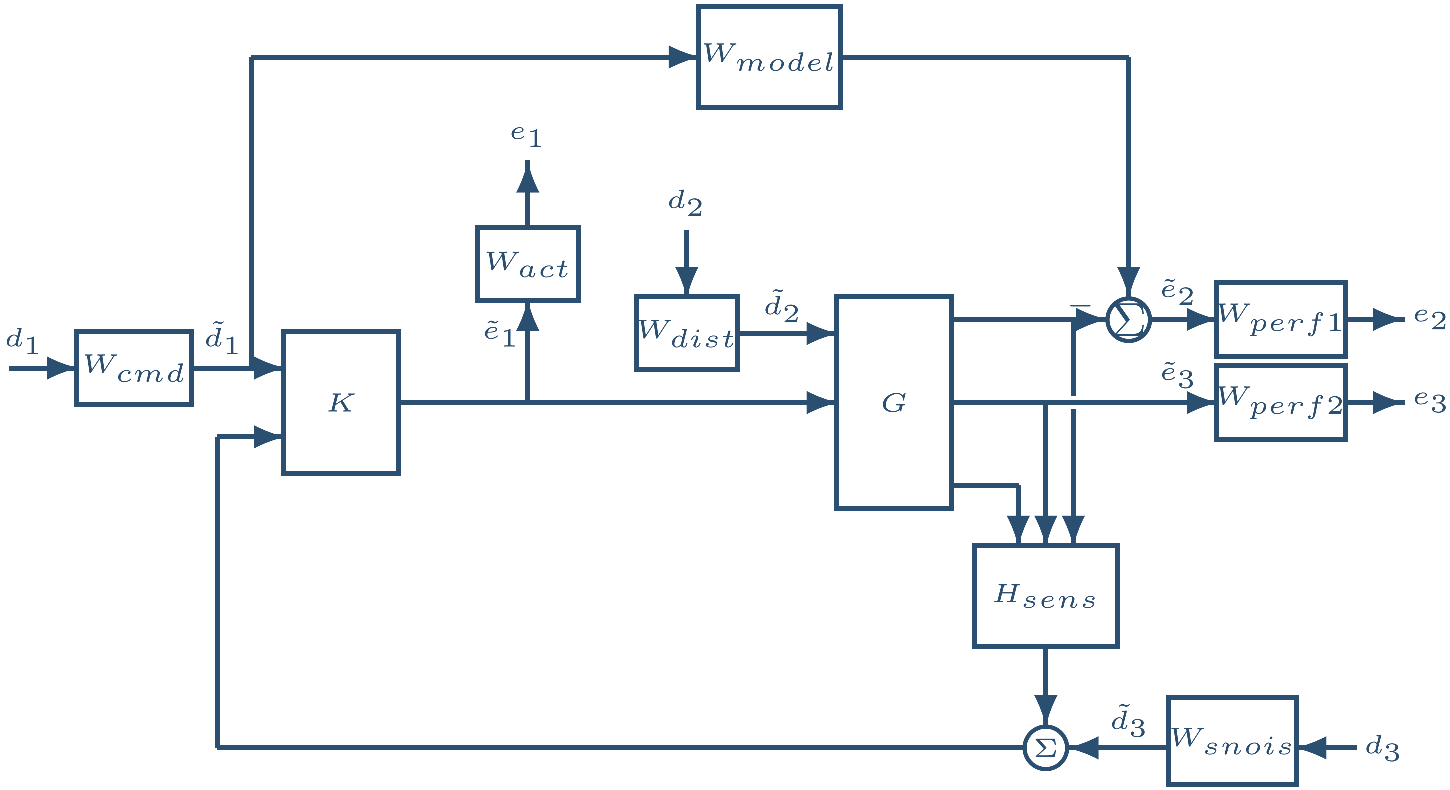
In this context, performance objectives must be a weighted norm

$$\|W_L T W_R\|$$

W_L and W_R can be frequency dependent, to account for bandwidth constraints and spectral content of exogenous signals.



Closed-loop performance objectives as weighted closed-loop transfer functions which are to be made small through feedback. Here's an example which includes many relevant terms.



The mathematical objective of $\mathcal{H}_\infty$ control is to make the closed-loop MIMO transfer function T_{ed} satisfy

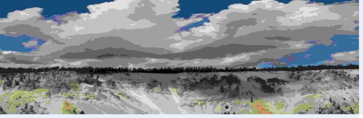
$$\|T_{ed}\|_\infty < 1.$$

MIMO Performance Signals

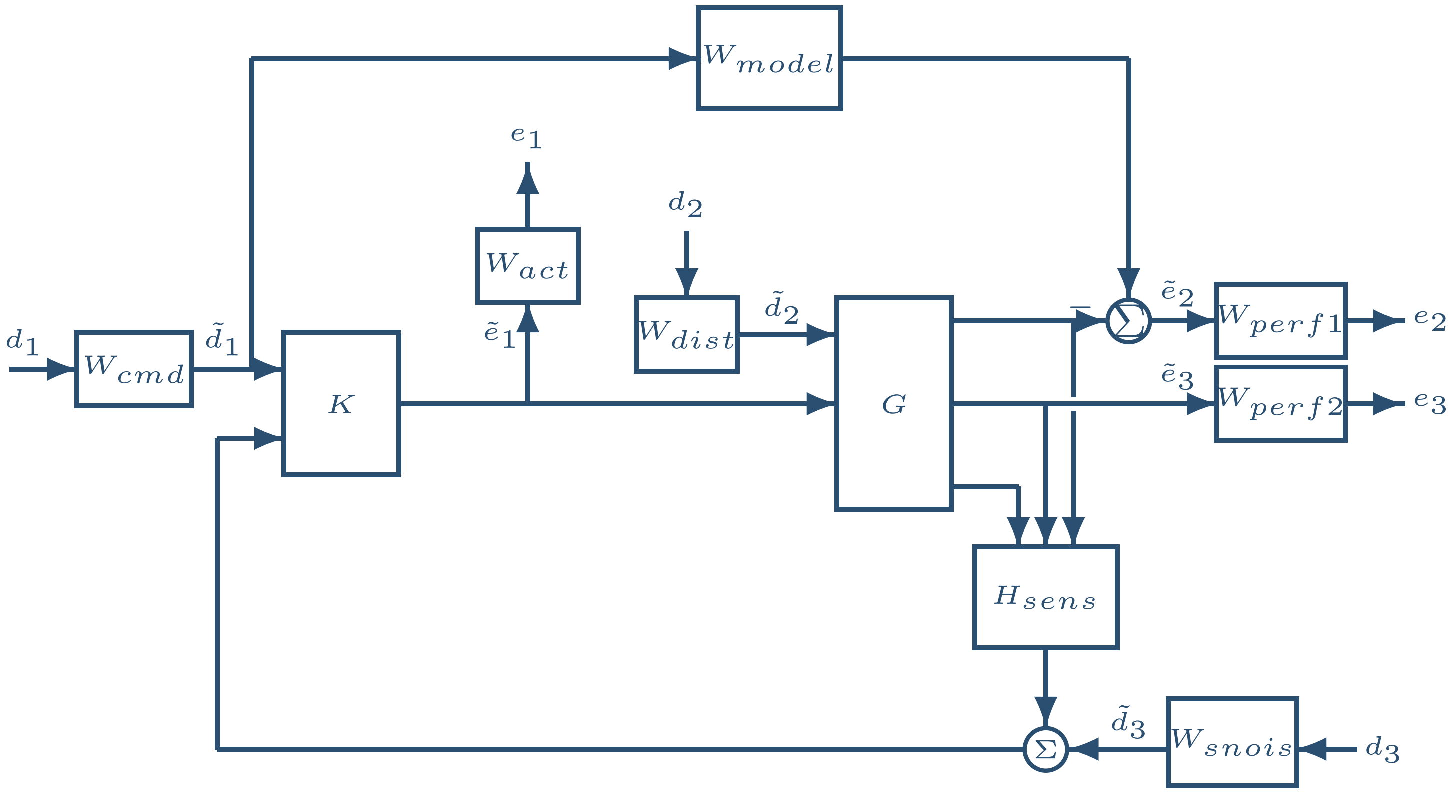
Weighting functions are used to scale the input/output transfer functions.

Interpretation of signals and weighting functions are

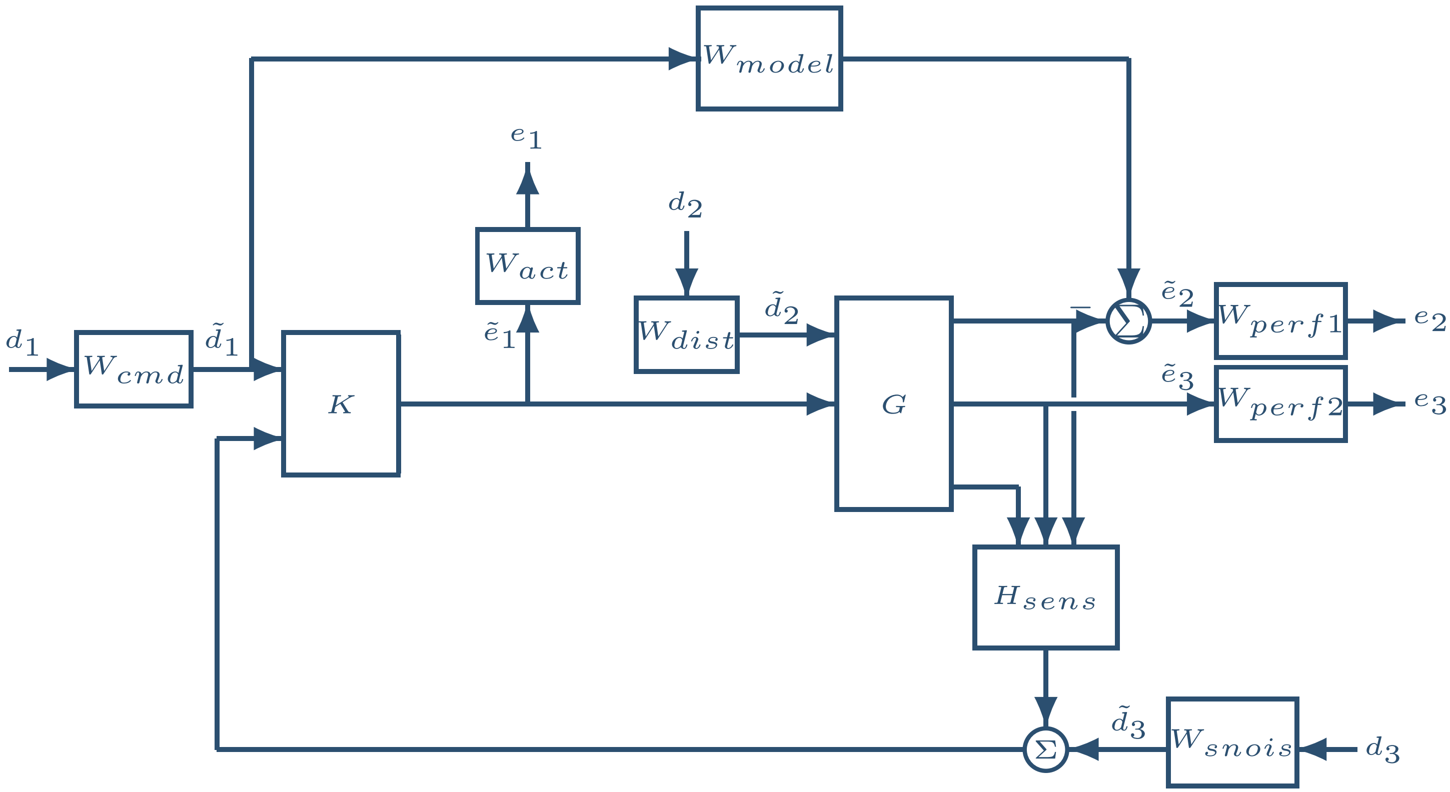
Signal	Meaning
d_1	Normalized reference command
$\tilde{d}_1$	Typical reference command
d_2	Normalized exogenous disturbances
$\tilde{d}_2$	Typical exogenous disturbances
d_3	Normalized sensor noise
$\tilde{d}_3$	Typical sensor noise
e_1	Weighted control signals
$\tilde{e}_1$	Actual control signals
e_2	Weighted tracking errors
$\tilde{e}_2$	Actual tracking errors
e_3	Weighted plant errors
$\tilde{e}_3$	Actual plant errors



Interpretation of Weights and Models: W_{cmd}



- Used in problems requiring tracking of a reference command.
- W_{cmd} shapes (magnitude and frequency) the normalized reference command signals into the actual (or typical) reference signals that we expect to occur.
- In typical servo-problems, W_{cmd} is flat at low frequency and rolls off at high frequency

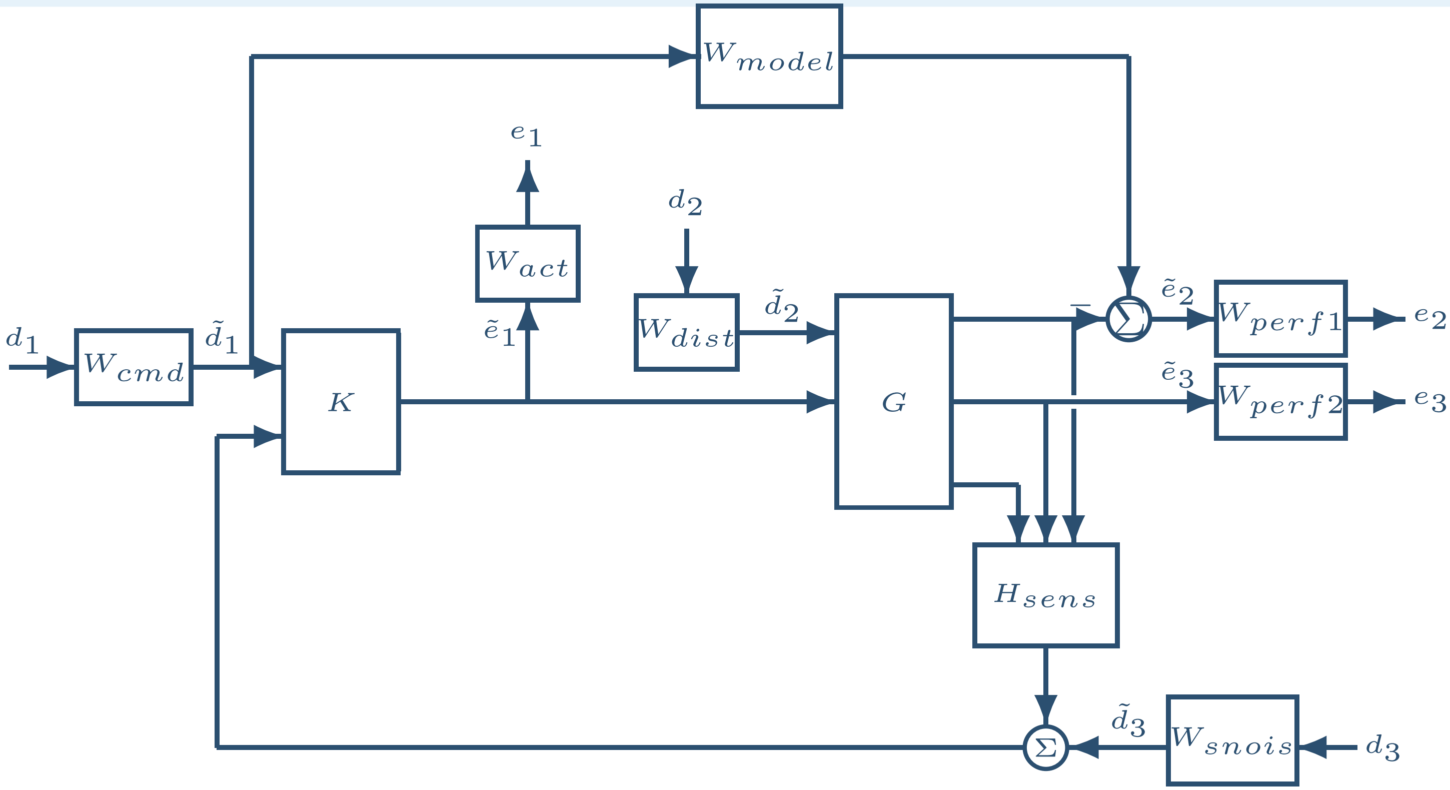


- For example, in a flight control problem, fighter pilots can (and will) generate stick input reference commands up to a bandwidth of about 2Hz. Say the stick has maximum travel of 3 inches. Pilot commands would then be modeled as normalized signals passed through a first order filter

$$W_{cmd} = \frac{3}{\frac{1}{2 \cdot 2\pi} s + 1}$$

Disturbance Rejection
Norms
$\mathcal{H}_\infty$ Interpretation
MIMO Performance
MIMO Signals
Control Problem
H_∞ Control
$\mathcal{H}_\infty$ History
$\mathcal{H}_\infty$ Design
D-K Iteration

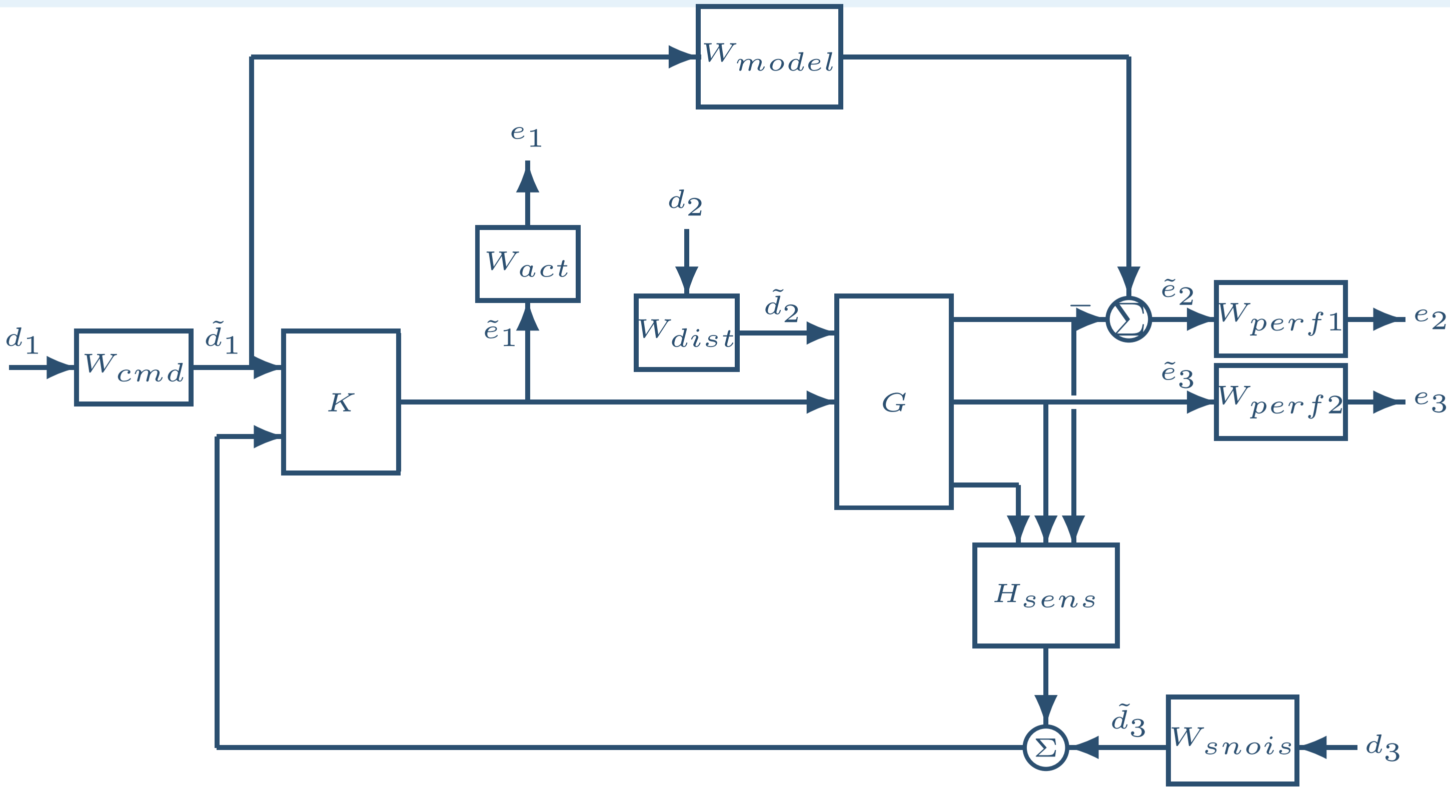
Interpretation of Weights and Models: W_{model}



- Represents a desired ideal model for the closed-loop system
- Used in problems with tracking requirements.
- Example: for "good" command tracking response, we might desire our closed-loop system to respond as well damped second-order system, so choose specific ω and ζ and define

$$W_{model} = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2}$$

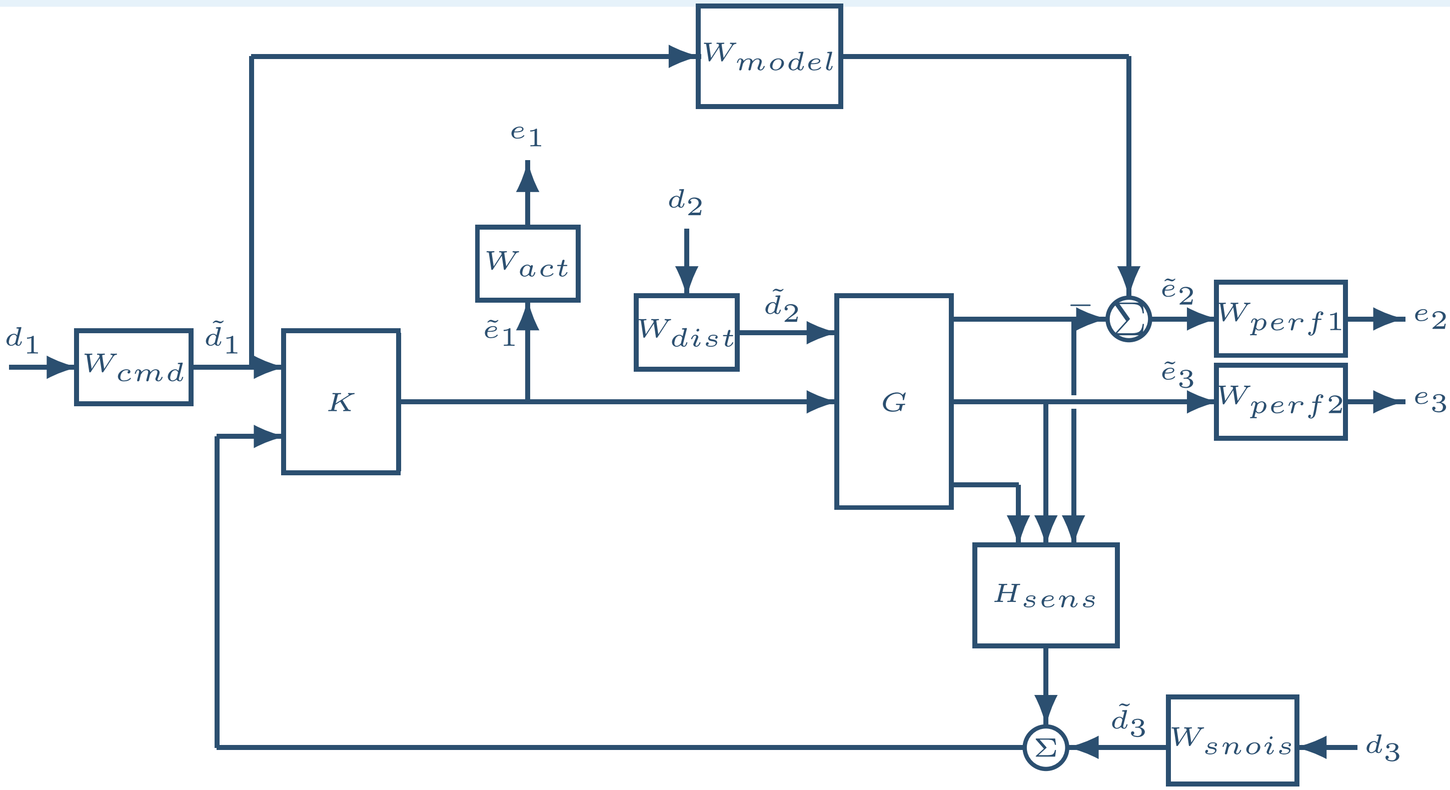
Interpretation of Weights and Models: W_{model} (cont'd)



- Example: Unit conversions might be necessary too. In the fighter pilot example, suppose roll-rate is being commanded, and 10°/second response is desired for each inch of stick motion. In these units, appropriate model is

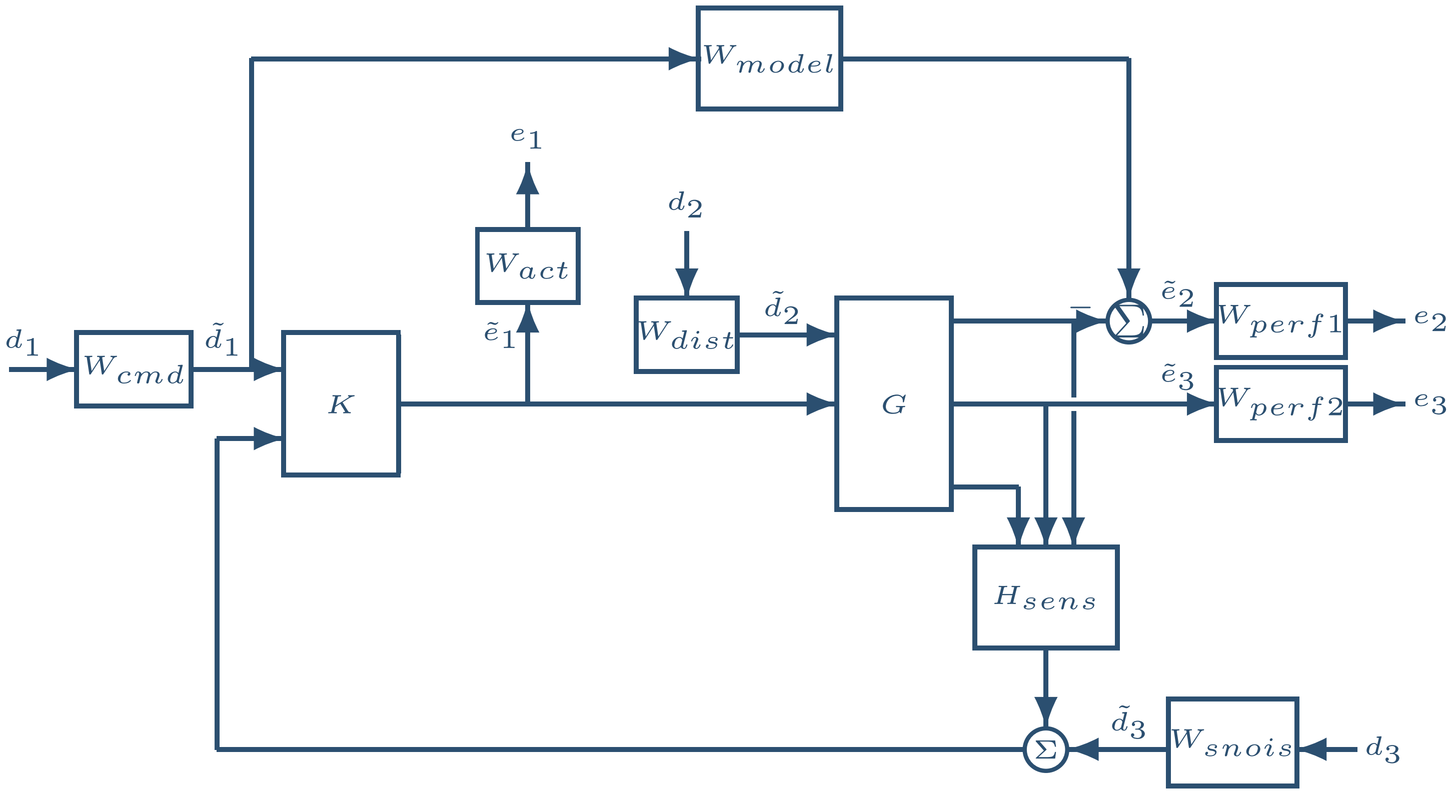
$$W_{model} = 10\frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2}$$

Interpretation of Weights and Models: W_{dist}



- Shapes the frequency content and magnitude of the exogenous disturbances effecting the plant
- Example: electron microscope
  - Dominant performance objective: mechanically isolate the microscope from outside mechanical disturbances, e.g. ground excitations, sound (pressure) waves, air currents
  - Capture spectrum and relative magnitudes of these disturbances via weighting matrix W_{dist}.

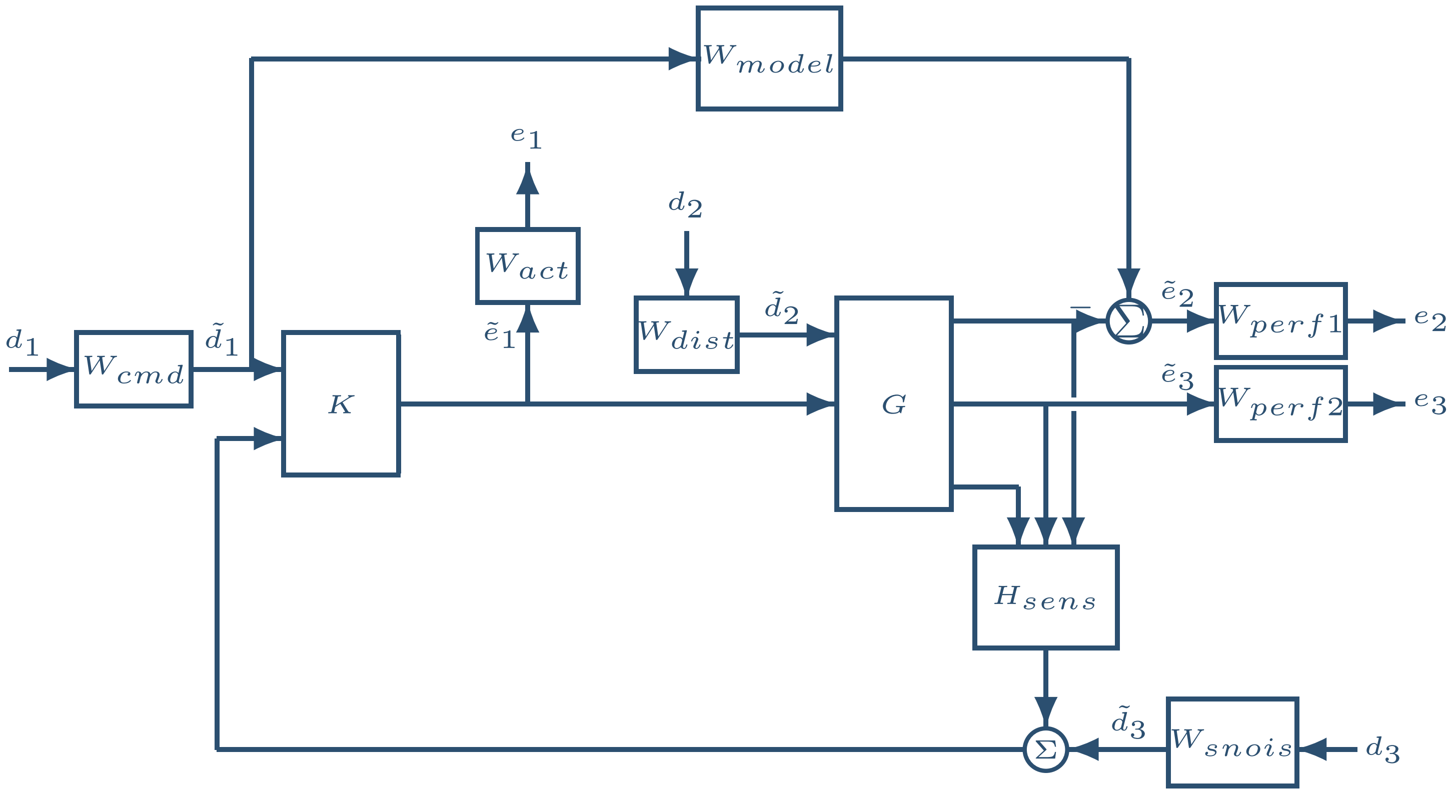
Interpretation of Weights: W_{perf1} and W_{perf2}



W_{perf1} weights the difference between the response of the plant and the response of the ideal model, W_{model}. Often we desire

- accurate matching of the ideal model at low frequency
- while requiring less accurate matching at higher frequency

Interpretation of Weights: W_{perf1} and W_{perf2}

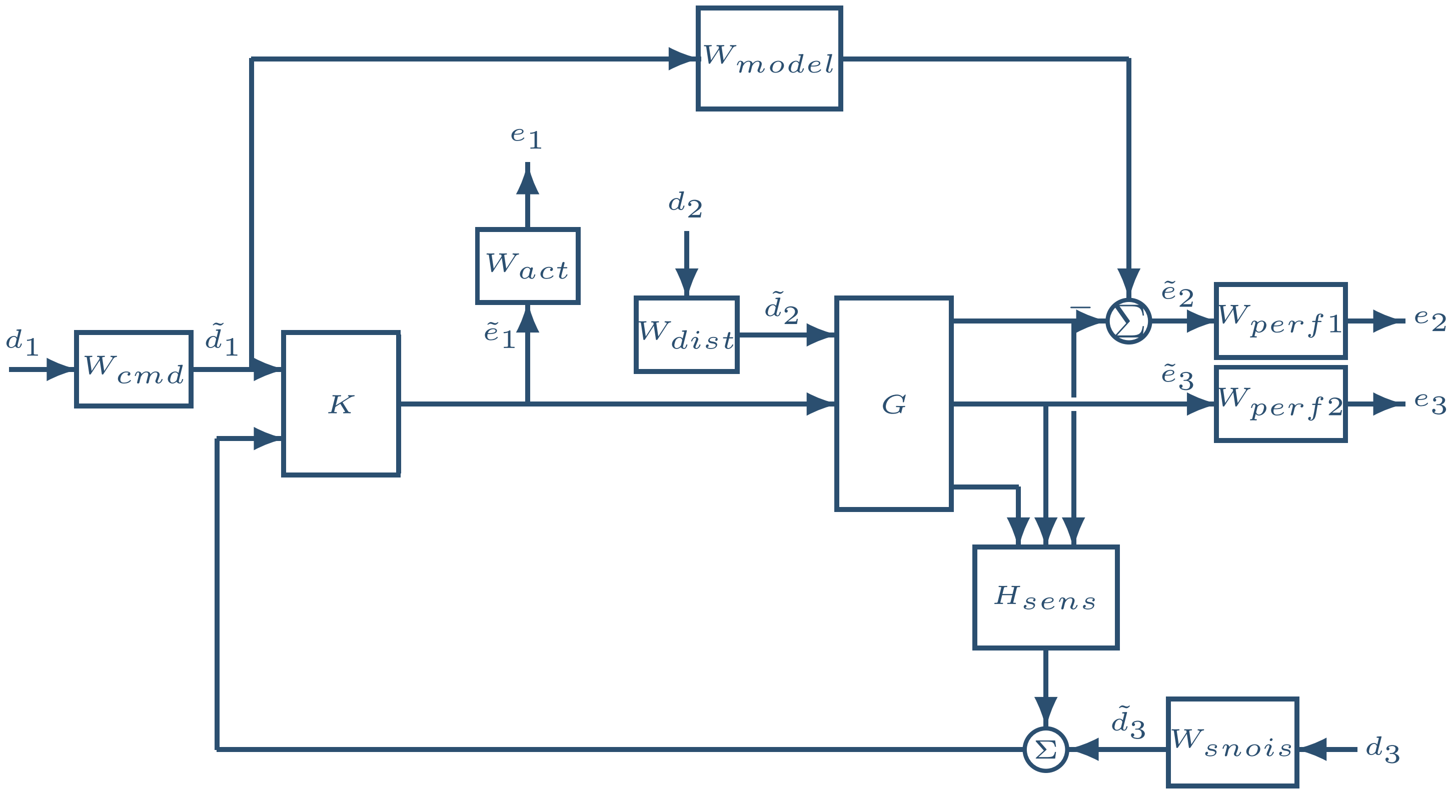


The inverse of the weight should be related to the allowable size of tracking errors, in the face of the reference commands and disturbances described by W_{ref} and W_{dist}.

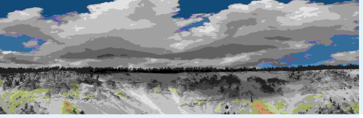
W_{perf2} penalizes variables internal to the process G, such as

- actuator states which are internal to G, or
- other variables that are not part of the tracking objective.

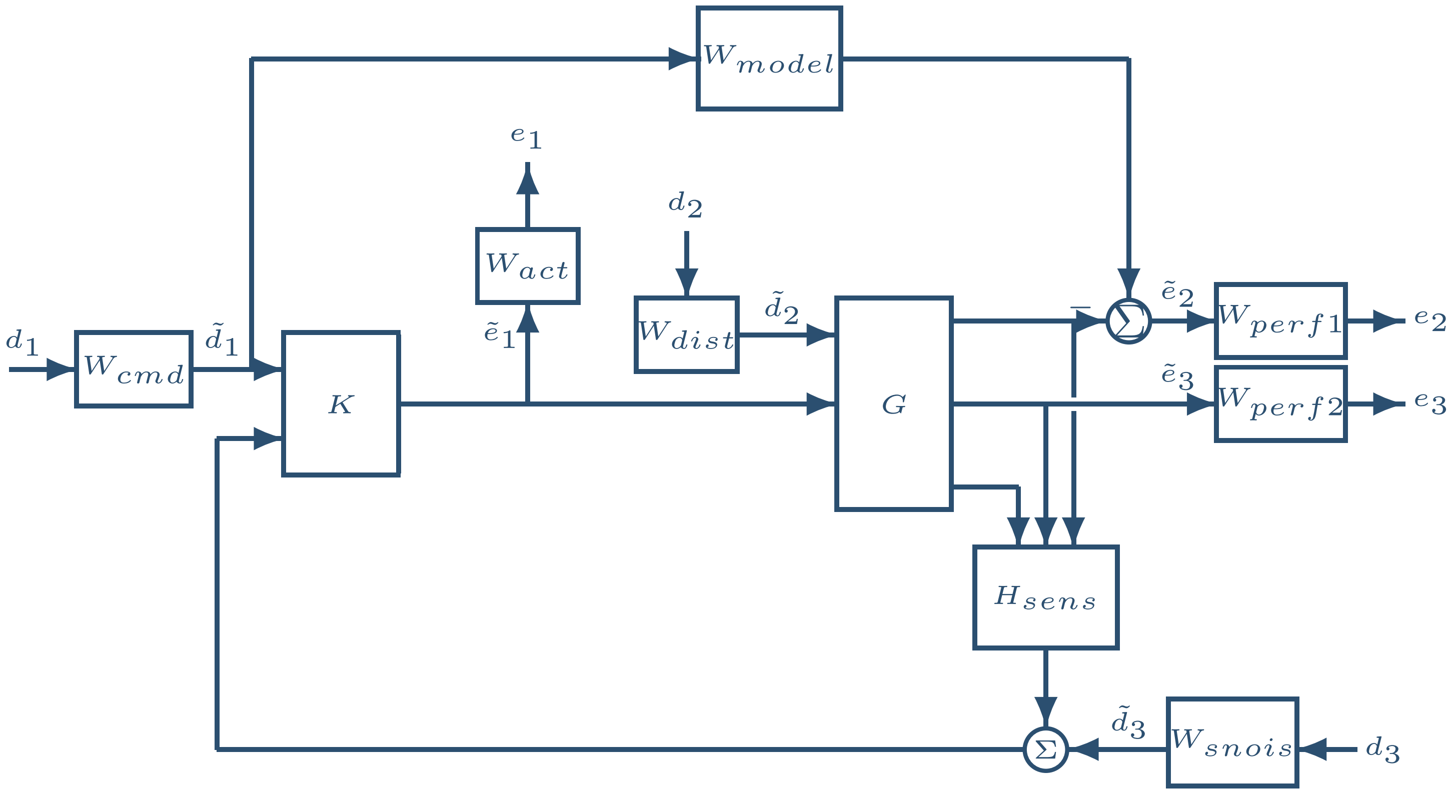
Interpretation of Weights and Models: W_{act}



- Used to shape the penalty on control signal usage
- Penalize limits the deflection/position, deflection rate/velocity, etc., response of the control signals, in the face of the tracking and disturbance rejection objectives already defined
- Each control signal is usually penalized independently.

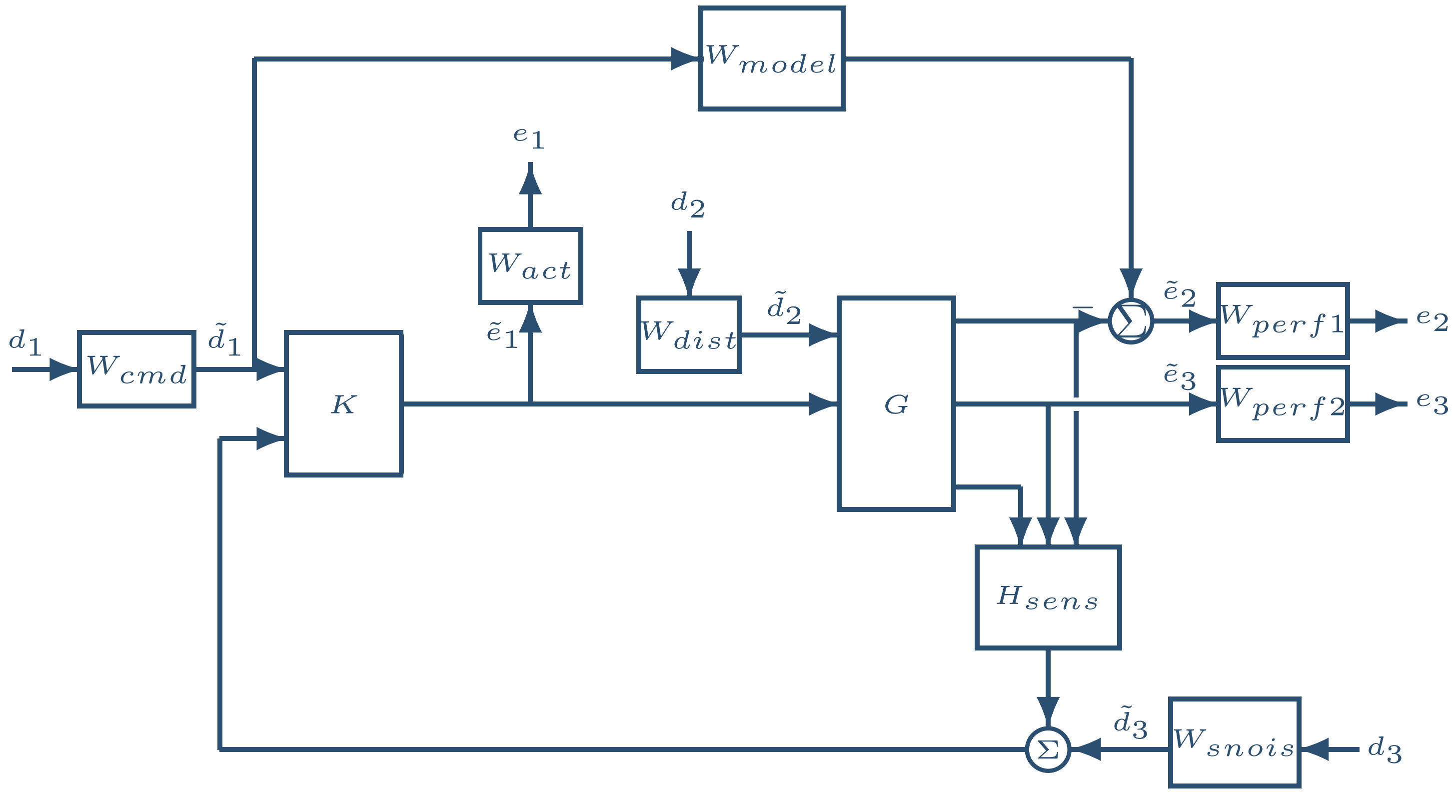


Interpretation of Weights and Models: W_{snois}



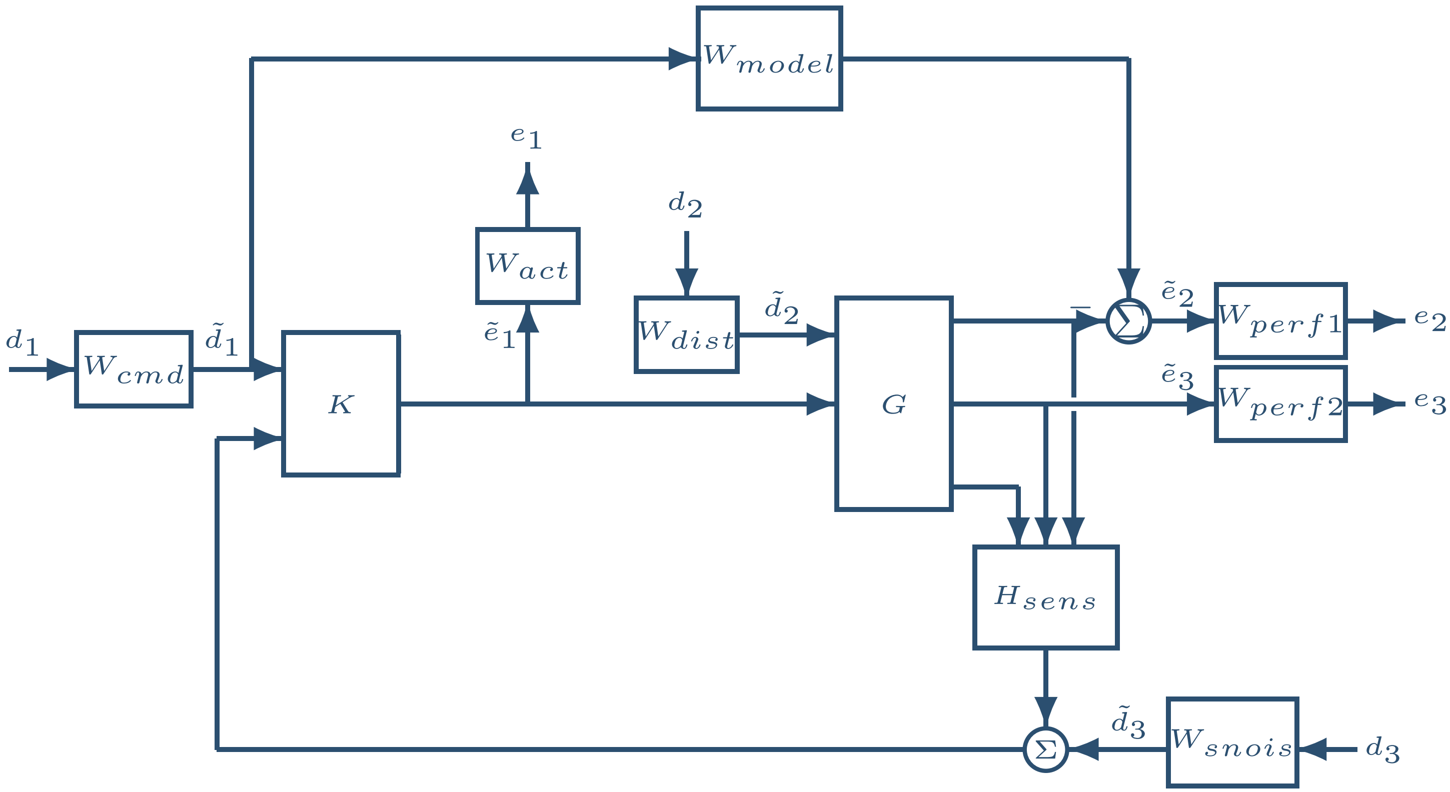
- Represents frequency content of sensor noise
- Derived from laboratory experiments or based on manufacturer measurements
- Example: medium grade accelerometers have substantial noise at low frequency and high frequency. Therefore the corresponding W_{snois} weight would be larger at low and high frequency and have a smaller magnitude in the mid-frequency range.

Interpretation of Weights and Models: W_{snois} (cont'd)

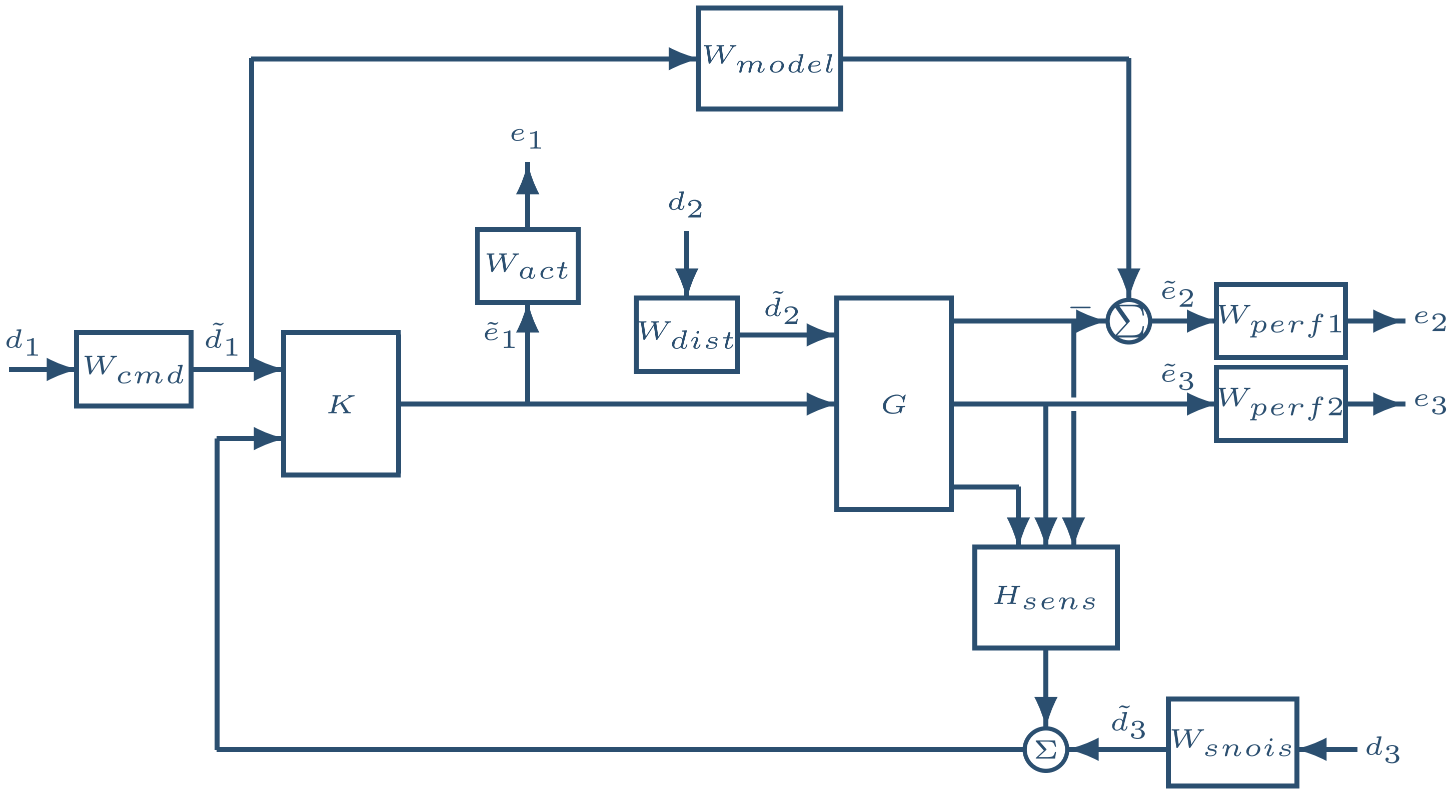


- Example: Displacement or rotation measurements are often quite accurate at low frequency or in steady-state but respond poorly as frequency increases. Weighting function for this sensor would be small at low frequency, gradually increase in magnitude as a first or second system and level out at high frequency.

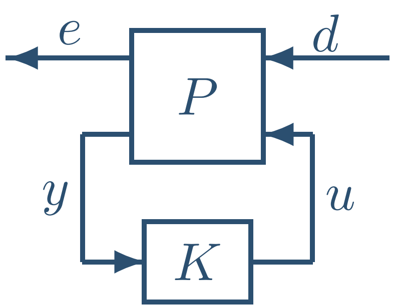
Interpretation of Weights and Models: H_{sens}

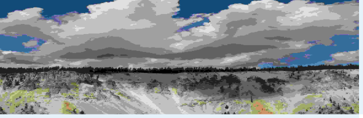


- Represents a model of the sensor dynamics or an external anti-aliasing filter
- Based on physical characteristics of the individual sensor components



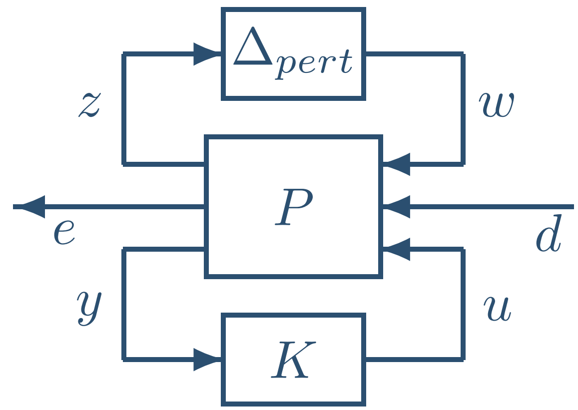
Everything that is not the controller, K, comprises the generalized plant, P





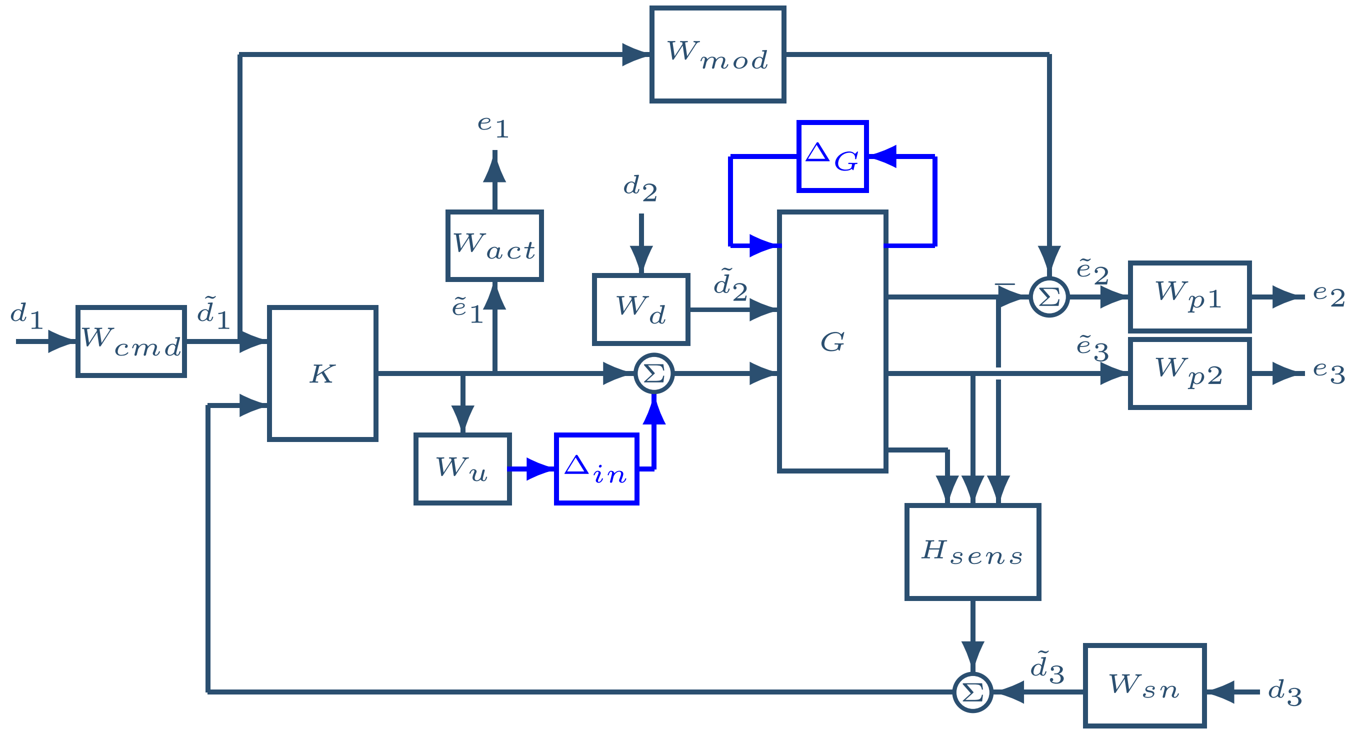
μ-Synthesis via D-K Iteration

D-K Iteration Problem Formulation



- P is the open-loop interconnection containing nominal plant model, performance and uncertainty weighting functions.
- Three sets of inputs: perturbation inputs w, disturbances d, and controls u.
- Three sets of outputs: perturbation outputs z, errors e and measurements y.
- $\Delta_{pert} \in \boldsymbol{\Delta}_{pert}$, which parametrizes all of the assumed model uncertainty in the problem.
- K is the controller.

Robust Control: Design K to optimize the closed-loop performance objectives *in the presence of the assumed model uncertainty*.



as robust disturbance rejection: Design K to make the closed-loop MIMO transfer function, T_{ed}, small in the presence of model uncertainty.

D-K Iteration Design Objective

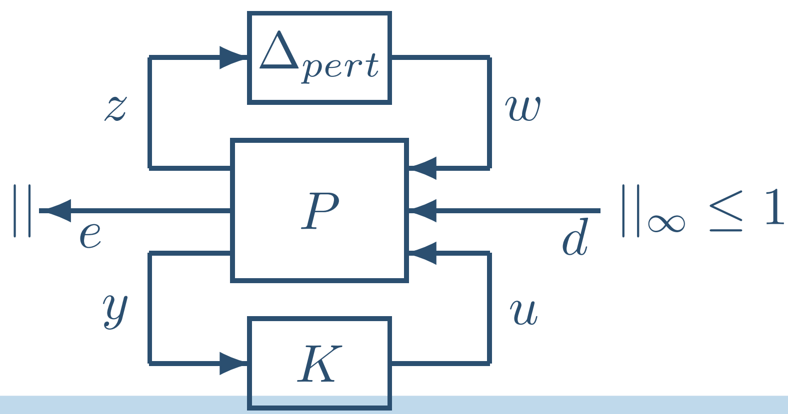
The set of systems to be controlled is described by the LFT

$$\left\{ F_U\left(P, \Delta_{pert}\right) : \Delta_{pert} \in \mathbf{\Delta}_{pert}, \max_{\omega} \|\Delta_{pert}(j\omega)\| \leq 1 \right\},$$

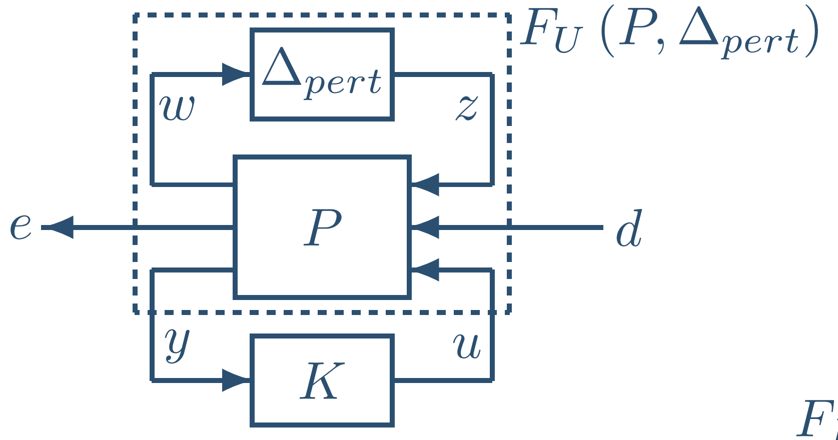
Design Objective:

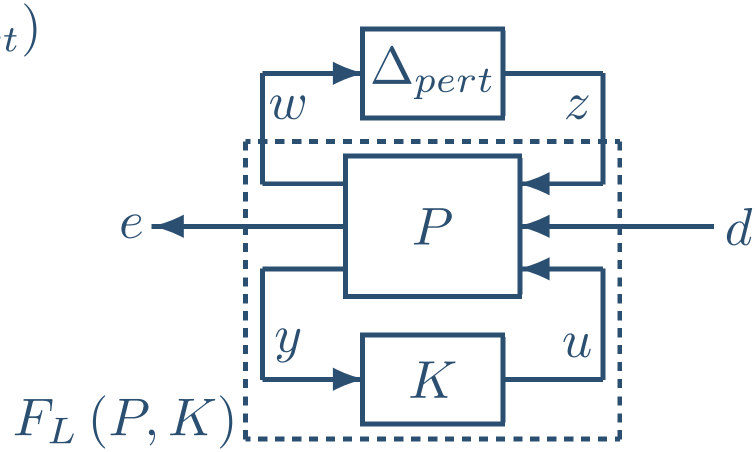
Find a controller K, such that for all $\Delta_{pert} \in \mathbf{\Delta}_{pert}$, the closed-loop system is stable and satisfies

$$||F_L[\underbrace{F_U\left(P, \Delta_{pert}\right)}_{\text{perturbed plant}}, K]||_\infty \leq 1.$$



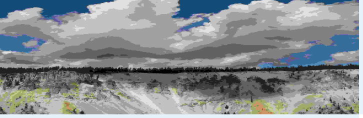
D-K Iteration Design Objective (cont'd)





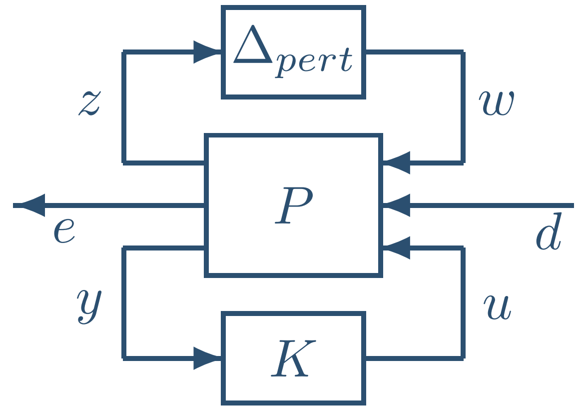
Robust performance test on $F_L(P, K)$ with respect to an augmented uncertainty structure,

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \Delta_{pert} & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta_{pert} \in \mathbf{\Delta}_{pert}, \ \Delta_F \in \mathbf{C}^{n_d \times n_e} \right\}.$$

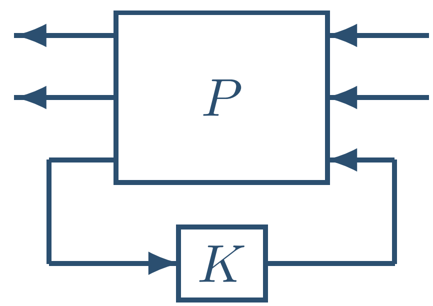


D-K Iteration Design Objective (cont'd)

Theorem: For all $\Delta_{pert} \in \mathbf{\Delta}_{pert}$, $||\Delta_{pert}||_\infty \leq 1$, the system



is stable, and has $||T_{d\leftarrow e}||_\infty \leq 1$ if and only if



is stable and $\max_\omega \; \mu_{\mathbf{\Delta}}(F_L(P,K)(\jmath\omega)) \leq 1$.

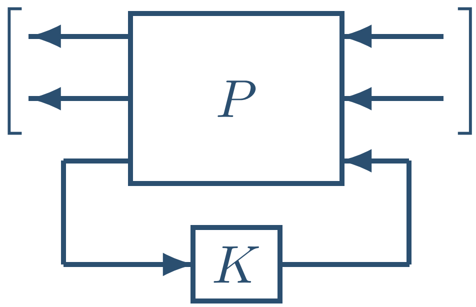
μ-Synthesis

Minimize, over all stabilizing controllers K, the peak value of $\mu_{\mathbf{\Delta}}(\cdot)$ of the closed-loop transfer function $F_L(P, K)$.

$$\min_{\substack{K \\ \text{stabilizing}}} \max_{\omega} \mu_{\mathbf{\Delta}}(F_L(P,K)(j\omega))$$

Pictorially, this is

$$\min_{\substack{K \\ \text{stabilizing}}} \max_{\omega \in \mathbf{R}} \mu_{\mathbf{\Delta}}$$



μ-Synthesis via Upper Bound

For tractability, replace $\mu_{\mathbf{\Delta}}(\cdot)$ by its upper bound,

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathbf{D}_{\mathbf{\Delta}}} \bar{\sigma}\left(DMD^{-1}\right)$$

where $\mathbf{D}$ is the set of matrices with the property that $D\Delta = \Delta D$ for every $D \in \mathbf{D}, \Delta \in \mathbf{\Delta}$. Under many situations, the bound is usually nearly equal. The design problem becomes

$$\min_{\substack{K \\ \text{stabilizing}}} \max_{\omega} \min_{D_\omega \in \mathbf{D}_{\mathbf{\Delta}}} \bar{\sigma}\left[D_\omega F_L(P,K)(\jmath\omega)D_\omega^{-1}\right]$$

D_ω is chosen from the set of scalings, $\mathbf{D}$, <u>independently</u> at every ω.

$$\min_{\substack{K \\ \text{stabilizing}}} \min_{D_\cdot, D_\omega \in \mathbf{D}_{\mathbf{\Delta}}} \max_{\omega} \bar{\sigma}\left[D_\omega F_L(P,K)(\jmath\omega)D_\omega^{-1}\right]$$

$$\min_{\substack{K \\ \text{stabilizing}}} \min_{D_\cdot, D_\omega \in \mathbf{D}_{\mathbf{\Delta}}} \left\|DF_L(P,K)D^{-1}\right\|_\infty$$

D-K Iteration

For simplicity, assume $\mathbf{\Delta}_{pert}$ only has full, unmodeled dynamics (ie., *complex*) blocks, say N of them, so that $\mathbf{\Delta}_{pert}$ is of the form

$$\mathbf{\Delta}_{pert} = \left\{ \text{diag}\left[\Delta_1, \Delta_2, \ldots, \Delta_N\right] : \Delta_i \in \mathbf{C}^{r_i \times c_i} \right\}$$

This rules out, for example, repeated, real-parameter uncertainty, but the methodology can be modified to address those types as well.

The set $\mathbf{\Delta}$ has the additional block (for the robust performance criterion)

$$\mathbf{\Delta} = \left\{ \text{diag}\left[\Delta_1, \Delta_2, \ldots, \Delta_N, \Delta_F\right] : \Delta_i \in \mathbf{C}^{r_i \times c_i}, \Delta_F \in \mathbf{C}^{n_d \times n_e} \right\}$$

The associated scaling set $\mathbf{D}$ is

$$\mathbf{D} = \{\text{diag}\left[d_1 I, d_2 I, \ldots, d_N I, I\right] : \; d_i > 0\}$$

D-K Iteration (cont'd)

Note that the elements of D can have any phase, and not change the value of $\bar{\sigma}\left(DMD^{-1}\right)$. For any positive d_i and real-valued θ_i,

$$\bar{\sigma}\left(\begin{bmatrix} d_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} M \begin{bmatrix} d_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}^{-1}\right)$$

$$=$$

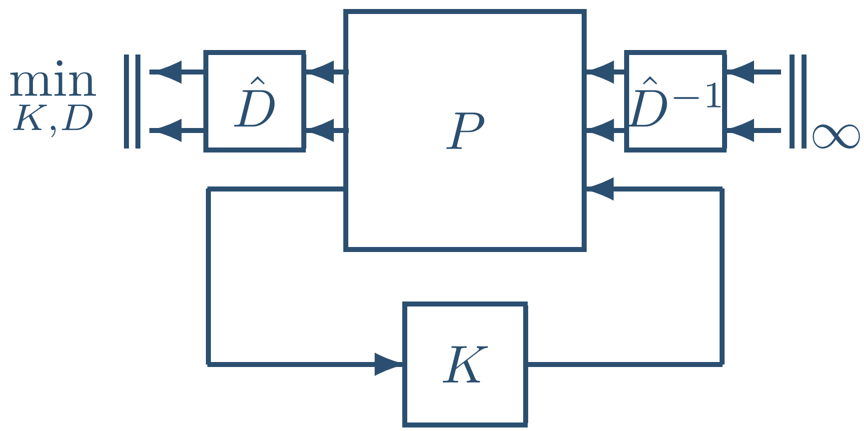
$$\bar{\sigma}\left(\begin{bmatrix} e^{\jmath\theta_1} d_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{\jmath\theta_N} d_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} M \begin{bmatrix} e^{\jmath\theta_1} d_1 I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{\jmath\theta_N} d_N I & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}^{-1}\right)$$

D-K Iteration (cont'd)

The new optimization is

$$\min_{\substack{K \\ \text{stabilizing}}} \quad \min_{\substack{\hat{D}(s)\in\mathbf{D} \\ \text{stable,min-phase}}} \quad \left\| \hat{D} F_L(P,K) \hat{D}^{-1} \right\|_\infty$$

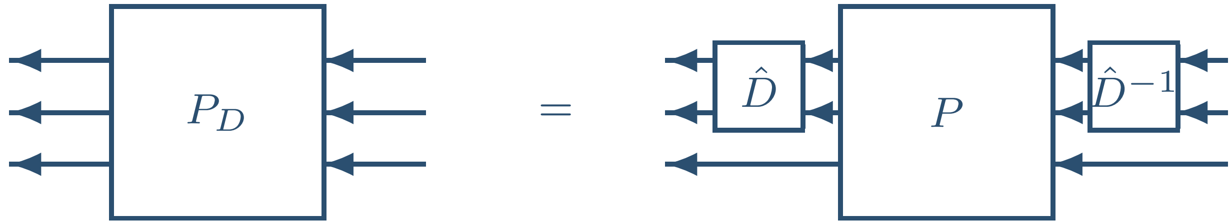
This optimization is currently "solved" by an iterative approach, referred to as "$D - K$ iteration." A block diagram depicting the optimization is



The steps of the iteration are as follows...

D-K Iteration: Holding D Fixed

Given, stable, minimum phase, real-rational $\hat{D}(s)$, define



- $F_L(P_D, K) = \hat{D} F_L(P, K) \hat{D}^{-1}$
- K stabilizes P_D if and only if K stabilizes P.
- P_D is real-rational

Then, solving the optimization

$$\min_{\substack{K \\ \text{stabilizing}}} \left\| \hat{D} F_L(P, K) \hat{D}^{-1} \right\|_\infty$$

is equivalent to

$$\min_{\substack{K \\ \text{stabilizing}}} \left\| F_L(P_D, K) \right\|_\infty$$

D-K Iteration: Holding K Fixed

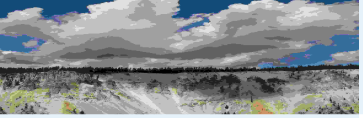
Optimization over D is carried out in a two-step procedure:

1. Finding the optimal frequency-dependent scaling matrix D at a large, but finite set of frequencies (this is the upper bound calculation for μ)

   - Given a stabilizing controller, $K(s)$, solve the minimization (upper bound for μ)

$$\min_{D_\omega \in \mathbf{D}} \bar{\sigma}\left[D_\omega F_L(P,K)(\jmath\omega)D_\omega^{-1}\right]$$

at M frequencies $(\omega_1, \omega_2, \ldots, \omega_M)$.



D-K Iteration: Holding K Fixed

2. Fit this optimal frequency-dependent scaling with a stable, minimum-phase, real-rational transfer function $\hat{D}$

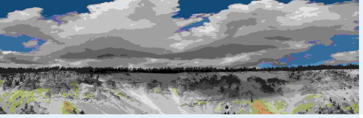
- This minimization is done over the real, positive D_ω from the set $\mathbf{D}$ using the μ upper bound.
- Recall that the addition of phase to each $d_i(\omega)$ does not affect the value of $\bar{\sigma}\left[D_\omega F_l(P,K)(\jmath\omega)D_\omega^{-1}\right]$. Important aspect of the scaling d_i is *its magnitude*, $|d_i(j\omega)|$.
- Bode integral formulae to determine the phase $\theta_i(\omega)$ of the stable, minimum-phase function L_i that satisfies for all ω.

$$|L_i(j\omega)| = d_i(\omega)$$

- A real-rational transfer function $\hat{d}_i(s)$ is found such that

$$\hat{d}_i(j\omega_k) \approx \underbrace{e^{j\theta_i(\omega_k)}}_{\text{phase}} \underbrace{d_i(\omega_k)}_{\text{magnitude}}$$

- $\hat{D}(s) = \mathrm{diag}\left[\hat{d}_1(s)I, \hat{d}_2(s)I, \ldots \hat{d}_{F-1}(s)I, I\right]$ and absorbed into P to yield P_D.



D-K Iteration Summary

Iterate between:

- Hold D fixed, find K
- Hold K fixed and find D.

Shortcomings

- Approximated $\mu_{\mathbf{\Delta}}(\cdot)$ by its upper bound. This is not a serious problem since the value of μ and its upper bound are often close.
- Restricted D's dependence on frequency to real, rational functions. Only a mild restriction, since rational functions can uniformly approximate continuous functions on finite intervals.
- Joint minimization of (D, K) is performed coordinate-wise. The $D - K$ iteration is not guaranteed to converge to to a global, or even local minimum. This is a serious problem, and represents the biggest limitation of the design procedure.

In spite of these drawbacks, the $D - K$ iteration control design technique appears to work well on many engineering problems.

D-K Coordinate Optimization Issue

DK iteration may have convergence problems. The example is due to Doyle and Chu (1985 CDC). Define

$$R := \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, U := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, V := \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} : \delta_i \in \mathbf{C} \right\}, \mathbf{D} := \left\{ \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} : d > 0 \right\}$$

The $D - K$ iteration replaces μ with the upper bound (in this case, 2 complex scalars, the upper-bound equals μ), leaving

$$\min_{Q \in \mathbf{R}} \min_{D \in \mathbf{D}} \bar{\sigma} \left[D \left(R + UQV \right) D^{-1} \right].$$

- For fixed $Q > 0$, the optimal D is $d_{\text{opt}} = \sqrt{Q}$, while for fixed d, the optimal Q is d^2.
- The desired optimum (minimum over both d and Q) is (actually an infimum in this case) is achieved as $d \to 0$, and $Q = 0$.

D-K Iteration Applications

- Space shuttle flight control system
- B-2, YF-22, HARV (F-18), F-14
- Inclusion μ robustness analysis tests into next generation MIL specifications and handling quantities models.
- Missile autopilots: IRIS-T (JHUAPL, Germany)
- Flexible structures (NASA, JPL, Civil Engineering)
- Earth moving equipment (Caterpillar, Kamatsu)
- Compact disk players (Philips)
- Thin-film manufacturing (3M)
- Active suspension (Ford)
- Tokamac (Switzerland)
- Satellites (JAXA, ESA), Launch Vehicles (Ariane)
- Wind Turbines (NREL)
- Aeroservoelastic vehicle (Air Force, Body Freedom Flutter, X-56A)
- Supercavitating Vehicles (UMN)
- Small UAVs Control and Fault Detection (UMN, SZTAKI)
- Air Data Fault Detection (Goodrich/UTC)

Simple design examples

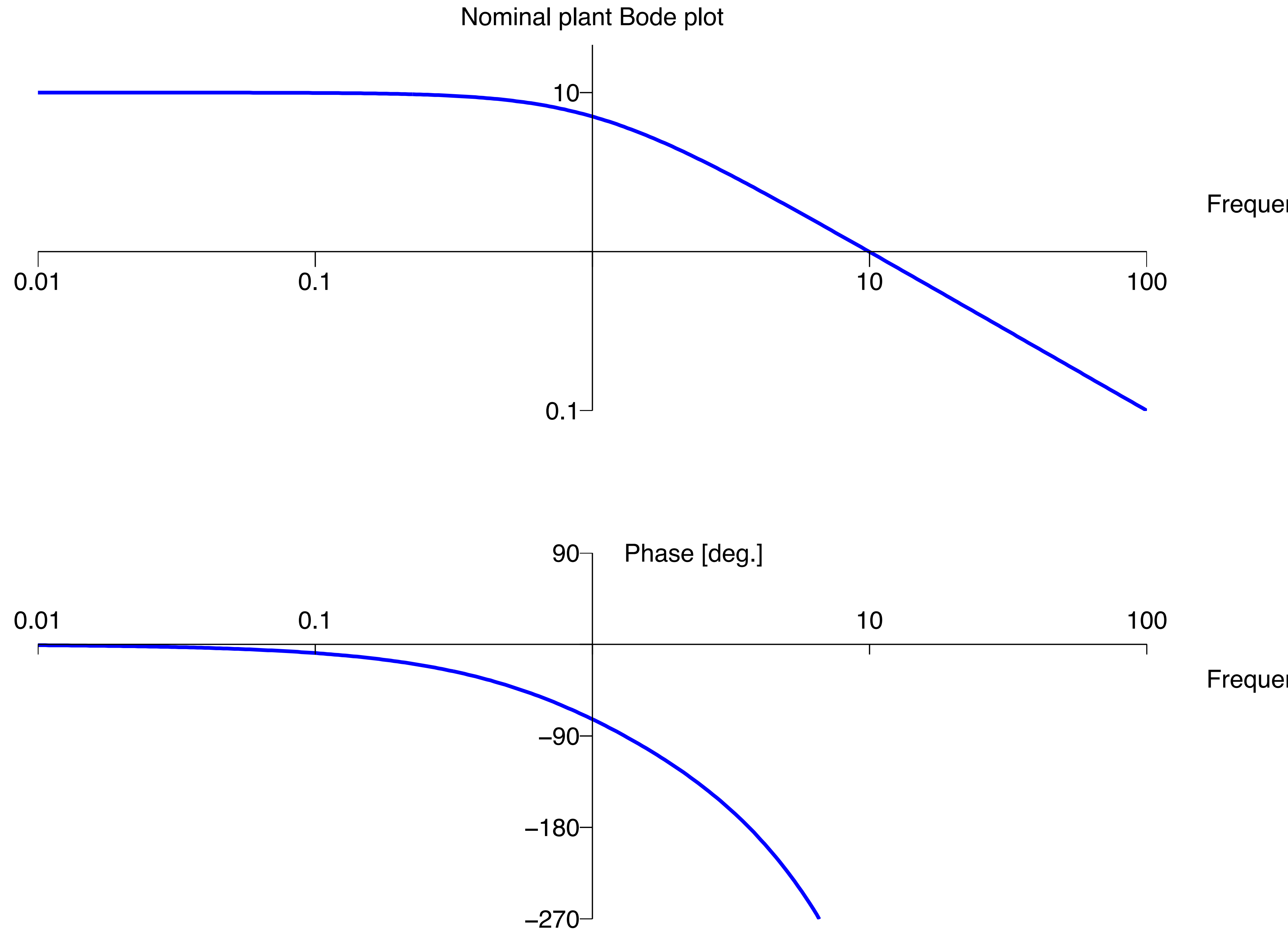
1. Robust model for a system with uncertain gain, time-constant and delay
2. Design a loopshaping controller (PI)
3. Analyze nominal performance, robust stability and robust performance
4. Perform 1 step of a D-K iteration (with a constant D scale) to improve robustness
5. Repeat the robustness analysis
6. Redesign the controller using H-infinity loopshaping
7. Repeat the robustness analysis

Design example: uncertain first order system

Nominal case

$$P = \frac{K e^{-\lambda s}}{1 + \tau s}$$

$$K = 10$$
$$\tau = 1.0$$
$$\lambda = 0.5$$



Design example: uncertain first order system

Nominal case $$P = \frac{K e^{-\lambda s}}{1 + \tau s}$$

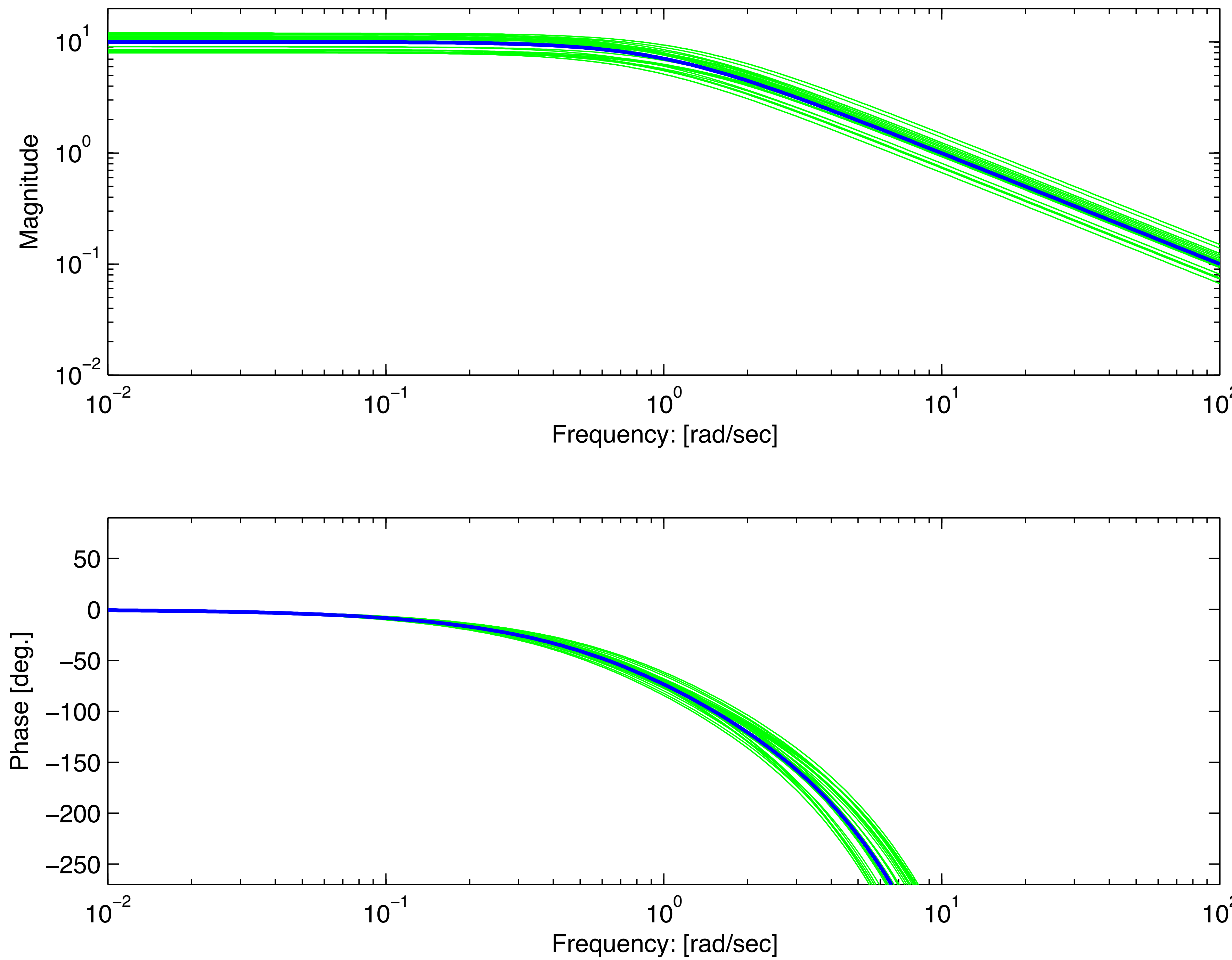
$$K = 10$$
$$\tau = 1.0$$
$$\lambda = 0.5$$

```
Kbnds = [8.5,11.5];
lambdabnds = [0.425,0.575];
taubnds = [0.85,1.15];

Kbnds = [8, 12];
lambdabnds = [0.4, 0.6];
taubnds = [0.8,1.2]';

% The nominal is defined as the midpoint.  This isn't necessarily
% optimal but it is reasonable in this case.

Knom = (Kbnds(1)+Kbnds(2))/2;
lambdanom = (lambdabnds(1)+lambdabnds(2))/2;
taunom = (taubnds(1)+taubnds(2))/2;

s = tf('s');
Pnom = exp(-lambdanom*s)*Knom/(1+taunom*s);
```

Perturbed case: randomly generated plants for the set

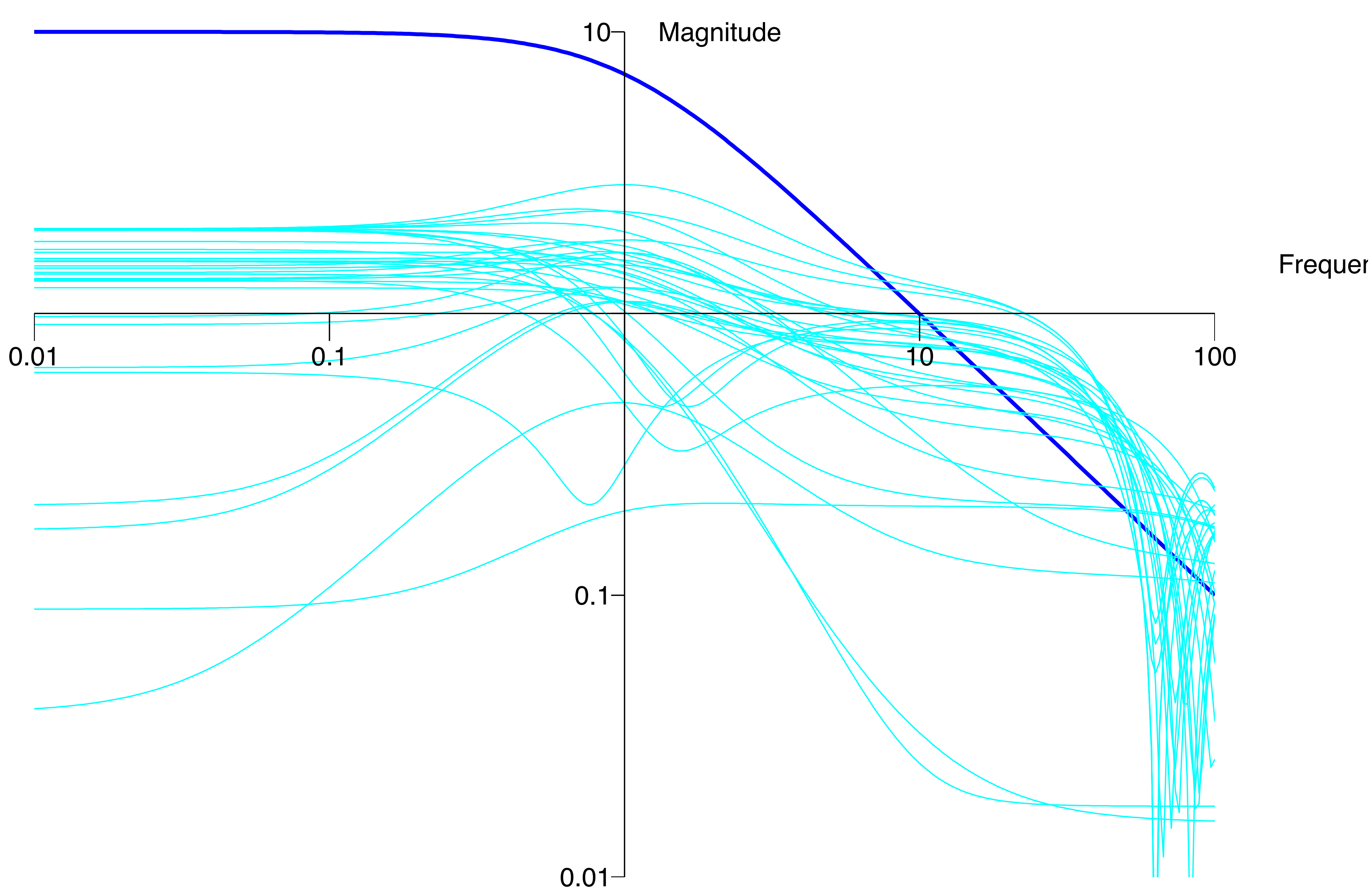
$$P = \frac{K e^{-\lambda s}}{1 + \tau s}$$

$$K \in [8.5, 11.5]$$
$$\tau \in [0.85, 1.15]$$
$$\lambda \in [0.425, 0.575]$$



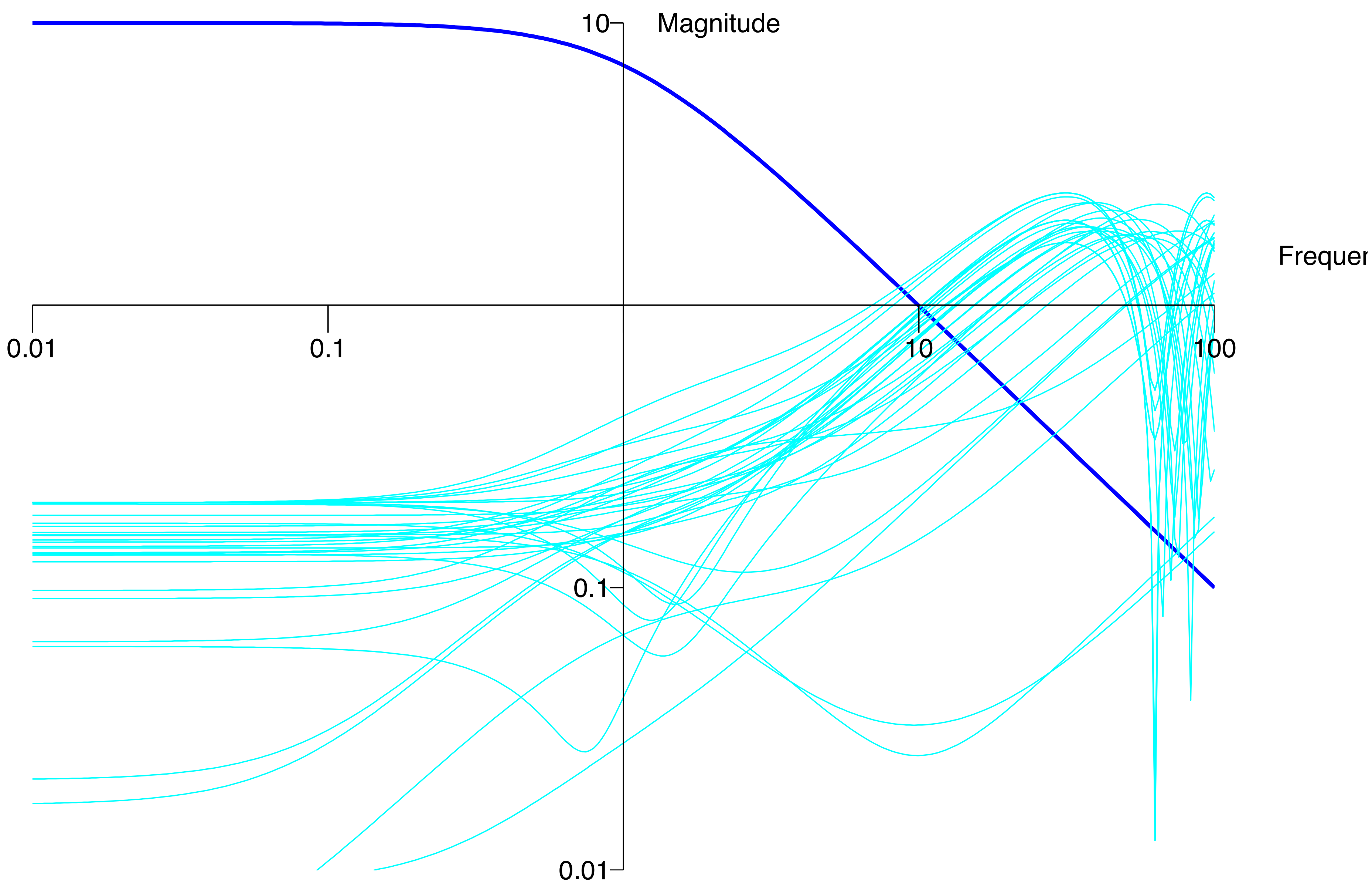
Absolute model errors

Error as a function of frequency: $|P_{\rm nom}(j\omega) - P(j\omega)|$



Relative model errors

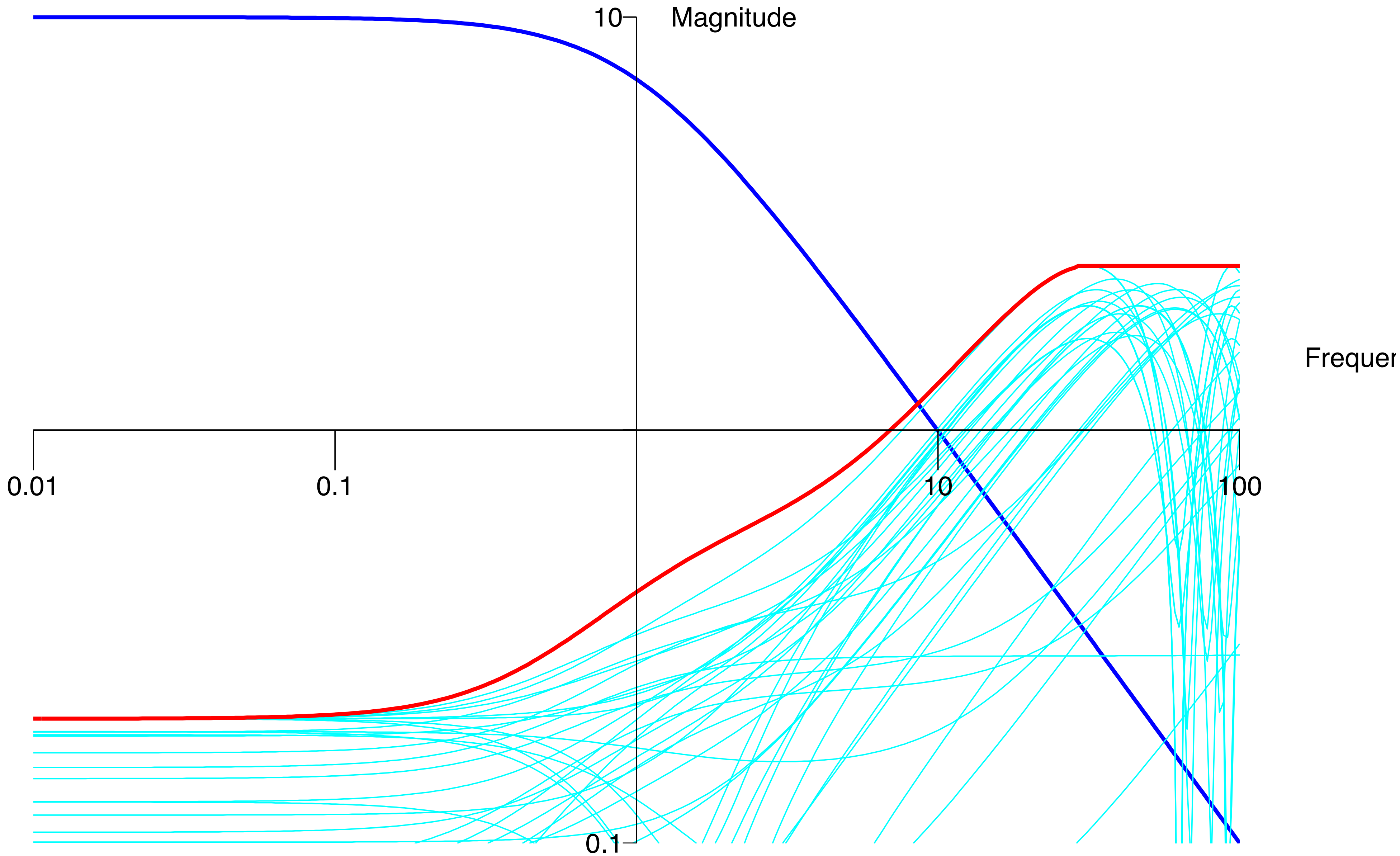
Error as a function of frequency: $\dfrac{|P_{\rm nom}(j\omega) - P(j\omega)|}{|P_{\rm nom}(j\omega)|}$



Relative model error bound

$$\left| \frac{G(j\omega) - P_{\text{nom}}(j\omega)}{P_{\text{nom}}(j\omega)} \right| \leq |W_{\text{m}}(\omega)|.$$

(See Laughlin *et al.* for the $W_{\text{m}}(s)$ formula)



Relative model error bound

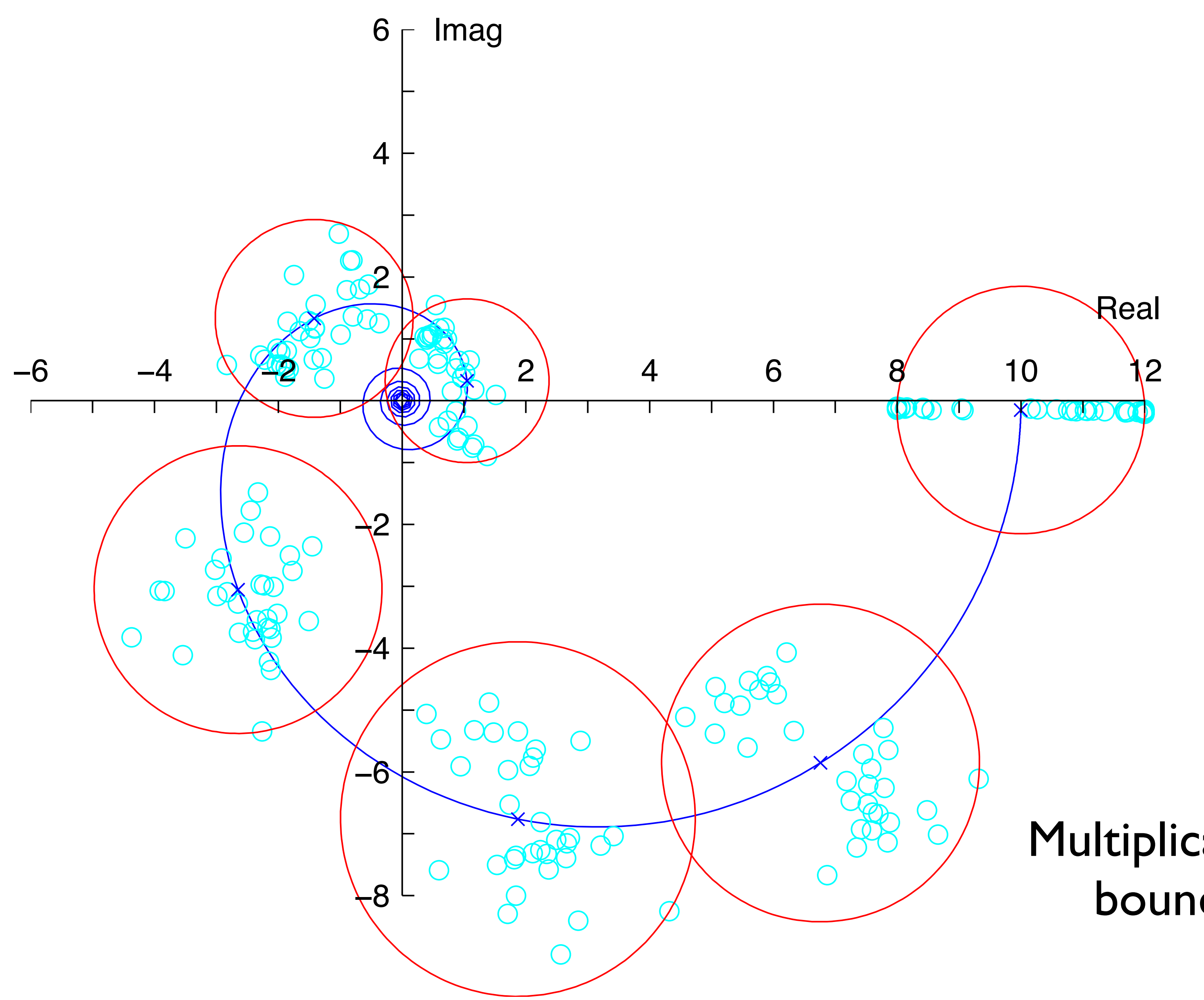
$$\left| \frac{G(j\omega) - P_{\rm nom}(j\omega)}{P_{\rm nom}(j\omega)} \right| \leq |W_{\rm m}(\omega)|.$$

(See Laughlin *et al.* for the $W_{\rm m}(s)$ formula)

```
% In order to design a controller we need a real-rational bound Wm.
% This one is a close fit to the above bound.

Wm = (1+ru)*((1 + taunom*s)/(1 + min(taubnds)*s))*...
(1 - (ru*lambdanom)*s)/(1 + (ru*lambdanom)*s) - 1;
Wm_w = squeeze(freqresp(Wm,omega));

% A simpler bound with more high frequency perturbations is given:
Wm = ru*(1 + s/0.5)/(1 + s/50);
Wm = Wm*(1 + s/8)/(1 + s/2);
Wm_w = squeeze(freqresp(Wm,omega));
```

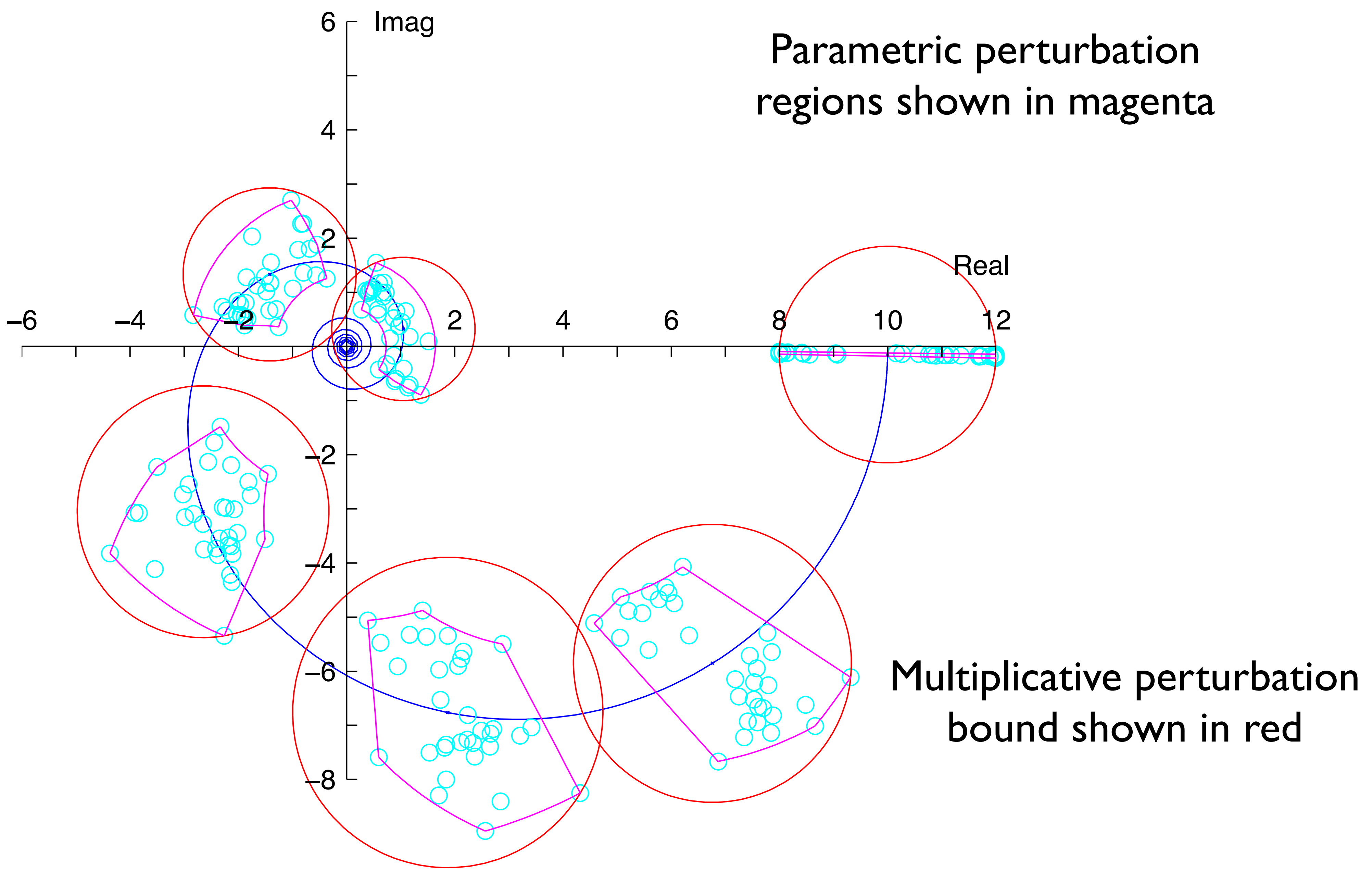
Relative model errors and bounding disks

Perturbed case: randomly generated plants for the set

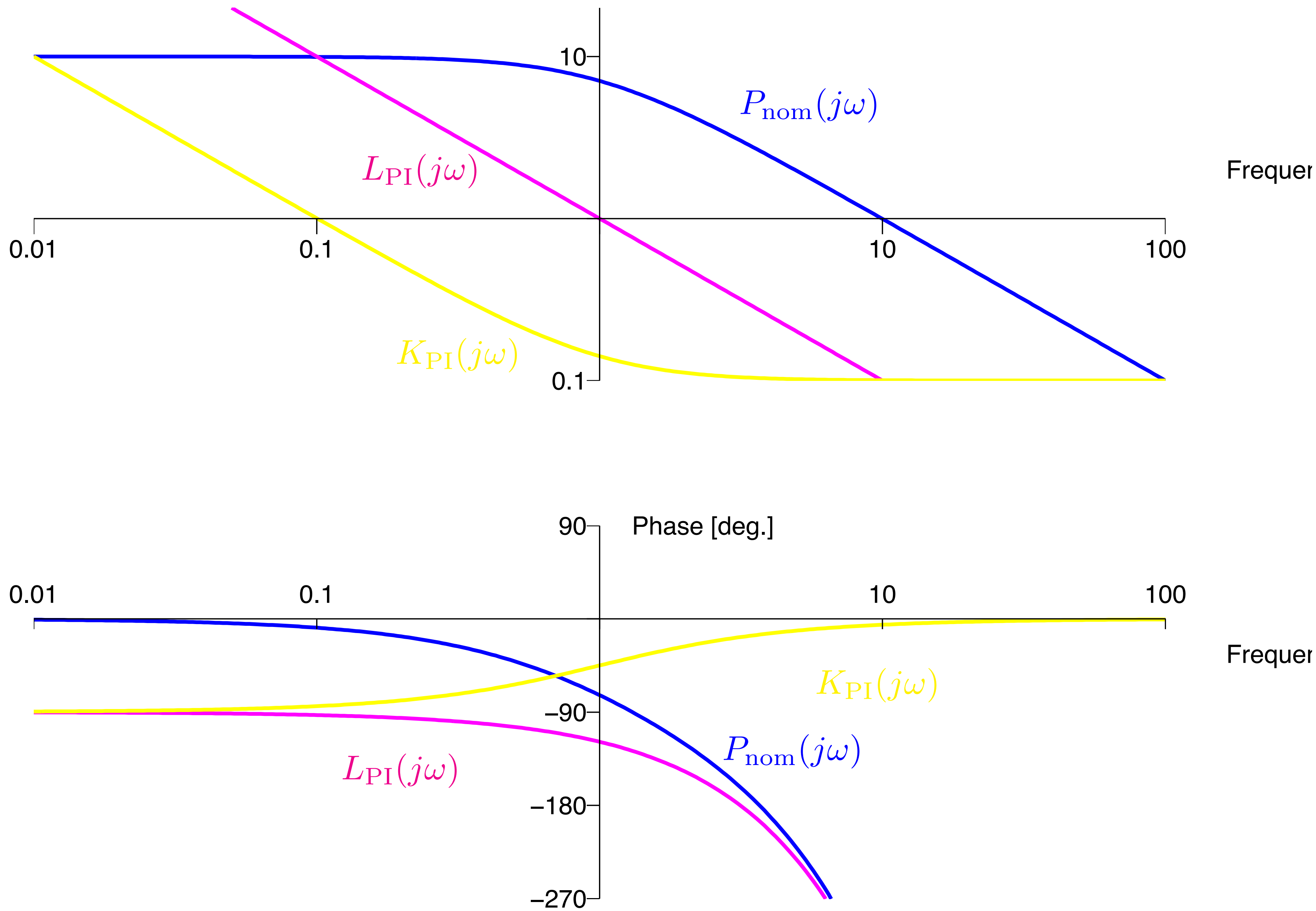


Multiplicative perturbation
bound shown in red

Perturbed case: randomly generated plants for the set



Nominal loopshaping design

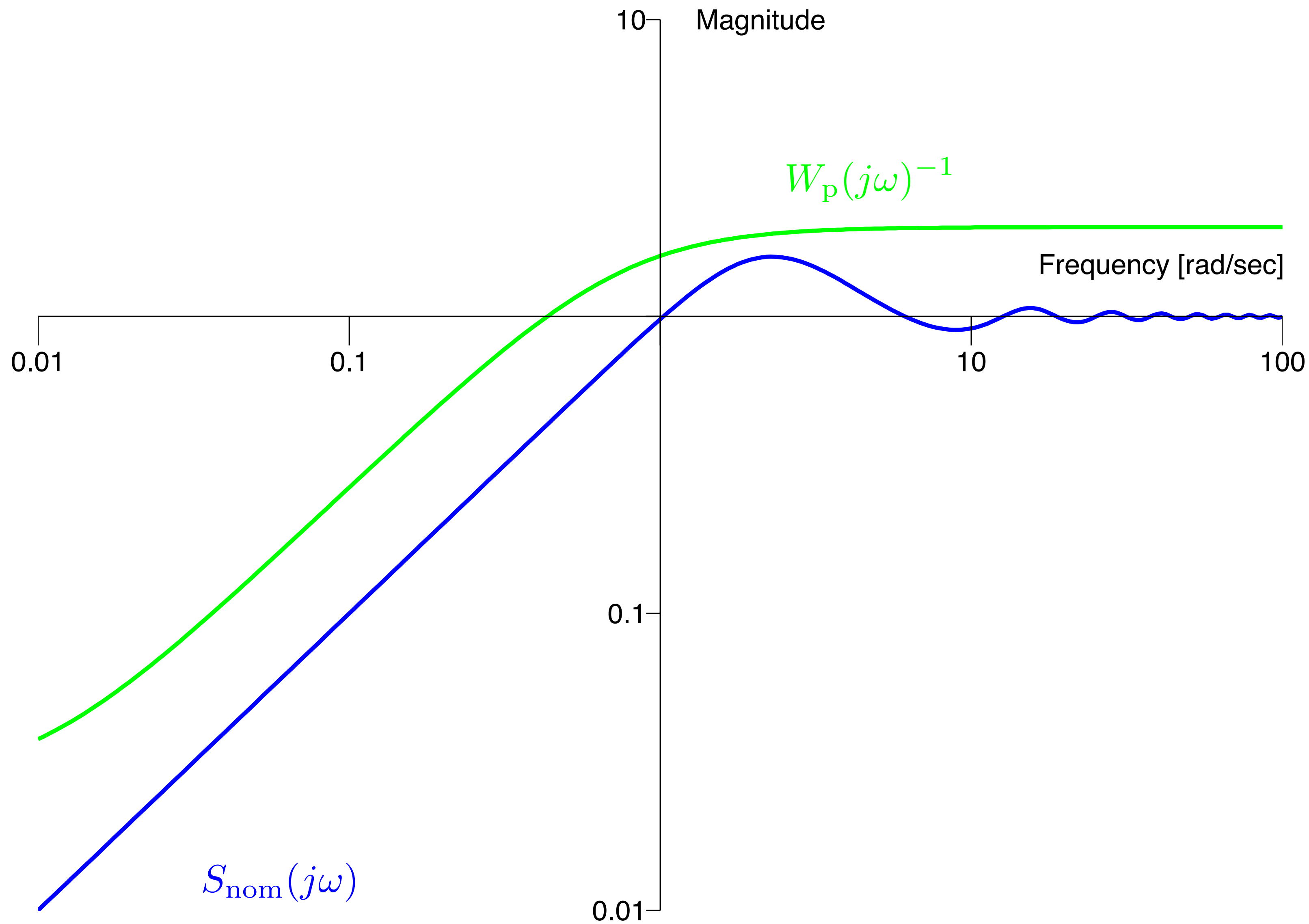


Nominal performance: weighted sensitivity

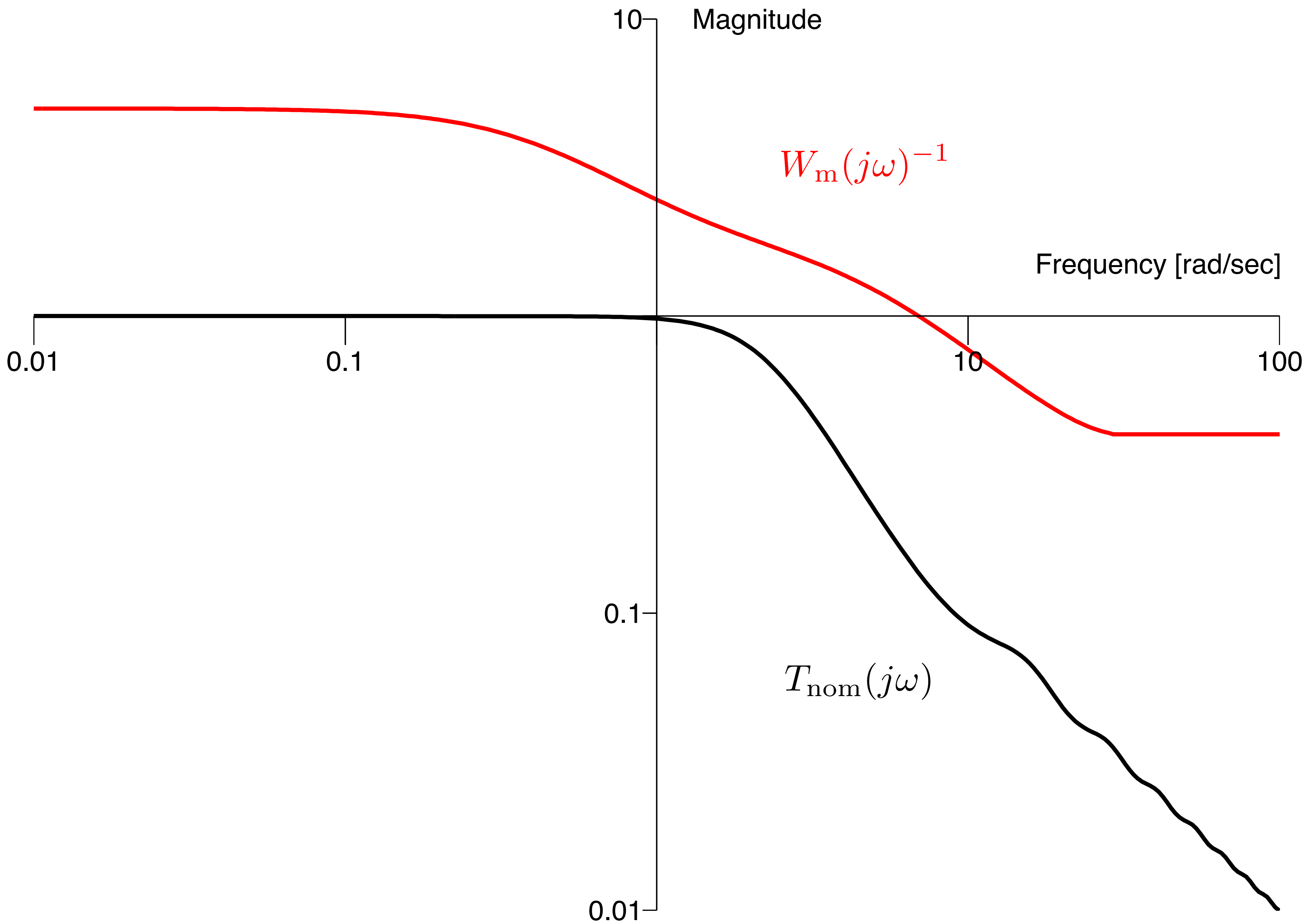
```
Snom_w = 1 ./ (1 + Lnom_w);
Tnom_w = 1 - Snom_w;

% Specifiy a performance bound:

Wp = (s+0.75)/(2*s + 0.02);
Wp_w = squeeze(freqresp(Wp,omega));
invWp = 1/Wp;
invWp_w = squeeze(freqresp(invWp,omega));

% We now check that |Wp(jw)* S(jw)| < 1.  If so, the nominal performance
% specification has been achieved.
```

PI design: nominal performance specification

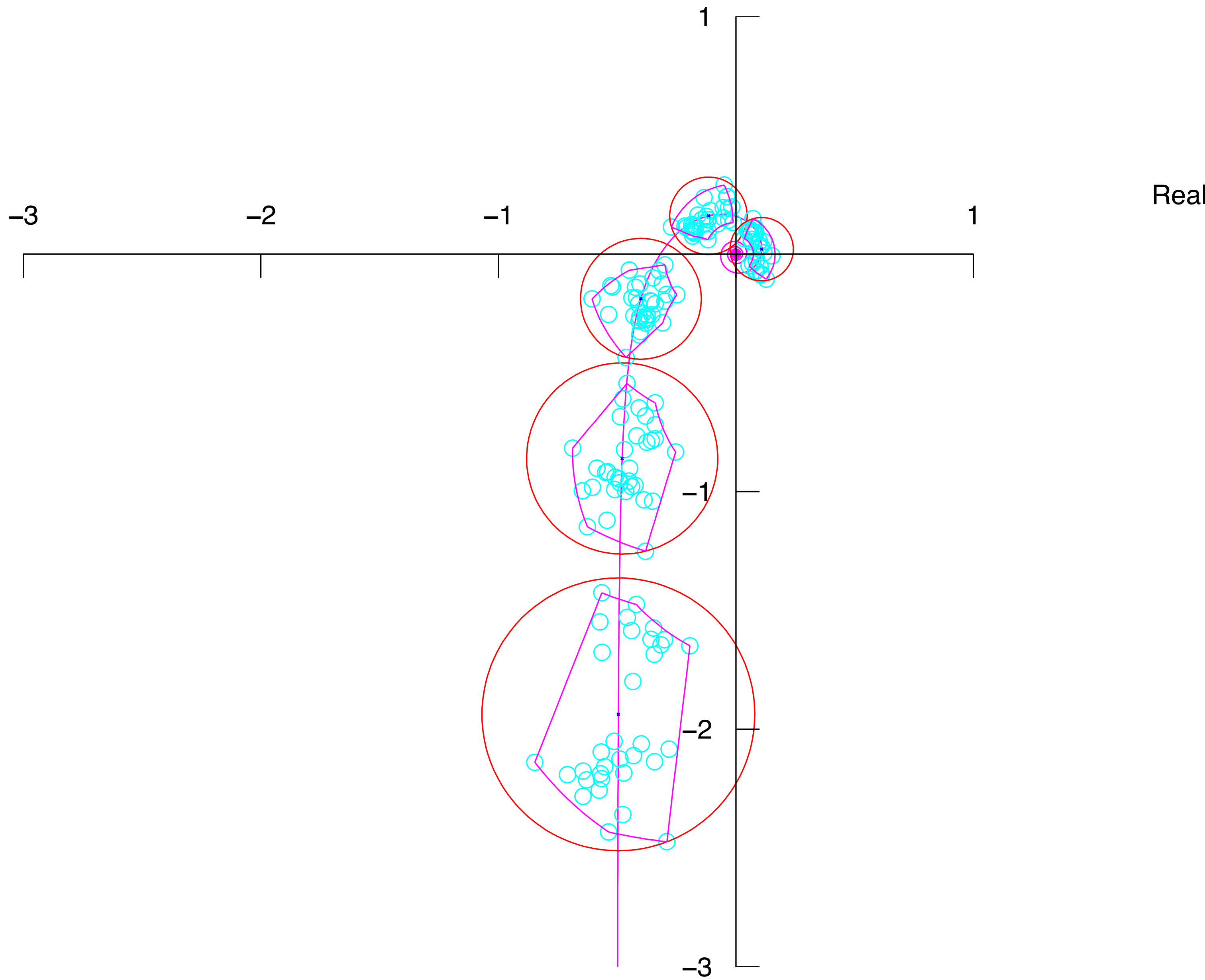
Nominal performance: weighted sensitivity



Robust stability: weighted complementary sensitivity



Robust stability: the perturbation disks never touch (or include) the -1 point.

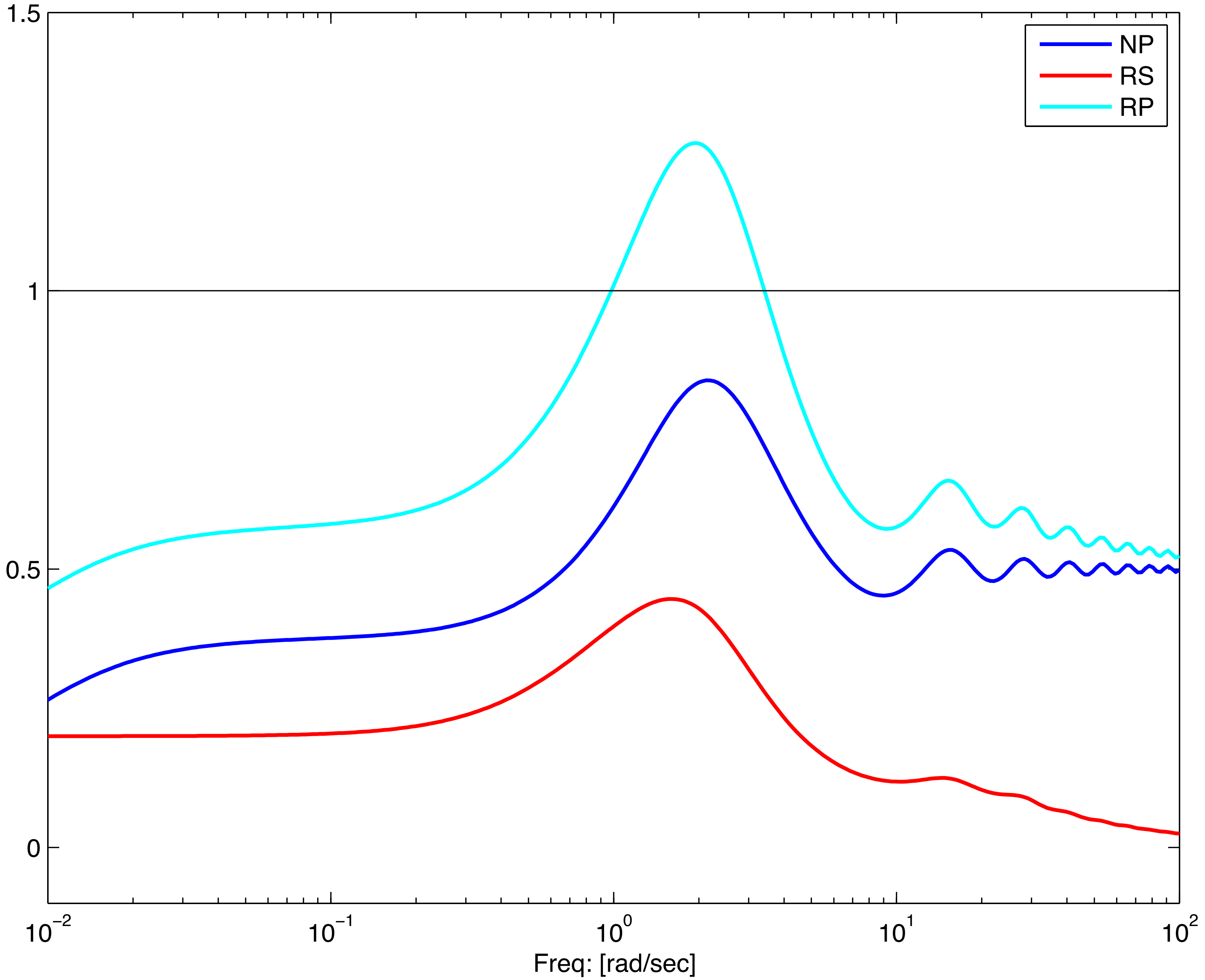


PI design: robust performance analysis via the structured singular value

```
% number of performace outputs.
ne = 1;
% number of exogenous inputs.
nw = 1;
% robust stability perturbation definition.
RSblk = [1,1];
% robust performance perturbation definition.
RPblk = [RSblk;
          nw,ne];
```

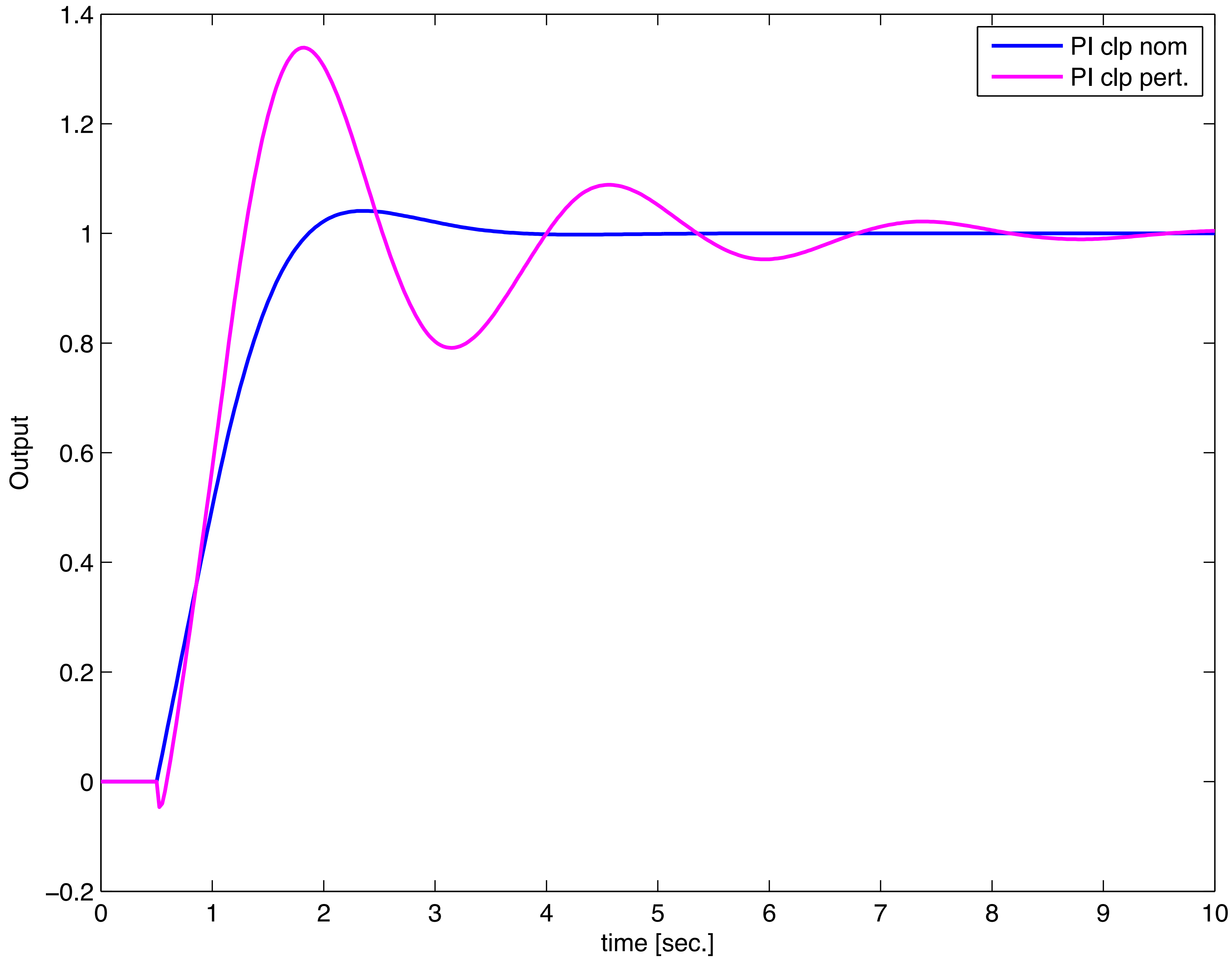
```
% Now look at robust stability and robust performance

   [RSbnds,RSmuinfo] = mussv(Pclp_w(1,1,:),RSblk);

   [RPbnds,RPmuinfo] = mussv(Pclp_w,RPblk);
```

PI design: robust performance



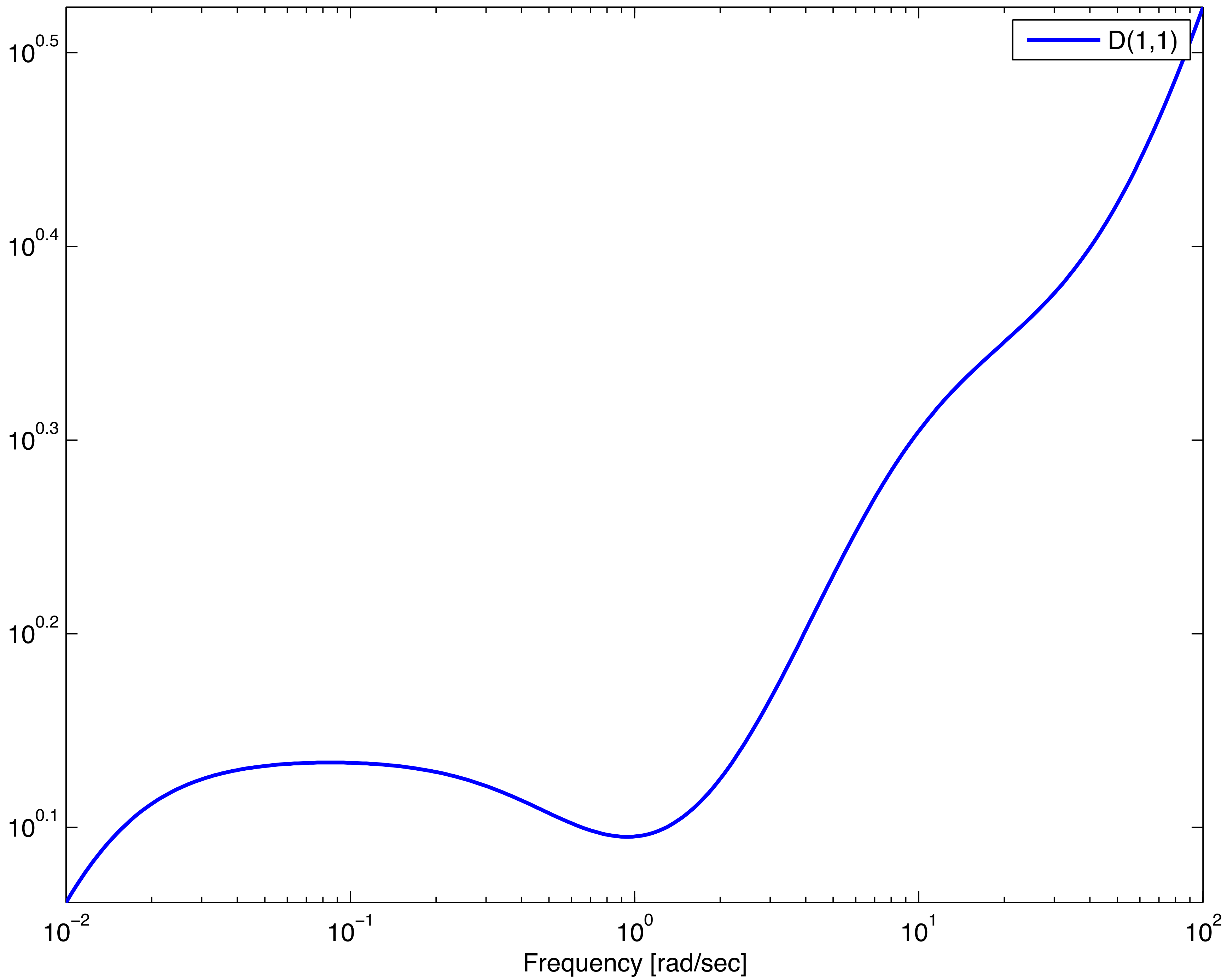
Nominal and worst case step responses

PI design: step responses



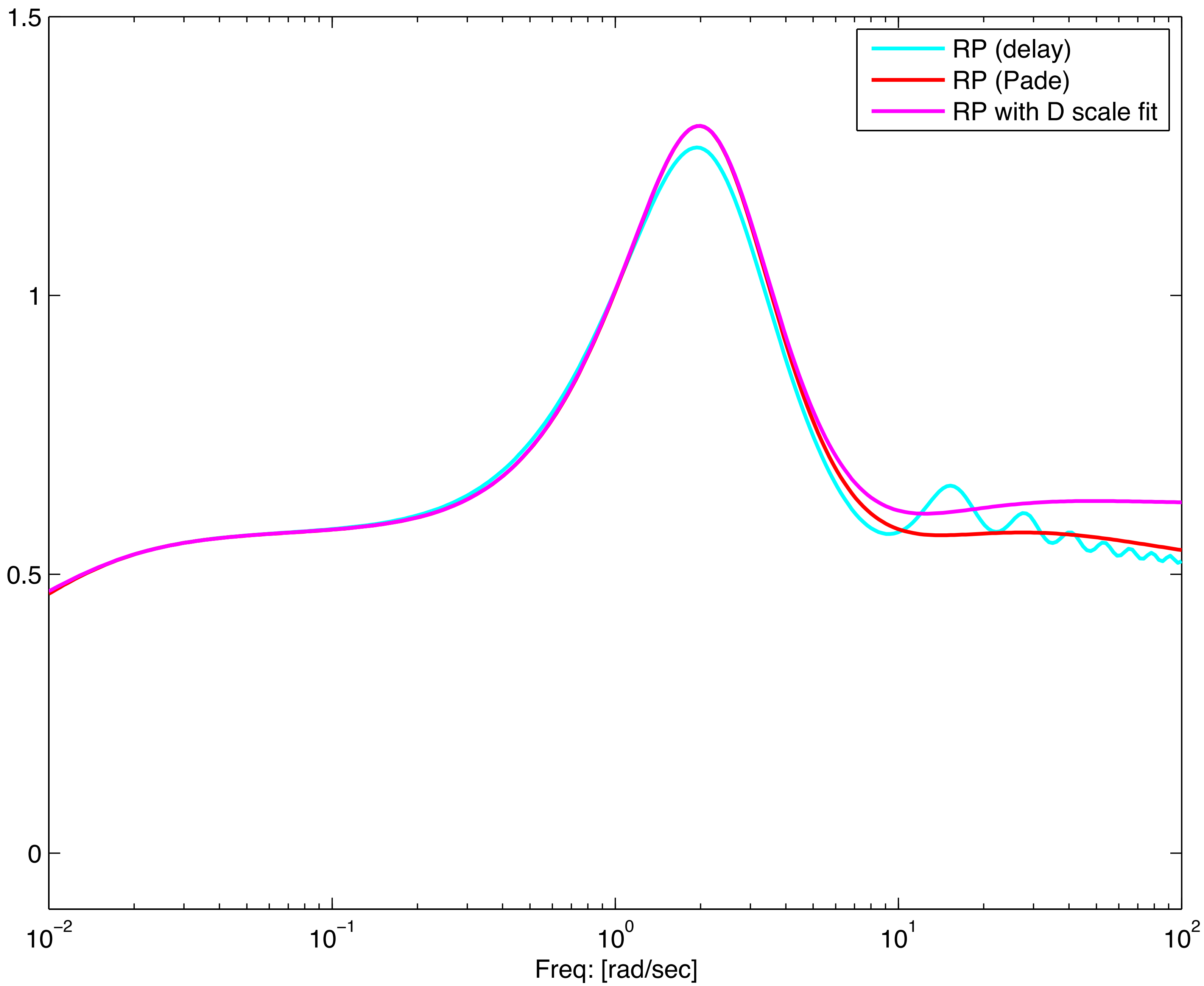
D scaling (normalized) for robust performance

D-scale for the robust performance analysis



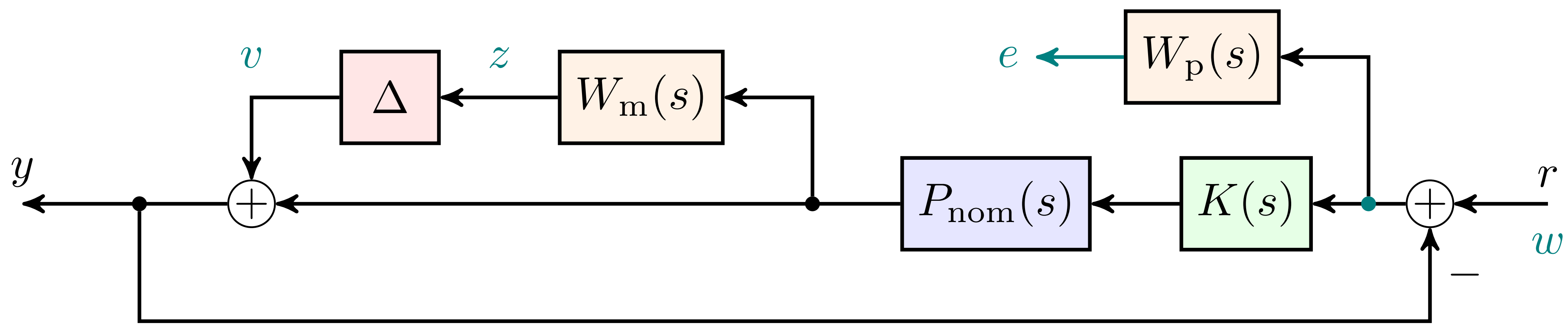
D scale fitting: upper bound comparison

Comparing $\sigma_{\max}\left(D_{\text{fit}}(j\omega)F_l(P_{\text{ic}}(j\omega),K(j\omega))D_{\text{fit}}(j\omega)^{-1}\right)$ with $\mu\left(F_l(P_{\text{ic}}(j\omega),K(j\omega))\right)$



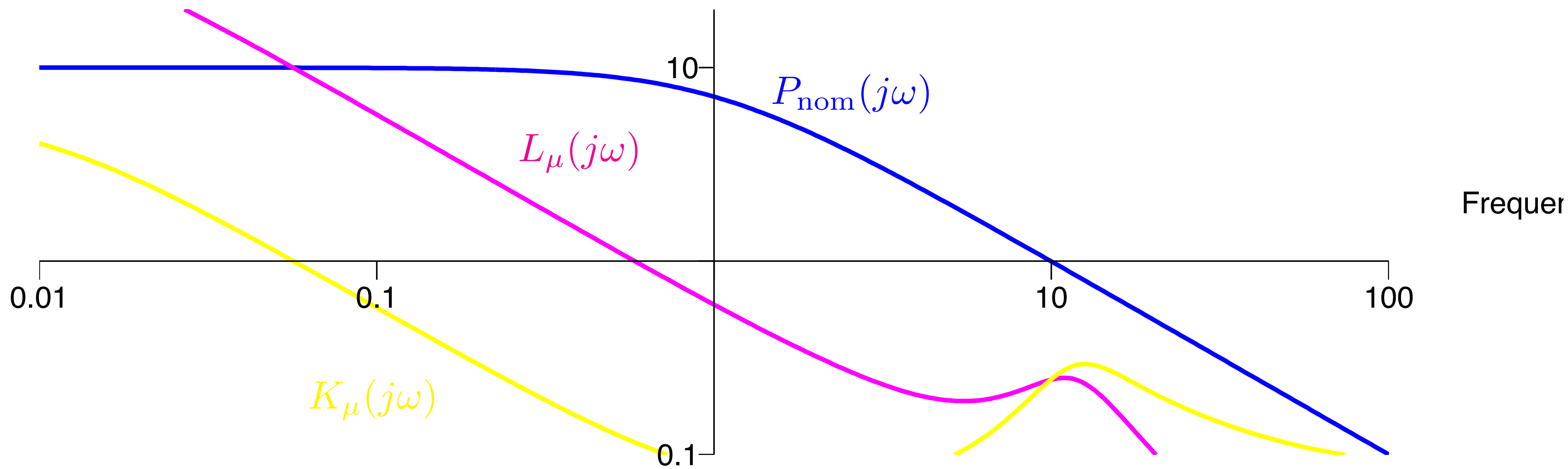
Both frequency domain and state-space interconnections are shown

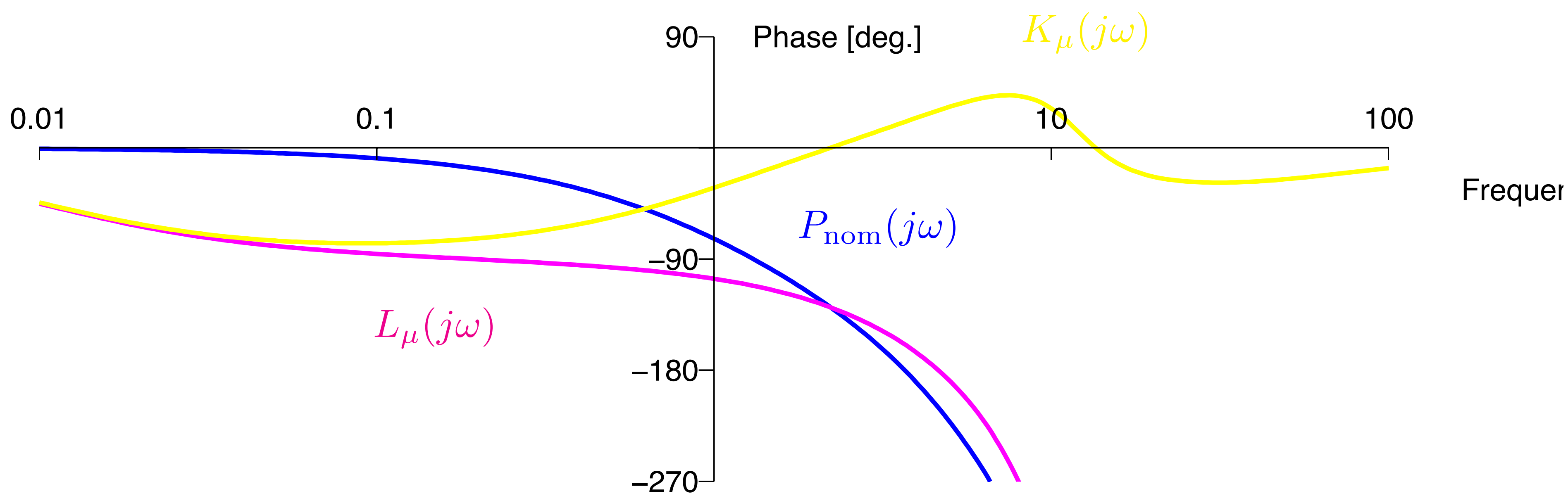
H-infinty design structure




```
dscalePic_ss = daug(D1scale,1) * Pic_ss * daug(invD1scale,1);
nu = 1;
ny = 1;

[Kmu1,Gmu1,gamma1,info1] = hinfsyn(dscalePic_ss,ny,nu,...
                  'GMAX',1.6,...
                  'METHOD','ric',...
                  'DISPLAY','on',...
                  'TOLGAM',0.1);
```

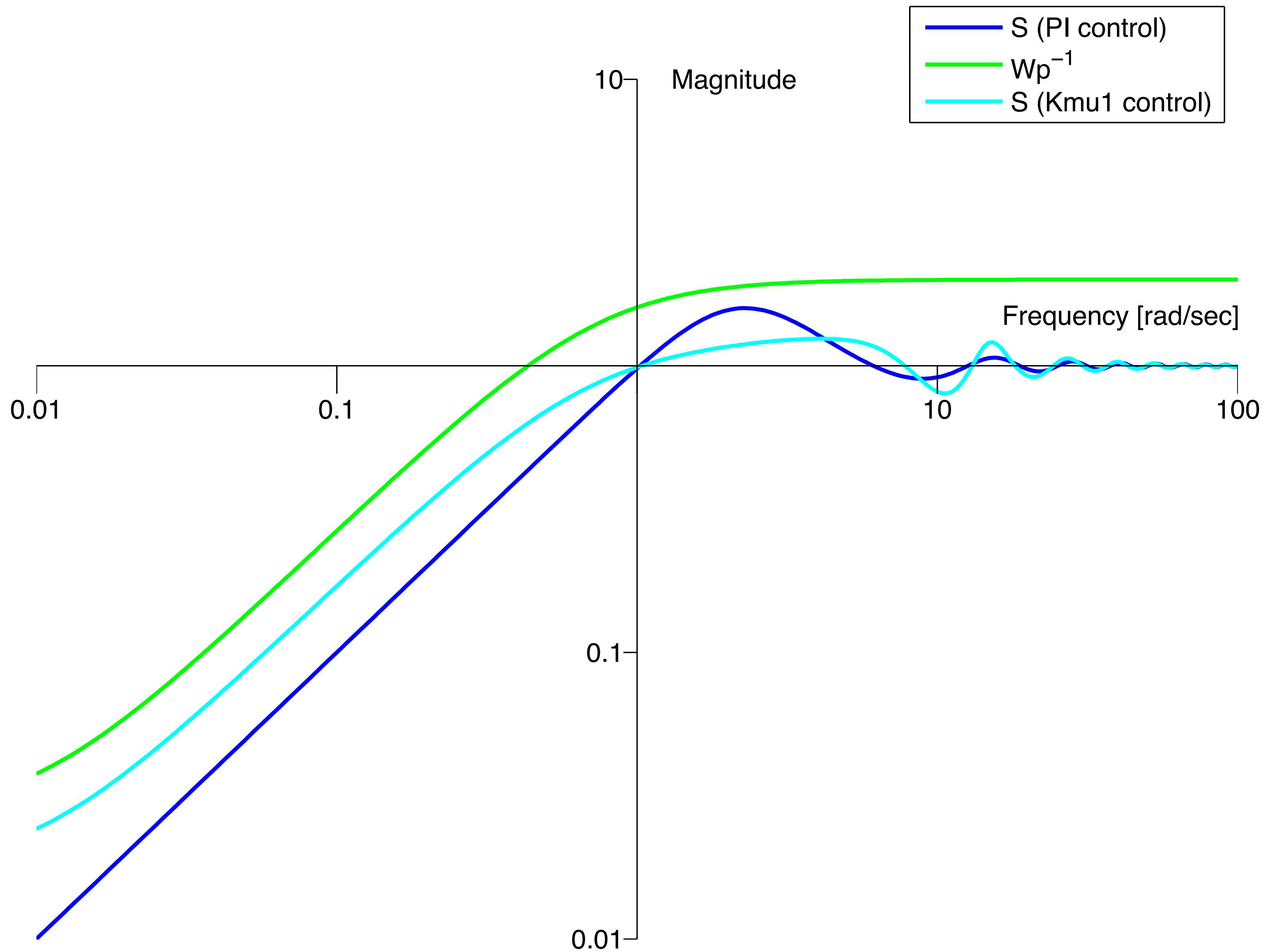
Mu controller: Bode analysis





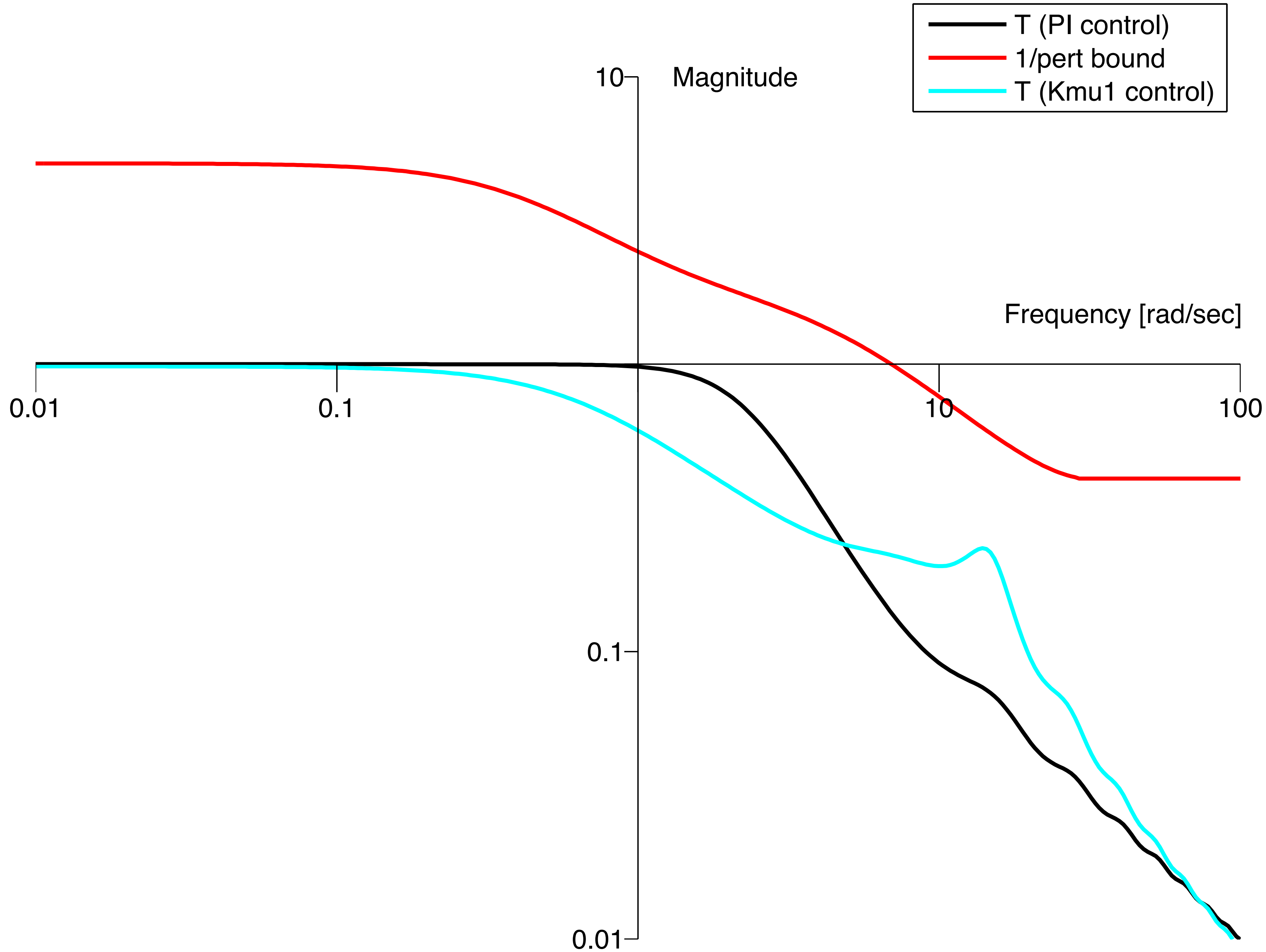
Mu controller: nominal performance

Nominal performance comparison: $K_{\mu}(s)$ and $K_{\rm PI}(s)$



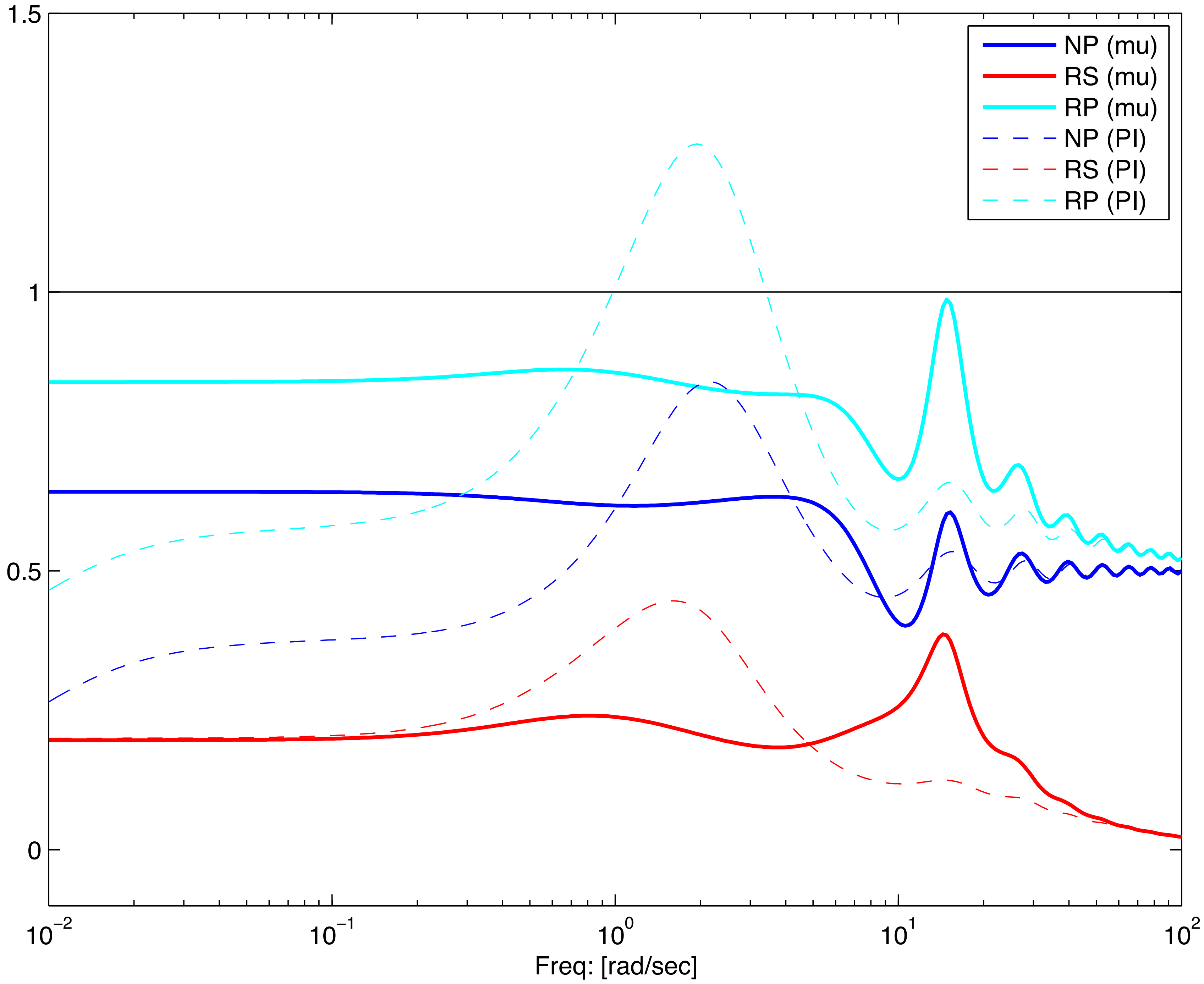
Mu controller: robust stability

Robust stability comparison: $K_\mu(s)$ and $K_{\rm PI}(s)$



Mu controller: NP, RS and RP

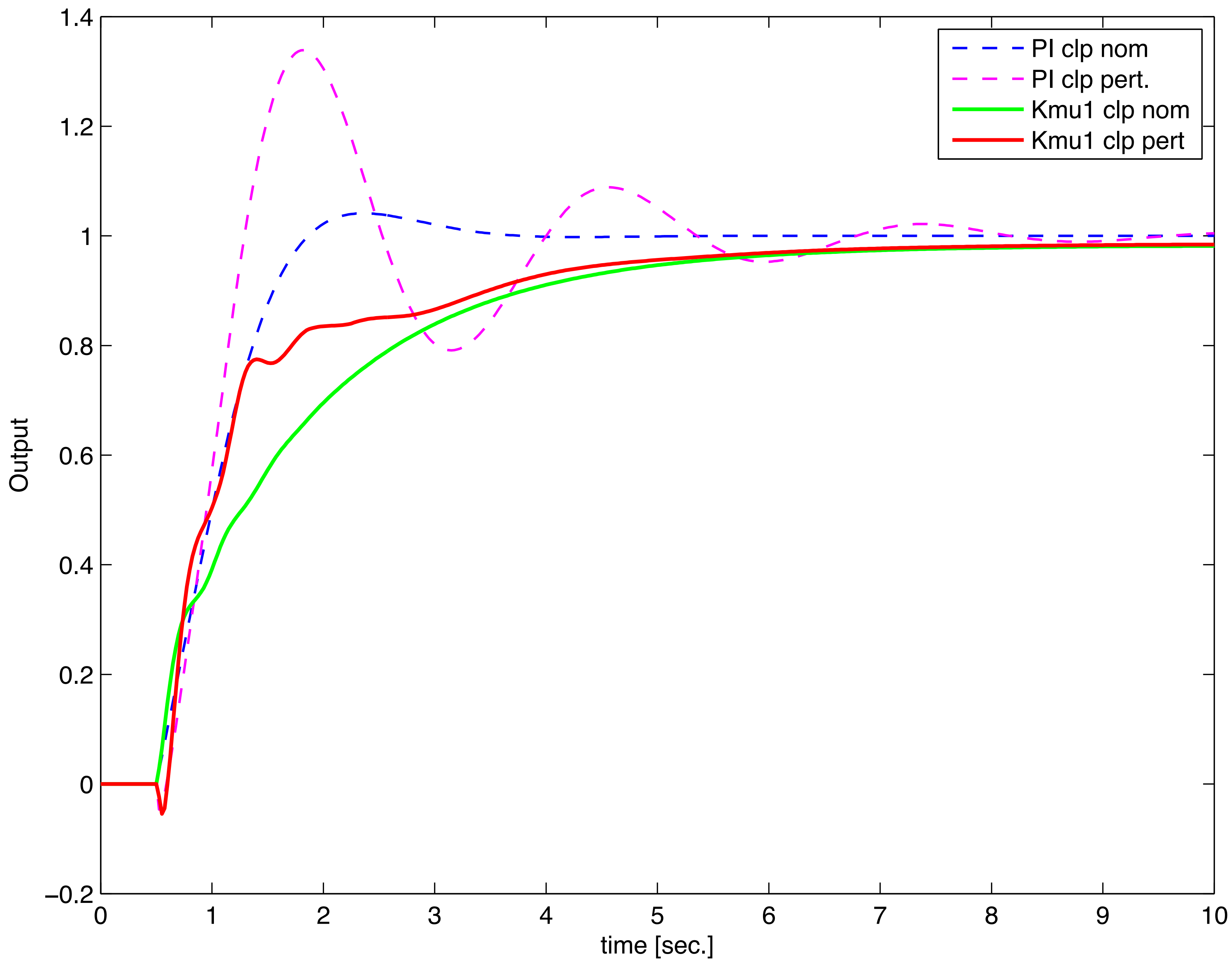
Robust performance comparison: $K_{\mu}(s)$ and $K_{\mathrm{PI}}(s)$



Mu controller: nominal and perturbed step responses

Step response comparison: $K_{\mu}(s)$ and $K_{\mathrm{PI}}(s)$

Worst-case perturbation is calculated for the PI controller



Next steps…

1. Include explicit actuation penalty (and penalize high frequency control action)

2. Include weighted noise on the measured signal.

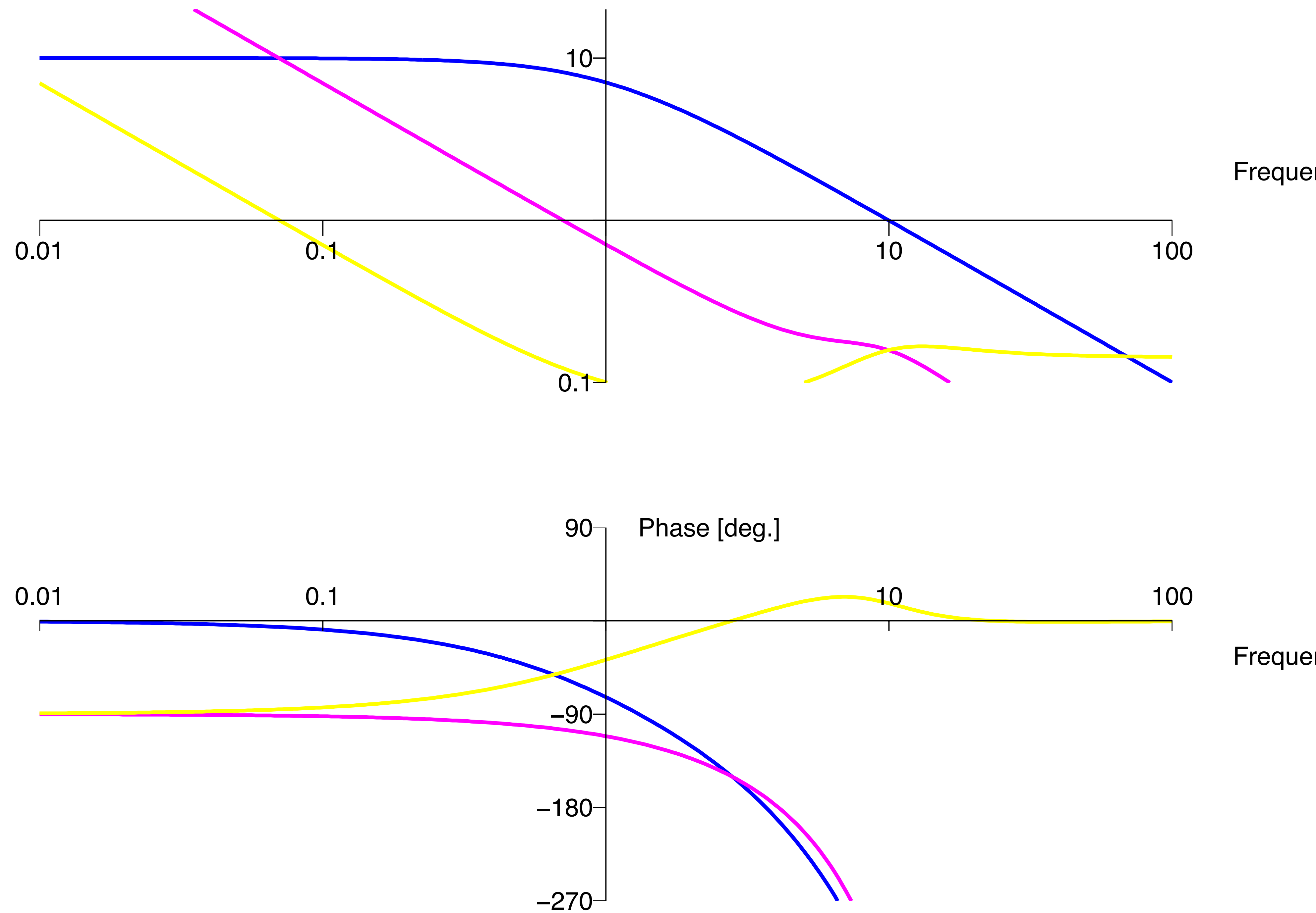
3. Provide both the reference and measurement to the controller (2-DOF structure).

4. Use H-infinity loop shaping to improve the robustness margins

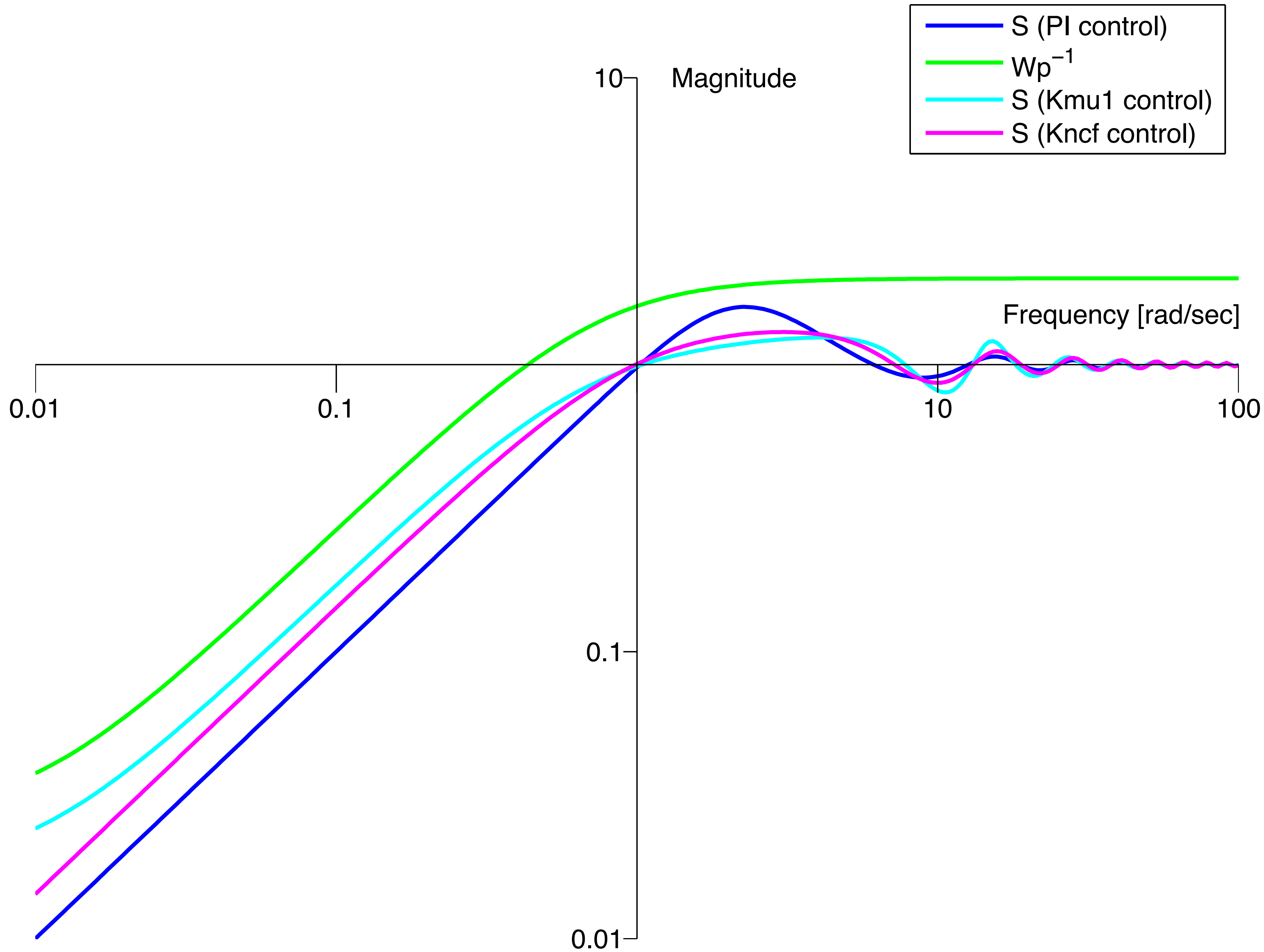
```
W2 = Cpi;
[Kncf_neg,Clpncf,gamma,info] = ncfsyn(Pnom_ss2,1,W2);

% account for unity gain positive feedback in ncfsyn
Kncf = -Kncf_neg;
```

```
gamma = 1.7444e+00
```

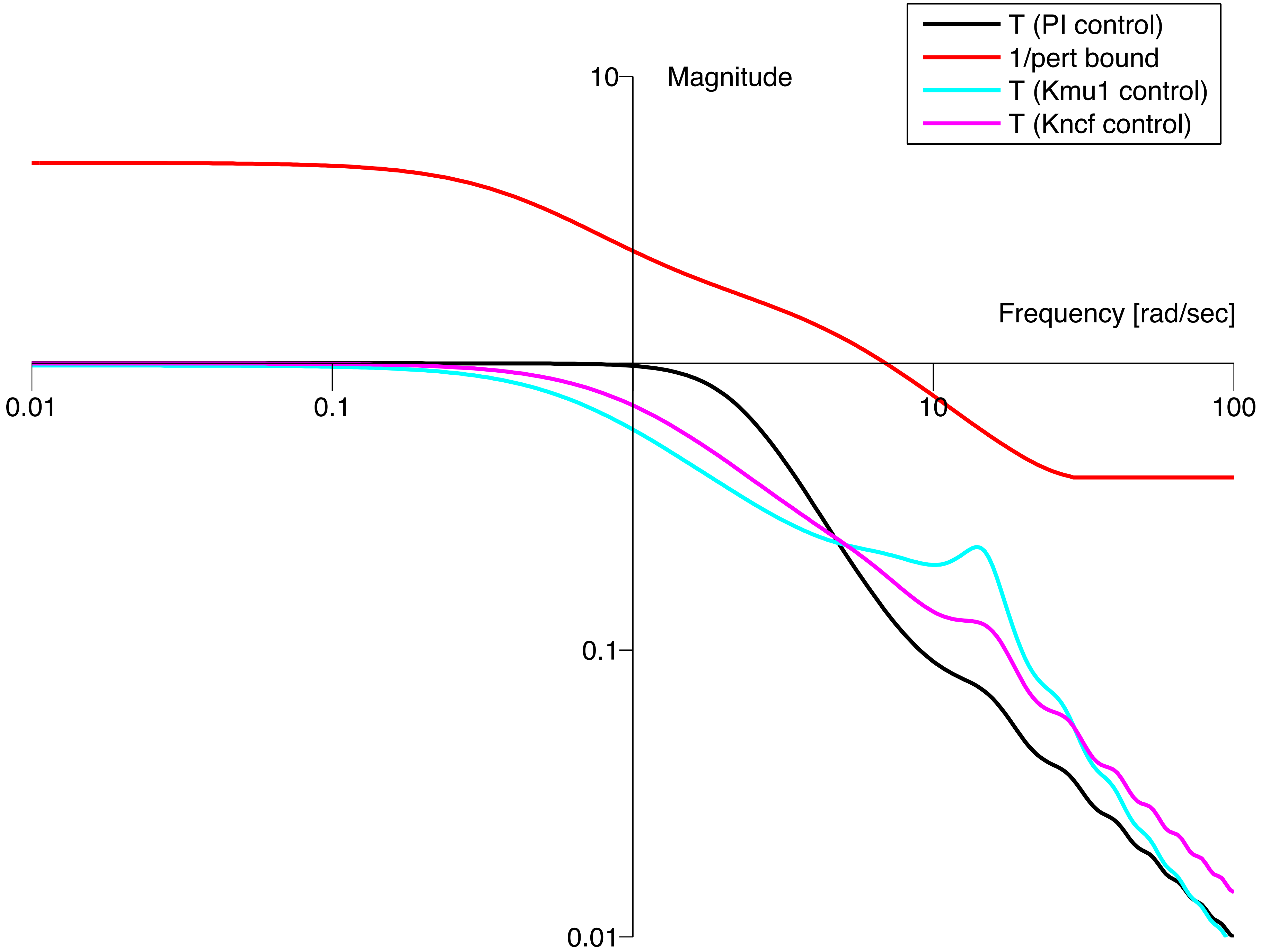
H-infinity loopshaping



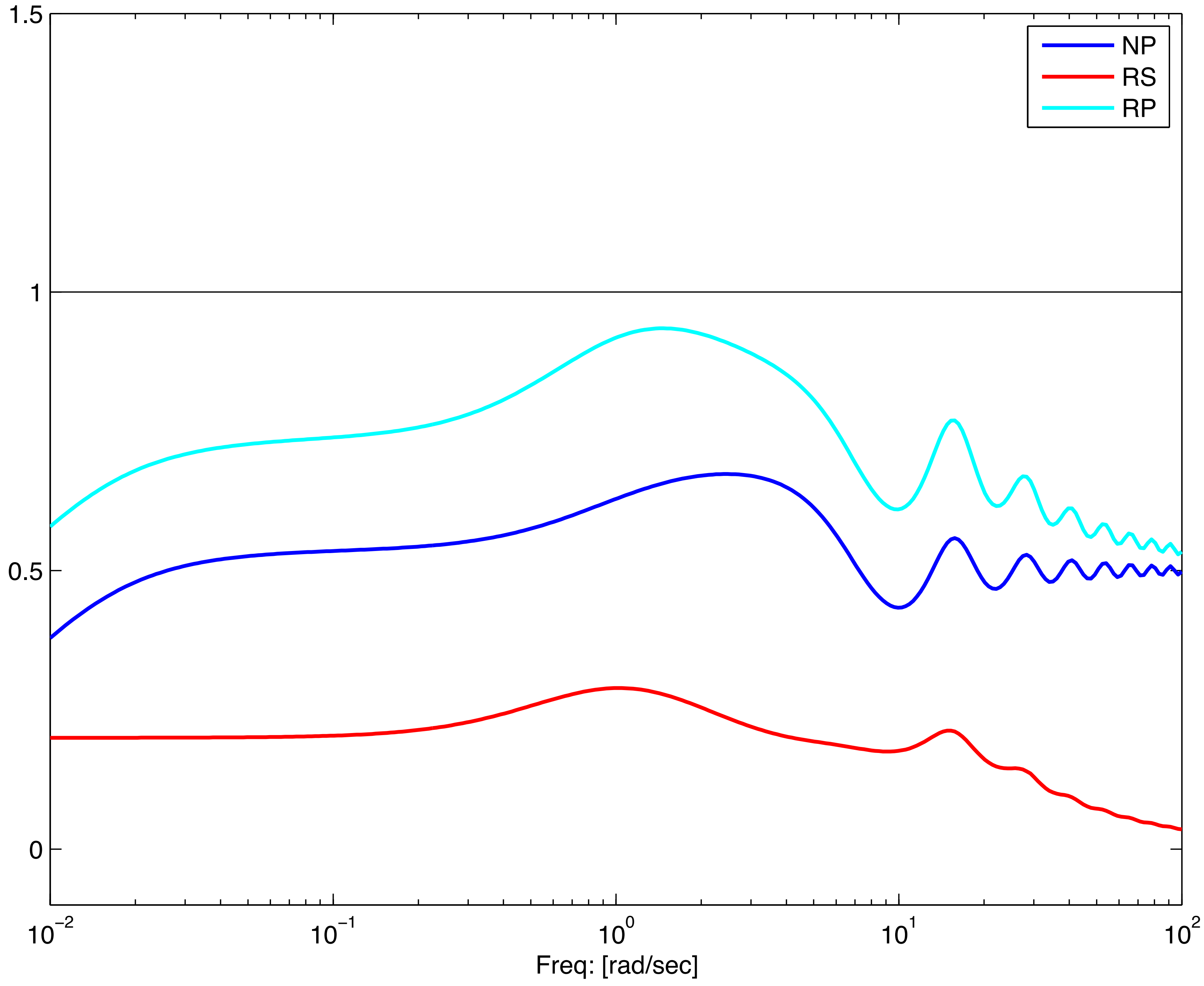
Nominal performance comparison: $K_{\mathrm{PI}}(s)$, $K_{\mu}(s)$, and $K_{\mathrm{NCF}}(s)$



Robust stability comparison: $K_{\mathrm{PI}}(s)$, $K_{\mu}(s)$, and $K_{\mathrm{NCF}}(s)$

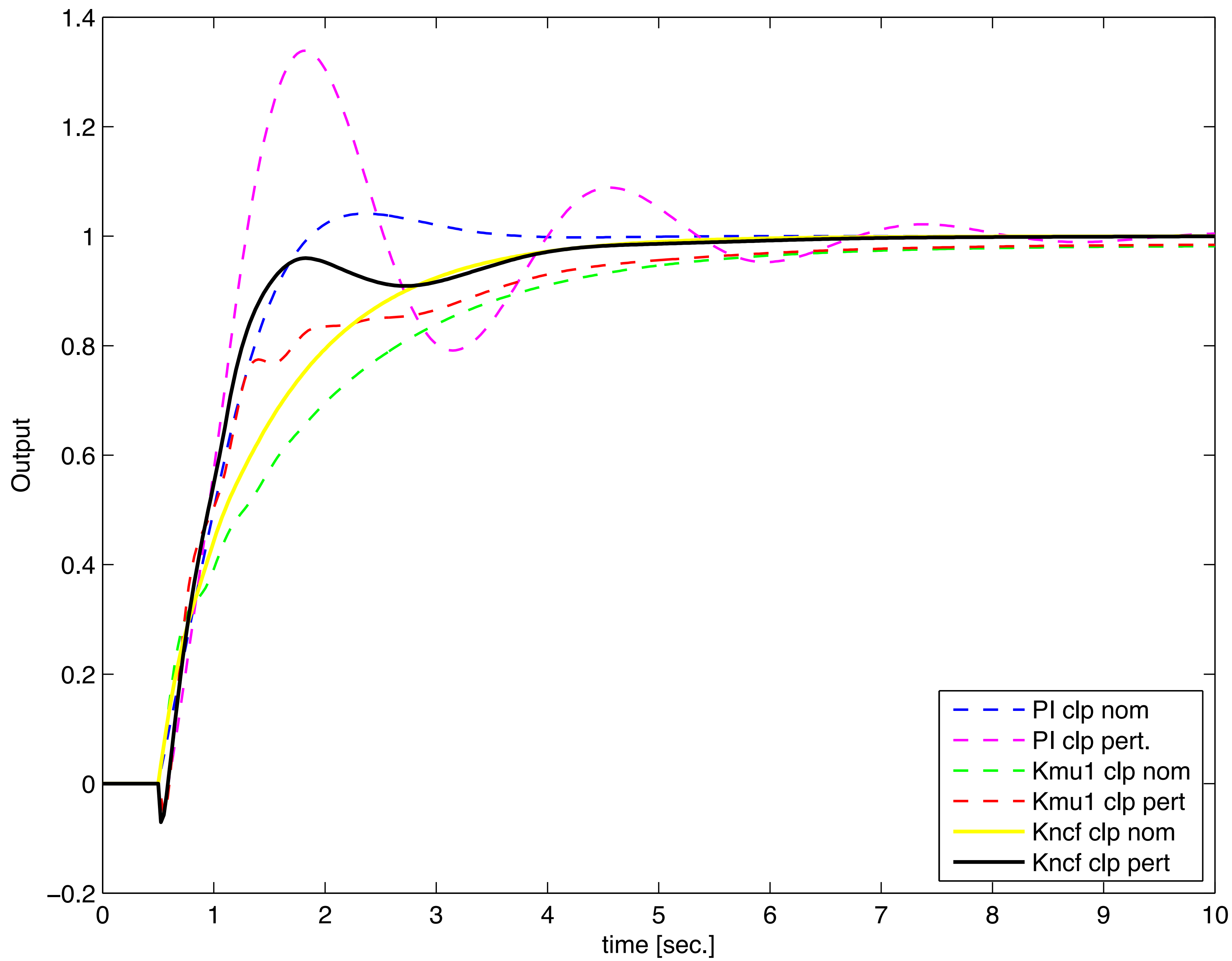


Robust performance comparison: $K_{\mathrm{PI}}(s)$, $K_{\mu}(s)$, and $K_{\mathrm{NCF}}(s)$



Step response comparison: $K_{\mathrm{PI}}(s)$, $K_{\mu}(s)$, and $K_{\mathrm{NCF}}(s)$

Worst-case perturbation is calculated for the PI controller



Re-entry Vehicle Flight Control System

Re-entry Vehicle Flight Control System

- Re-entry Vehicle
- Re-entry Vehicle Lateral-Directional Equations of Motion
- Aerodynamic Coefficient Uncertainty
- Control Problem Formulation
  - Requirements, Problem Formulation
- $\mathcal{H}_\infty$ and μ Synthesis Controllers
  - Robust Stability, Robust Performance and Worst-Case Analysis
- Summary

Re-entry Vehicle (RV)

The re-entry vehicle is similar to the X-38 emergency crew return vehicle (CRV) for the International Space Station.*

- CRV glides from orbit unpowered, steerable parafoil parachute for landing.
- Full lifting body flight control system (FCS)
  - Differential body flaps and a rudder for lateral directional control.
  - Symmetric body flaps for longitudinal motion control.
- Aerodynamic coefficients measured in wind tunnel: Nominal with range of variation.

Goal

- Determine the stability robustness and worst-case performance of the re-entry vehicle FCS in the presence of uncertainty in the aerodynamic coefficients.

∗ J. Shin, G.J. Balas, and A.K. Packard, "Worst case analysis of the X-38 crew return vehicle flight control system," *AIAA Journal of Guidance, Dynamics and Control*, vol. 24, no. 2, March-April 2001, pp. 261-269.

RV Lateral-Directional Model

Assumptions: pitch rate is constant, separation of rigid body motion axes.

$$I_{xx}\dot{p} - I_{xz}\dot{r} = l + (I_{yy}r + I_{xz}p - I_{zz}r)q \quad (1)$$

$$-I_{xz}\dot{p} + I_{zz}\dot{r} = n + (I_{xx}p - I_{xz}r - I_{yy}p)q \quad (2)$$

$$\dot{\phi} = p + \tan(\theta)r \quad (3)$$

$$\dot{\beta} = Y_b\beta + (\frac{w_0}{V} + Y_p)p + (Y_r - \frac{u_0}{V})r + Y_{da}da + Y_{dr}dr + \frac{g\cos(\gamma)}{V}\phi \quad (4)$$

g is gravity,, w_0 is $V\sin(\alpha)$, u_0 is $V\cos(\alpha)$, γ is flight path angle.

The roll moment, l, and yaw moment, n, can be written as a function of lateral-directional nondimensional derivatives:

$$l = (QSb)(Cl_{\beta cg}\beta + \frac{b}{2V}Cl_{pcg}p + \frac{b}{2V}Cl_{rcg}r + Cl_{dacg}da + Cl_{drcg}dr)$$

$$n = (QSb)(Cn_{\beta cg}\beta + \frac{b}{2V}Cn_{pcg}P + \frac{b}{2V}Cn_{rcg}r + Cn_{dacg}da + Cn_{drcg}dr) \quad (5)$$

The subindex cg represents the re-entry vehicle center of the gravitational point.

RV Lateral-Directional Model cont'd

The derivatives at the center of the gravitational point are derived from the derivatives at re-entry vehicle aerodynamic center

$$Cl_{icg} = Cl_i - \frac{Z_f}{b} Cy_i$$
$$Cn_{icg} = Cn_i + \frac{X_f}{b} Cy_i, \qquad i = \beta,\ p,\ r,\ da,\ dr$$

where Z_f (ft) and X_f (ft) are positions of the center of the gravitational point of re-entry vehicle from the aerodynamic point.

Combining equations leads to roll rate and yaw rate equations:

$$\dot{p} = D_I[l + \frac{q}{I_{xx}}(I_{yy}r + I_{xz}p - I_{zz}r) + \frac{I_{xz}}{I_{xx}}\{n + \frac{q}{I_{zz}}(I_{xx}p - I_{xz}r - I_{yy}p)\}]$$
$$\dot{r} = D_I[\frac{I_{xz}}{I_{zz}}\{l + \frac{q}{I_{xx}}(I_{yy}r + I_{xz}p - I_{zz}r)\} + n + \frac{q}{I_{zz}}(I_{xx}p - I_{xz}r - I_{yy}p)]$$

where D_I is $D_I = (1 - \frac{I_{xz}I_{xz}}{I_{xx}I_{zz}})^{-1}$.

RV Aerodynamic Coefficient Uncertainties

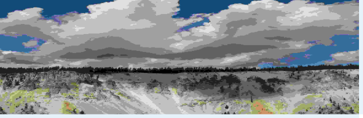
Re-entry vehicle aerodynamic data have uncertainties in these nondimensional stability derivatives: Cl_β, Cl_{da}, Cl_{dr}, Cy_β, Cy_{da}, Cy_{dr}, Cn_β, Cn_{da}, and Cn_{dr}. Uncertainty in stability derivatives can be described by a nominal aerodynamic derivative with a bounded range of possible values. For example Cl_β can be described as

$$Cl_\beta := Cl_{\beta_{\min}} \leq Cl_\beta \leq Cl_{\beta_{\max}}$$

Within the Robust Control Toolbox, the uncertain parameter Cl_β would be represented as a `ureal` object

```
CLbeta = ureal('CLbeta',CLbetaNom,'Range',[CLbetaMin CLbetaMax]);
```

where `CLbetaNom` corresponds to the nominal value of Cl_β and `CLbetaMin` and `CLbetaMax` correspond to $Cl_{\beta_{\min}}$ and $Cl_{\beta_{\max}}$ respectively. All 9 stability derivatives are represented as uncertain real parameters (`ureal` objects) in the analysis.



RV Lateral-Directional Linearized Model

RV FCS
RV
RV Lat-Dir Model
RV Uncertainty
RV Lat-Dir Linearized
RV Control

The output variables are β, p, r, ϕ and N_y,

$$N_y = N_{ycg} + x_a \dot{r} - z_a \dot{p}. \tag{6}$$

where x_a (ft) and z_a (ft) are the positions of the acceleration sensor. The equations of the linearized lateral-directional motion are (coefficients in blue are uncertain):

$$\begin{bmatrix} \dot\beta \\ \dot p \\ \dot r \\ \dot\phi \end{bmatrix} = \begin{bmatrix} A_{\dot\beta\beta} & A_{\dot\beta p} & A_{\dot\beta r} & \frac{g\cos(\gamma)}{V} \\ A_{\dot p\beta} & A_{\dot p p} & A_{\dot p r} & 0 \\ A_{\dot r\beta} & A_{\dot r p} & A_{\dot r r} & 0 \\ 0 & 1 & \tan(\theta) & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} B_{\dot\beta da} & B_{\dot\beta dr} \\ B_{\dot p da} & B_{\dot p dr} \\ B_{\dot p da} & B_{\dot p dr} \\ B_{\dot r da} & B_{\dot r dr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} da \\ dr \end{bmatrix}$$

$$\begin{bmatrix} \beta \\ p \\ r \\ \phi \\ N_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_{n_y\beta} & C_{n_y p} & C_{n_y r} & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ D_{n_y da} & D_{n_y dr} \end{bmatrix} \begin{bmatrix} da \\ dr \end{bmatrix}$$

(See the M-file `RVlatmodel.m`)

Elements of state space model are (blue denotes uncertain):

$$A_{\dot{\beta}\beta} = \frac{QSCy_\beta}{massV}$$

$$A_{\dot{\beta}p} = \sin(\alpha) + \frac{QSbCy_p}{2massV^2}$$

$$A_{\dot{\beta}r} = \frac{QSbCy_r}{2massV^2} - \cos(\alpha)$$

$$A_{\dot{p}\beta} = a_1\{Cl_\beta + (\frac{I_{xz}X_f}{I_{zz}b} - \frac{Z_f}{b})Cy_\beta + \frac{I_{xz}}{I_{zz}}Cn_\beta\}$$

$$A_{\dot{p}p} = \frac{a_1}{2V}\{Cl_p + (\frac{I_{xz}X_f}{I_{zz}b} - \frac{Z_f}{b})Cy_p + \frac{I_{xz}}{I_{zz}}Cn_p + \frac{2I_{pp}q_0VI_{xx}}{QSb^2}\}$$

$$A_{\dot{p}r} = \frac{a_1}{2V}\{Cl_r + (\frac{I_{xz}X_f}{I_{zz}b} - \frac{Z_f}{b})Cy_r + \frac{I_{xz}}{I_{zz}}Cn_r + \frac{2I_{pr}q_0VI_{xx}}{QSb^2}\}$$

$$A_{\dot{r}\beta} = a_2\{Cn_\beta + (\frac{X_f}{b} - \frac{I_{xz}Z_f}{I_{xx}b})Cy_\beta + \frac{I_{xz}}{I_{xx}}Cl_\beta\}$$

$$A_{\dot{r}p} = \frac{a_2}{2V}\{Cn_p + (\frac{X_f}{b} - \frac{I_{xz}Z_f}{I_{xx}b})Cy_p + \frac{I_{xz}}{I_{xx}}Cl_p + \frac{2I_{rp}q_0VI_{zz}}{QSb^2}\}$$

$$A_{\dot{r}r} = \frac{a_2}{2V}\{Cn_r + (\frac{X_f}{b} - \frac{I_{xz}Z_f}{I_{xx}b})Cy_r + \frac{I_{xz}}{I_{xx}}Cl_r + \frac{2I_{rr}q_0VI_{zz}}{QSb^2}\}$$

$$B_{\dot{p}da} = a_1\{Cl_{da} + (\frac{I_{xz}X_f}{I_{zz}b} - \frac{Z_f}{b})Cy_{da} + \frac{I_{xz}}{I_{zz}}Cn_{da}\}$$

$$B_{\dot{p}dr} = a_1\{Cl_{dr} + (\frac{I_{xz}X_f}{I_{zz}b} - \frac{Z_f}{b})Cy_{dr} + \frac{I_{xz}}{I_{zz}}Cn_{dr}\}$$

$$B_{\dot{r}da} = a_2\{Cn_{da} + (\frac{X_f}{b} - \frac{I_{xz}Z_f}{I_{xx}b})Cy_{da} + \frac{I_{xz}}{I_{xx}}Cl_{da}\}$$

$$B_{\dot{r}dr} = a_2\{Cn_{dr} + (\frac{X_f}{b} - \frac{I_{xz}Z_f}{I_{xx}b})Cy_{dr} + \frac{I_{xz}}{I_{xx}}Cl_{dr}\}$$

$$B_{\dot{\beta}da} = \frac{QSCy_{da}}{massV}, \qquad B_{\dot{\beta}dr} = \frac{QSCy_{dr}}{massV}$$

$$C_{n_y\beta} = VA_{\dot{\beta}\beta} + X_aA_{\dot{r}\beta} - Z_aA_{\dot{p}\beta}$$

$$C_{n_yp} = \frac{QSbCy_p}{2massV} + X_aA_{\dot{r}p} - Z_aA_{\dot{p}p}$$

$$C_{n_yr} = \frac{QSbCy_r}{2massV} + X_aA_{\dot{r}r} - Z_aA_{\dot{p}r}$$

$$D_{n_yda} = \frac{QSCy_{da}}{mass} + X_aA_{\dot{r}da} - Z_aA_{\dot{p}da}$$

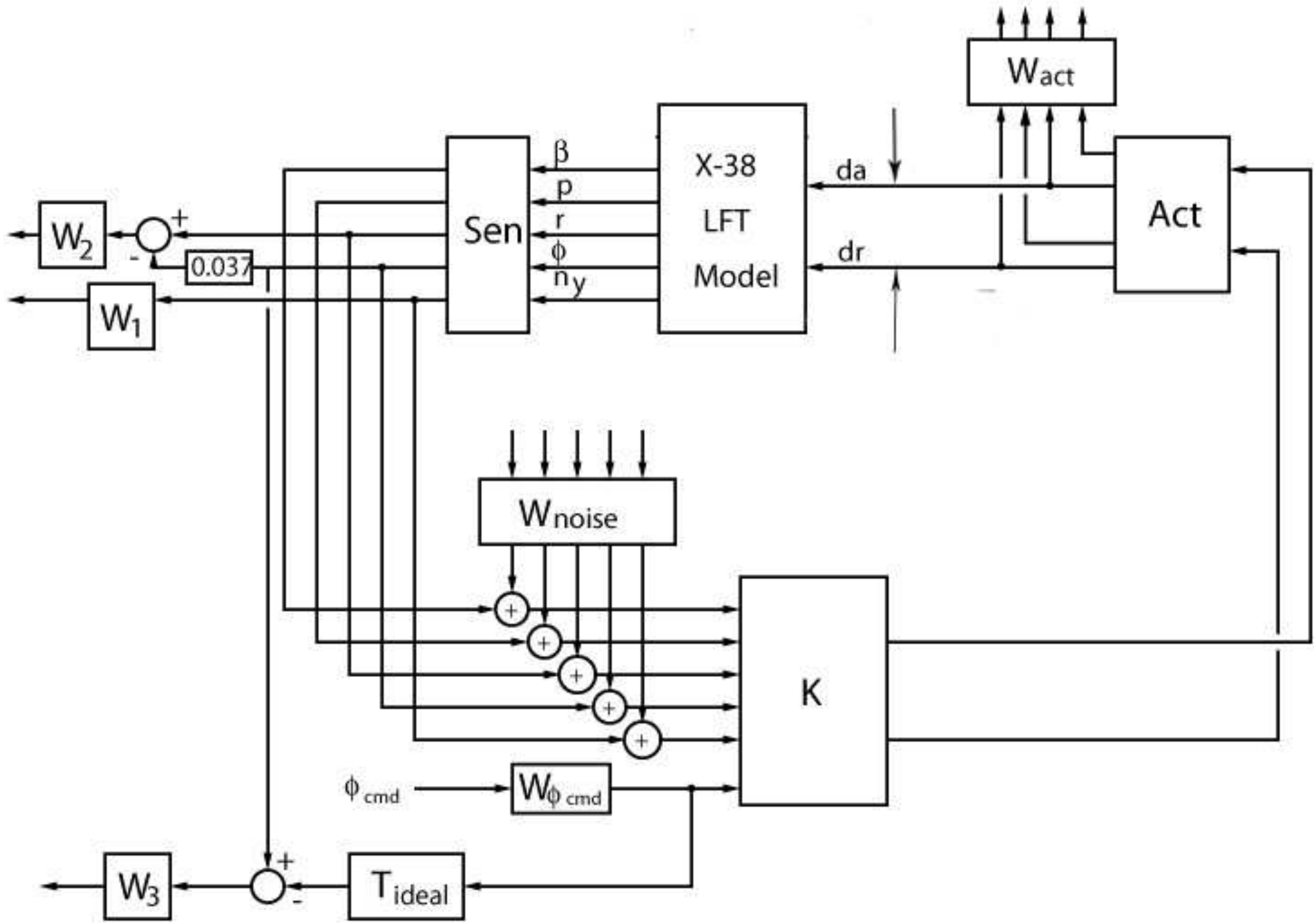
$$D_{n_ydr} = \frac{QSCy_{dr}}{mass} + X_aA_{\dot{r}dr} - Z_aA_{\dot{p}dr}$$

RV Lateral-Directional Control Problem Formulation

The lateral-directional axis control objectives are:

- Good low frequency tracking of ϕ commands (up to $1\ rad/s$), a coordinated turn, and small lateral accelerations.
- Robust to variations in aerodynamic coefficients, exogenous disturbances and sensor noise.

The performance and robustness objectives are characterized as a H_∞ norm minimization of weighted transfer functions.



Wact
X-38
LFT
Model
da
dr
Act
Sen
β
p
r
ϕ
ny
W2
W1
0.037
+
-
Wnoise
K
ϕ cmd
Wϕ cmd
W3
Tideal
+
-

RV Lateral-Directional Control Problem Formulation

H_∞ weighting functions:

- Ideal ϕ command response: $T_{\text{ideal}} = \frac{0.81}{s^2+1.8s+0.81}$
- ϕ command: $W_{\phi_{\text{cmd}}} = \frac{0.1s+1}{2s+1}$
- Minimize ϕ_{cmd} to ϕ_{err}: $W_3 = 10\frac{0.001s+1}{s+1}$
- Minimize n_y: $W_1 = 4$
- Coordinated turn: $W_2 = 5\frac{0.001s+1}{0.5s+1}$
- Input disturbances: 0.5
- Sensors noise (β, p, r, ϕ, n_y): $(0.15, 0.12, 0.05, 0.025, 0.2)$
- Actuator rates/deflections: $\frac{8}{30}, \frac{30}{30}$

(See the M-file `RV_wtolic.m`)

RV Lateral-Directional Control Problem Formulation

Angle rate gyros are modeled as $\frac{66}{s+66}$ and the N_y accelerometer modeled as $\frac{40}{s+40}$.

$$Sen = \begin{bmatrix} \frac{66}{s+66} I_{4\times 4} & 0 \\ 0 & \frac{40}{s+40} \end{bmatrix}$$

EMA actuators are modeled as a 2^{nd} order system, with a prefilter to smooth the discrete ZOH. A transport delay of 0.04 seconds is approximated by a first order Pade delay.

$$Act = \frac{50^2}{s^2 + 70.7s + 50^2} \frac{26^2}{s^2 + 36.8s + 26^2} \frac{50 - s}{20 + s} I_{2\times 2}$$

In the $\mathcal{H}_\infty$ control design, the actuator and time delay are approximated with a first-order lag and Pade approximation of $Act = \frac{20}{s+20} \frac{20-s}{20+s}$

RV Lateral-Directional Control Designs

$\mathcal{H}_\infty$ and μ-synthesis controllers were synthesized for the control problem interconnection shown on the previous slide. The uncertainty in the aerodynamic derivatives was eliminated from the $\mathcal{H}_\infty$ design and were included in the μ-synthesis control designs. The resulting controllers were:

- `Khinf` has 2 outputs, 6 inputs and 16 states.
- `Kmu` has 2 outputs, 6 inputs and 42 states.

`Kmu` was reduced using the `reduce` command with the balanced realization option selected. The reduced order controller, `Kmur` had 10 states. In the following analyses, the full order $\mathcal{H}_\infty$ and reduced order μ controllers are used.

RV Lateral-Directional Controller Analysis

Analyze each controller using a variety of analysis tools.

- `loopmargin`
  - Classical margins from `allmargin` (CM).
  - Disk margin (DM)
  - Multivariable margin (MM)
- `robuststab`, `robustperf`
- `wcgain`

H-infinity and mu Control Design and Analysis of Re-entry Vehicle

CAT/MUSYN shortcourse, May 2014

Contents

Figure: Re-entry Vehicle

A linear model is constructed for the lateral-directional dynamics of a re-entry vehicle. Nine aerodynamic derivatives are modeled as uncertain, real parameters. The uncertain lateral-directional state-space re-entry vehicle model, `RVunc`, has 4 states, 2 inputs, and 5 outputs. The states correspond to beta (rad), p (rad/s), r (rad/s), and phi (rad). The outputs are the states plus lateral acceleration, ny (ft/s^2). The inputs are deflections of the flaps, da (deg), and rudder, dr (deg).

```
RVunc = RVlatmodel;
RVunc
```

```
RVunc =

  Uncertain continuous-time state-space model with 5 outputs, 2 inputs, 4 states.
  The model uncertainty consists of the following blocks:
    Clb: Uncertain real, nominal = -0.115, variability = [-0.05,0.05], 1 occurrences
    Clda: Uncertain real, nominal = 0.0115, variability = [-0.0025,0.0025], 1 occurrences
    Cldr: Uncertain real, nominal = 0.023, variability = [-0.01,0.01], 1 occurrences
    Cnb: Uncertain real, nominal = 0.049, variability = [-0.02,0.02], 1 occurrences
    Cnda: Uncertain real, nominal = 0.012, variability = [-0.005,0.005], 1 occurrences
    Cndr: Uncertain real, nominal = -0.04, variability = [-0.01,0.01], 1 occurrences
    Cyb: Uncertain real, nominal = -0.189, variability = [-0.055,0.055], 1 occurrences
    Cyda: Uncertain real, nominal = 0.015, variability = [-0.003,0.003], 1 occurrences
    Cydr: Uncertain real, nominal = 0.04, variability = [-0.0035,0.0035], 1 occurrences

Type "RVunc.NominalValue" to see the nominal value, "get(RVunc)" to see all properties, and
"RVunc.Uncertainty" to interact with the uncertain elements.
```

```
RVunc.InputName
```

```
ans =
    'da'
    'dr'
```

```
RVunc.OutputName
```

```
ans =
    'beta'
    'p'
    'r'
    'phi'
    'ny'
```

```
RVunc.StateName
```

```
ans =
    'beta'
    'p'
    'r'
    'phi'
```

Weighted Open-Loop Interconnection

A weighted open-loop interconnection is now constructed for control design and analysis. The lateral-directional axis control objectives are

- Good low frequency tracking of ϕ commands (up to 1 rad/s), a coordinated turn, and small lateral accelerations.
- Robust to variations in aerodynamic coefficients, exogenous disturbances and sensor noise.

The performance and robustness objectives are characterized as H-infinity norm minimization of weighted transfer functions.

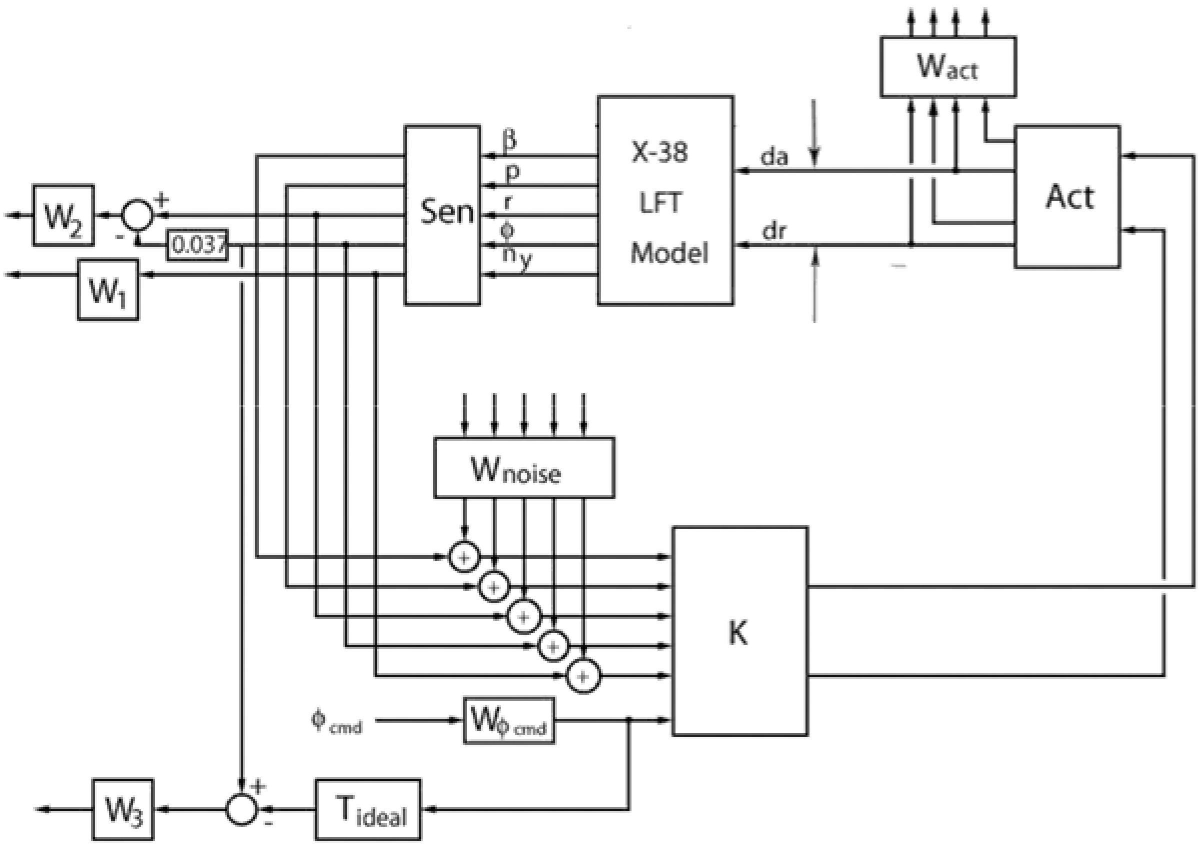


Figure: Re-entry Vehicle Control Interconnection Diagram

Sensor Models

The sensors models include: ny accelerometer, sideslip, β, roll rate, p, yaw rate, r, and bank angle, ϕ.

```
aflt = tf(1,[1/66 1]);
nyflt = 0.03108*tf(1,[1/40 1]);
sensors = blkdiag(aflt,aflt,aflt,aflt,nyflt);
```

Actuator Models

The aileron and rudder actuators are modeled as first order systems. The 0.05 sec computational time delay is represented as a 1st order Pade delay. Each actuator model has two outputs, rate and deflection. Both are penalized as generalized errors, and only the deflection is used as control to the rigid body.

```
act = ss([tf([20 0],[1 20]);tf(20,[1 20])])*tf([-1 20],[1 20]);
acts = blkdiag(act,act);
acts.InputName = {'AilCmd','RudCmd'};
acts.OutputName = {'AilRate','AilDefl','RudRate','RudDefl'};
```

Actuator weighting function: `Wact`

Weighting functions are used to translate desired requirements and objectives on the physical system into the norm-bounded H-infinity framework. The actuator weight `Wact` penalizes the actuator rates (8/30) and deflections (30/30).

```
wact = diag([8/30 30/30]);
Wact = blkdiag(wact,wact);
```

Bank angle tracking weights: `Tideal`, `Wphicmd`, `Wp3`

The bank angle tracking requirement is included as a model matching problem, `Tideal` represents the desired response from the pilot command to bank angle response. The error between the desired response and actual response is penalized with `Wp3`. The weight `Wphicmd` describes the typical spectra of the pilot bank angle commands.

```
Wphicmd  = tf([1/10 1], [1/0.5 1]);
Tideal = tf(0.81,[1 1.8 0.81]);
const = 0.037;
Wp3 = 10*tf([1/1000 1],[1/1 1]);
```

Coordinated turn weights: `Wp1`, `Wp2`

A coordinated turn is desired to minimize lateral acceleration. The weight `Wp2` is used to define the coordinated turn objective and `Wp1` is used to penalize lateral accelerations, `ny`.

```
Wp1 = 4;
Wp2 = 5*tf([1/1000 1],[1/2 1]);
```

Noise and disturbance weights: `Wn`, `Wdist`

The sensor noise weights, `Wn`, are defined a constants in the control problem formulation. The input disturbances are modeled using weight `Wdist`.

```
Wnb = 0.15;
Wnp = 0.12;
Wnr = 0.05;
Wnphi = 0.025;
Wnny = 0.2;
Wn = blkdiag(Wnb,Wnp,Wnr,Wnphi,Wnny);
Wdist = blkdiag(0.5,0.5);
```

Construct the weighted open-loop interconnection structure.

The uncertain, weighted open-loop interconnection, `rvdesolic`, is used for control

```
systemnames = 'RVunc sensors acts Wp1 Wp2 Wp3 const Tideal Wphicmd';
systemnames = [ systemnames ' Wact Wn Wdist'];
inputvar    = '[ phicmd; dist(2); noise(5); cmd(2) ]';
outputvar   = '[ Wp1; Wp2; Wp3; Wact; Wn+sensors; Wphicmd ]' ;
input_to_RVunc    = '[ Wdist(1)+acts(2); Wdist(2)+acts(4) ]';
input_to_sensors  = '[ RVunc ]';
input_to_Wp1      = '[ sensors(5) ]';
input_to_Wp2      = '[ sensors(3)-const ]';
input_to_const    = '[ sensors(4) ]';
input_to_Wp3      = '[ sensors(4)-Tideal ]';
input_to_Tideal   = '[ Wphicmd ]';
input_to_Wphicmd  = '[ phicmd ]' ;
input_to_acts     = '[ cmd ]' ;
input_to_Wact     = '[ acts ]' ;
```

```
input_to_Wn       = '[ noise ]' ;
input_to_Wdist    = '[ dist ]' ;
rvdesolic = sysic;
```

Synthesize a H-infinity Controller: Kh

The uncertainty in the aerodynamic derivatives is eliminated from the weighted interconnection structure, `rvdesolic` for the H-infinity design. The H-infinity controller is synthesized for the nominal weighted interconnection structure, `rvdesolic.Nominal`. The controller receives 5 measurements, β, p, r, ϕ and ny, as well as the weighted pilot bank angle command. The controller returns 2 inputs: elevon and rudder commands.

```
[Kh,clph,gam,hinfo] = hinfsyn(rvdesolic.Nominal,6,2,'Display','on');
```

```
Test bounds:      0.0000 <  gamma   <=      2.9018

  gamma    hamx_eig  xinf_eig  hamy_eig   yinf_eig   nrho_xy   p/f
    2.902   5.5e-01   0.0e+00   5.8e-01    0.0e+00    0.1399    p
    1.451   5.4e-01   0.0e+00   5.9e-01   -1.4e-16    1.4038#   f
    2.176   5.8e-01   0.0e+00   5.8e-01   -1.7e-16    0.2876    p
    1.814   5.6e-01   0.0e+00   5.8e-01   -1.5e-17    0.4982    p
    1.632   5.5e-01   0.0e+00   5.9e-01   -6.8e-17    0.7413    p
    1.542   5.5e-01   0.0e+00   5.9e-01   -1.3e-16    0.9698    p
    1.496   5.5e-01   0.0e+00   5.9e-01   -2.6e-18    1.1461#   f
    1.519   5.5e-01   0.0e+00   5.9e-01   -6.6e-17    1.0505#   f
    1.530   5.5e-01   0.0e+00   5.9e-01    0.0e+00    1.0086#   f
    1.536   5.5e-01   0.0e+00   5.9e-01   -3.7e-17    0.9888    p

 Gamma value achieved:     1.5359
```

The information displayed during the H-infinity design process indicates the conditions which were satisfied and violated during the iteration procedure. The H-infinity controller stabilizes the closed-loop system and achieves a closed-loop norm listed above.

The resulting central control from `hinfsyn` has the same number of states as the weighted interconnection structure (ie., the "generalized plant") used for the design, `rvdesolic.Nominal`. Verify this.

```
size(rvdesolic.Nominal)
```

```
State-space model with 13 outputs, 10 inputs, and 18 states.
```

```
size(Kh)
```

```
State-space model with 2 outputs, 6 inputs, and 18 states.
```

Confirm that the controller indeed stabilizes the generalized plant, and achieves the norm listed.

```
isstable(lft(rvdesolic.Nominal,Kh))
```

```
ans =
     1
```

```
norm(lft(rvdesolic.Nominal,Kh),inf)
```

```
ans =
   1.5359e+00
```

D-K Iteration Controller Design

The H-infinity controller previously synthesized ignored the aerodynamic coefficient uncertainty in the design process. In this section, a μ-controller will be synthesized for the uncertainty reentry vehicle using the $D-K$ iteration procedure.

The `dksynOptions` function is used to set the options for `dksyn`. The number of $D-K$ synthesis iteration is set to 3 and the D and G-scalings maximum orders are set to 3 and 2 respectively. Note that initially the real parameters are treated as complex parameters during the $D-K$ iteration synthesis process.

```
dopt = dksynOptions('NumberOfAutoIterations',3,'AutoScalingOrder',[3 2]);
[Kmu,~,MUBND] = dksyn(rvdesolic,6,2,dopt);
MUBND
```

```
MUBND =
   2.5076e+00
```

Comparisons: Nominal performance of `Kmu` versus `Kh`

The nominal performance of the `Kmu` controller is larger (i.e. worse) than the nominal performance achieved by the H-infinity controller.

```
nomgh = norm(clph,inf)
```

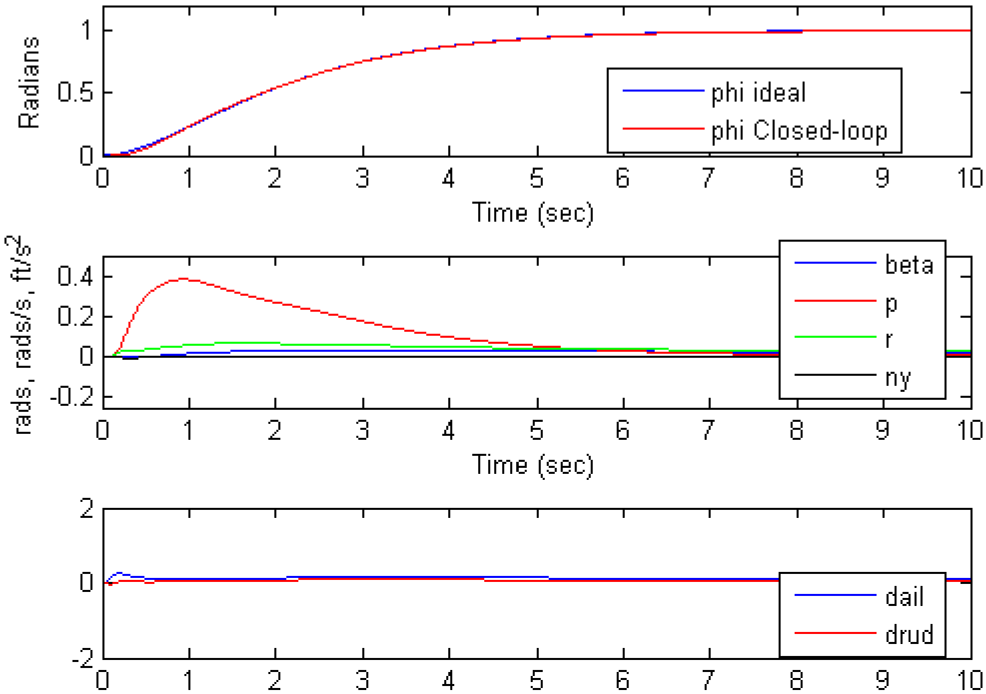
```
nomgh =
   1.5359e+00
```

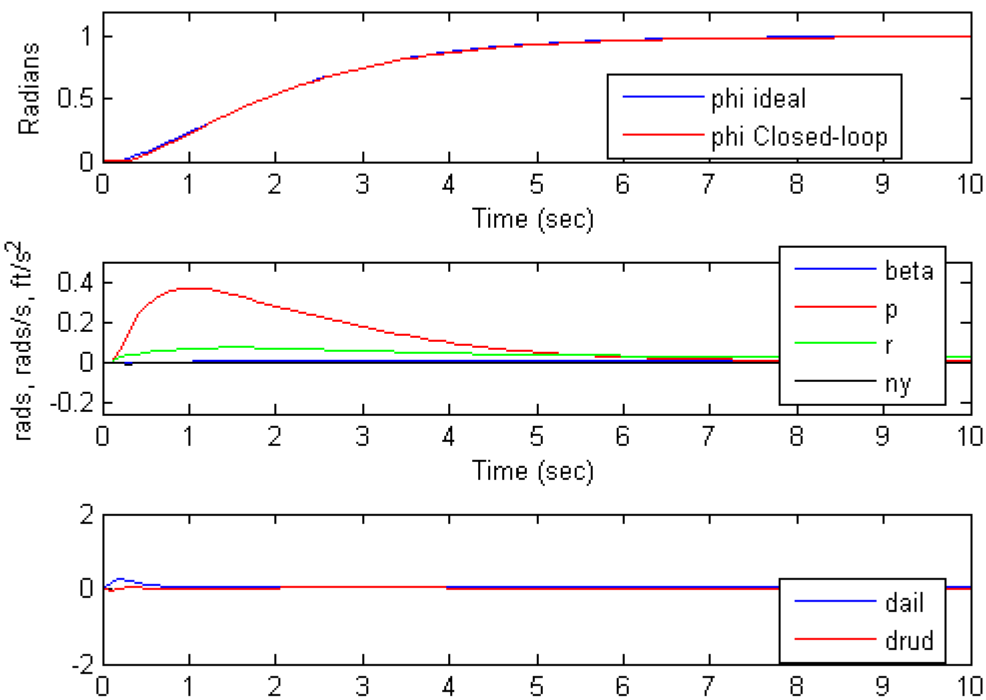
```
clpmu = lft(rvdesolic,Kmu);
nomgmu = norm(clpmu,inf)
```

```
nomgmu =
   2.5048e+00
```

The nominal time domain responses associated with the H-infinity and `Kmu` controllers are similar.

```
figure(1);
[ynom,tnom,TRclp] = RV_linsim(RVunc.Nom,Kh,10);

figure(2);
[ynom,tnom,TRclp] = RV_linsim(RVunc.Nom,Kmu,10);
```





Comparisons: Robust stability of `Kmu` versus `Kh`

The robust stability margins are computed for both closed-loop systems. The re-entry vehicle model only contains real parametric uncertainty. The efficiency of the robust stability algorithms is improved by adding a small amount of complex uncertainty to each real parameter uncertainty. Specifically, 3% complex uncertainty is added to each parameter using the COMPLEXIFY command. The Kmu controller acheives significantly larger stability margins as compared to the H-infinity controller.

```
om = logspace(-2,2,120);
clph = lft(rvdesolic,Kh);
clphg = ufrd(clph,om);
[stabmargh,destabunch] = robuststab( complexify(clphg,0.03) );
stabmargh

clpmu = lft(rvdesolic,Kmu);
clpmug = ufrd(clpmu,om);
[stabmargmu,destabuncmu] = robuststab( complexify(clpmug,0.03) );
stabmargmu
```

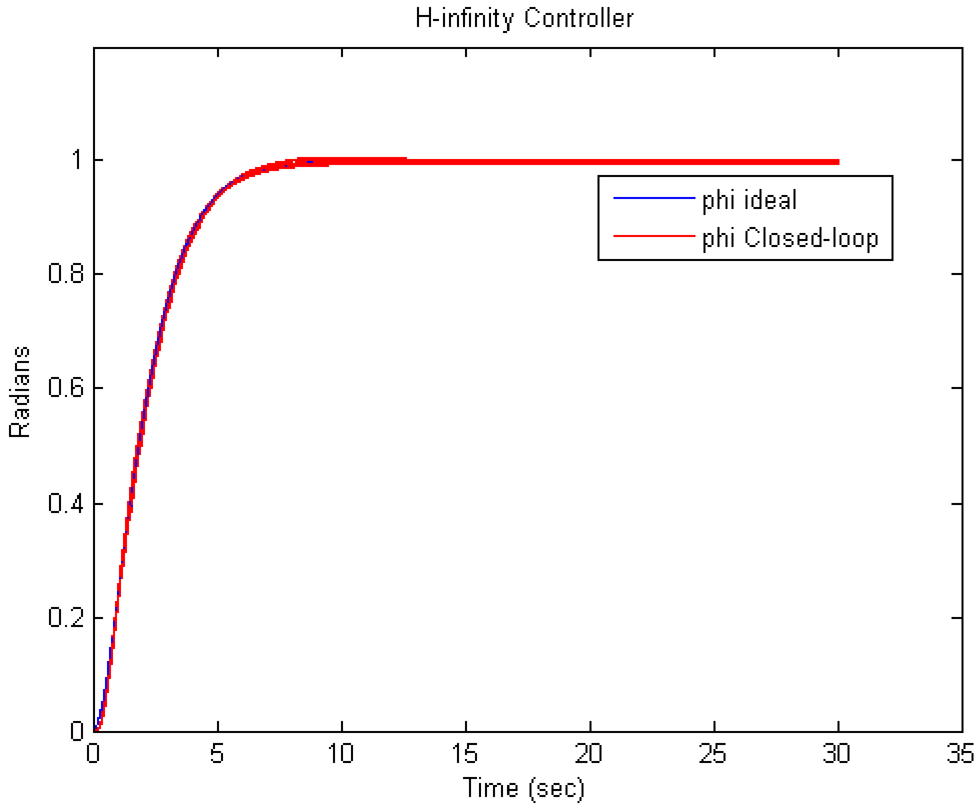
```
stabmargh =
               LowerBound: 6.6699e-01
               UpperBound: 7.1724e-01
    DestabilizingFrequency: 3.2554e-01
stabmargmu =
               LowerBound: 1.1639e+00
               UpperBound: 1.2646e+00
    DestabilizingFrequency: 3.5174e-01
```

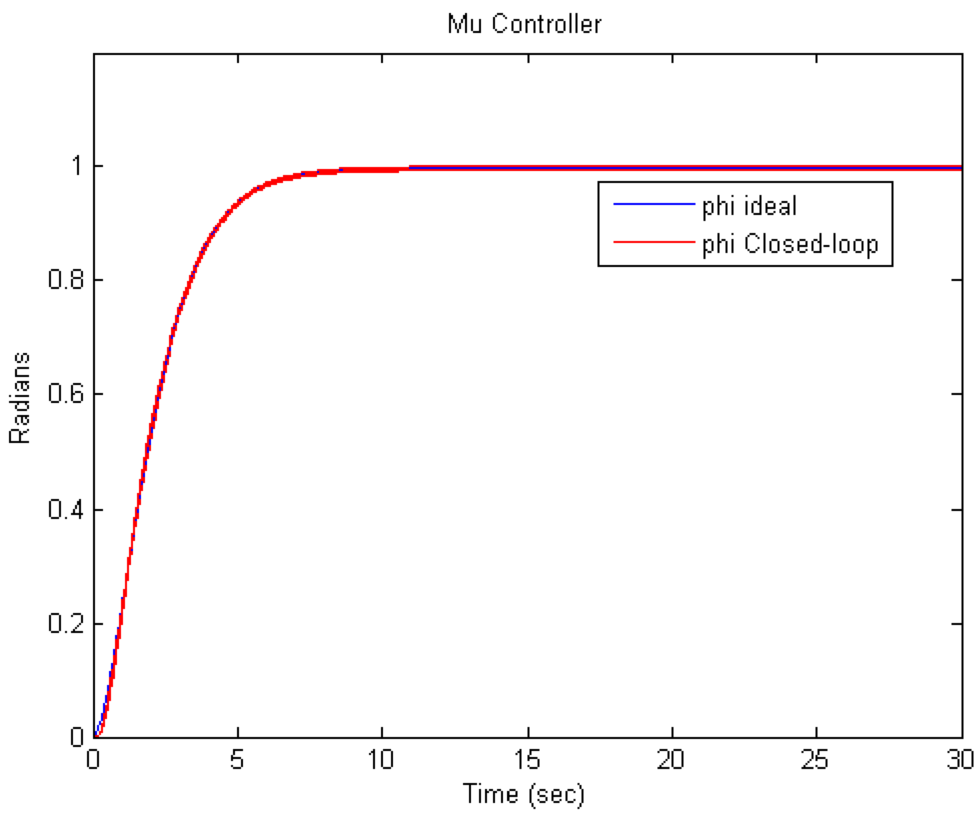
Comparisons: Monte Carlo time-domain responses for Kmu versus Kh

The code below generates many time responses by random sampling of the parameter uncertainties. Both controllers appear to have similar performance for these Monte Carlo simulations. There is slightly less variation in the mu controller responses. This is an indication of the robustness of the Kmu controller.

```
Nsim=25;
flg =0;
Tfinal = 30;
for i=1:Nsim,
    [ynom,tnom,TRclp] = RV_linsim(usample(RVunc),Kh,Tfinal,flg);
    figure(3);
    plot(tnom,ynom(:,1),'b', tnom,ynom(:,2),'r'); hold on;

    [ynom,tnom,TRclp] = RV_linsim(usample(RVunc),Kmu,Tfinal,flg);
    figure(4);
    plot(tnom,ynom(:,1),'b', tnom,ynom(:,2),'r'); hold on;
end

figure(3);
legend('phi ideal','phi Closed-loop','location','best')
title('H-infinity Controller');
xlabel('Time (sec)')
ylabel('Radians')
ylim([0 1.2]);
hold off;

figure(4);
legend('phi ideal','phi Closed-loop','location','best')
title('Mu Controller');
xlabel('Time (sec)')
ylabel('Radians')
ylim([0 1.2]);
```

```
hold off;
```





Worst-case gain of `Kmu`

The worst-case gain of the mu controller is computed. The worst-case gain for the H-infinity controller is not computed since this controller does not achieve robust stability over the uncertainty set.

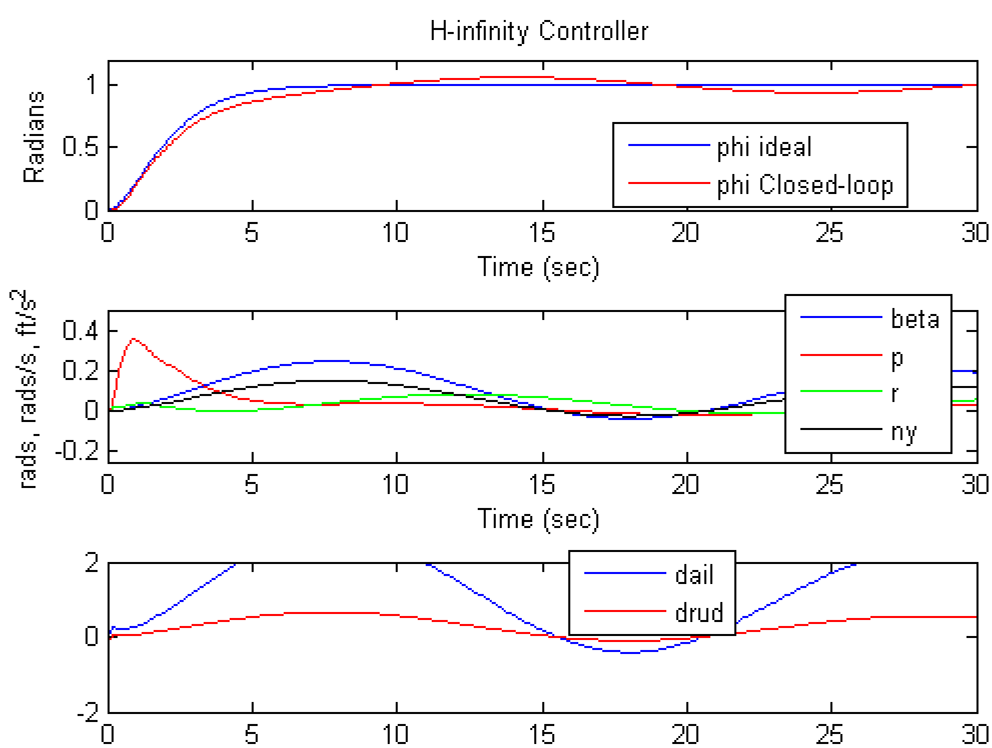
```
[wcgmu,wcumu] = wcgain(clpmug, wcgopt);
wcgmu
```

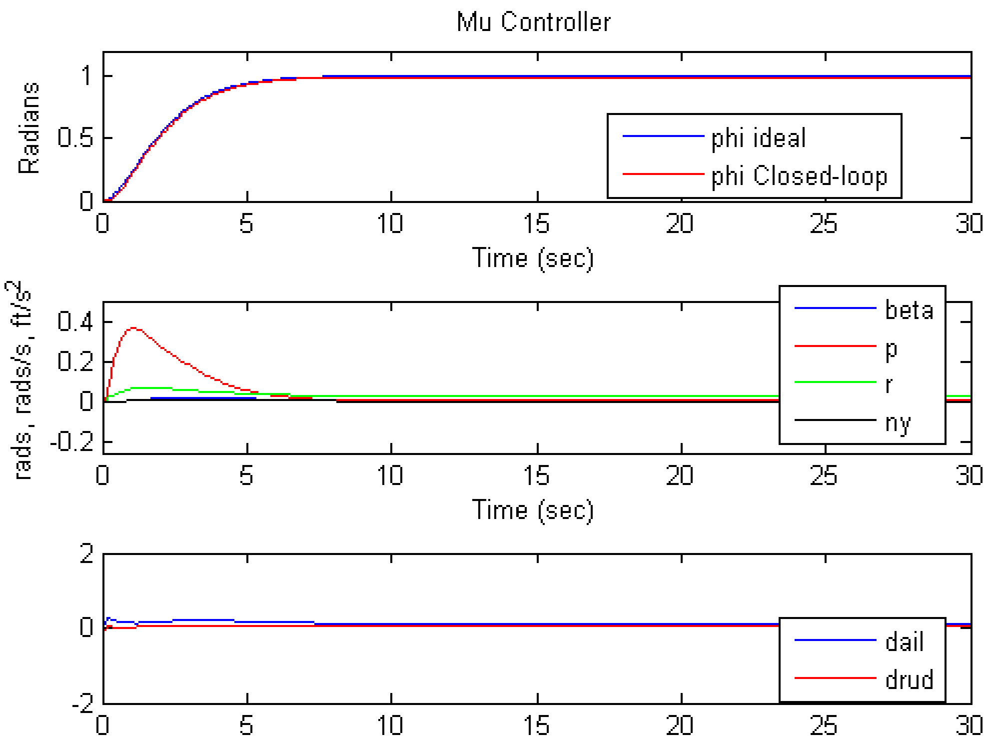
```
wcgmu =
           LowerBound: 1.3237e+01
           UpperBound: 1.3237e+01
    CriticalFrequency: 4.7937e-01
```

Comparisons: Time Domain Simulations of `Kmu` versus `Kh`

The perturbations obtained from the robust stability and worst-case gain analyses can be further investigated in the time domain. First simulate both controllers using the destabilizing perturbation found for the H-infinity controller. The ICOMPLEXIFY command is used to remove the small complex terms introduced by the COMPLEXIFY command. Note that the H-infinity controller oscillates at the destabilizing frequency returned in "stabmargh". The performance of the `Kmu` controller with this uncertainty is relatively unchanged relative to the nominal performance.

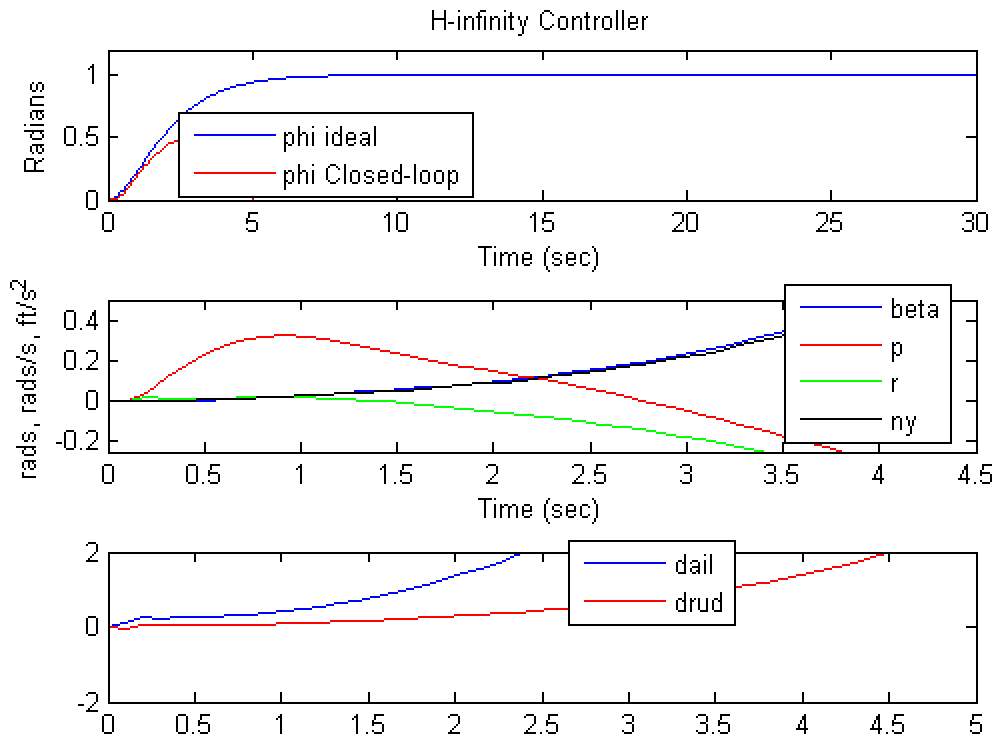
```
Tfinal = 30;
destabunchREAL = icomplexify( destabunch );
figure(5)
[ynom,tnom,TRclp] = RV_linsim(usubs(RVunc,destabunchREAL),Kh,Tfinal);
subplot(311); title('H-infinity Controller');

figure(6)
[ynom,tnom,TRclp] = RV_linsim(usubs(RVunc,destabunchREAL),Kmu,Tfinal);
subplot(311); title('Mu Controller');
```

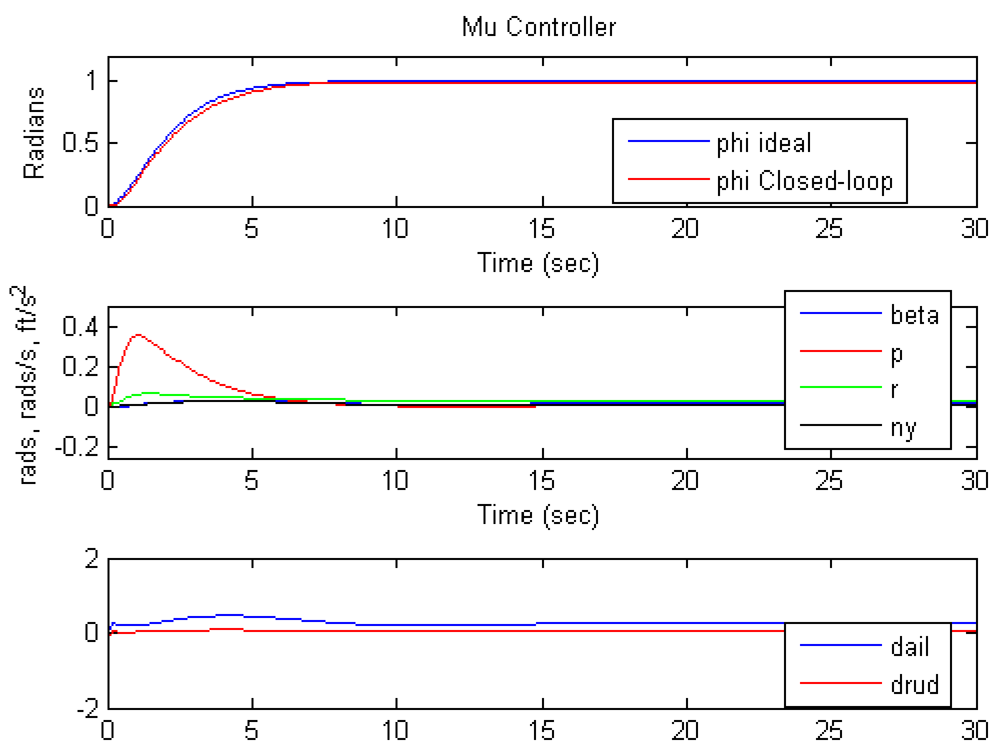




Next simulate both controllers using the worst-case perturbation computed for the Kmu controller. The instability of the H-infinity controller is evident with this perturbation. The performance of the mu controller on this worst-case perturbation is still relatively similar to the nominal performance. This is an indication of the robustness achieved by the Kmu controller.

```
figure(7)
[ynom,tnom,TRclp] = RV_linsim(usubs(RVunc,wcumu),Kh,Tfinal);
subplot(311); title('H-infinity Controller');

figure(8)
[ynom,tnom,TRclp] = RV_linsim(usubs(RVunc,wcumu),Kmu,Tfinal);
subplot(311); title('Mu Controller');
```





Conclusions

A H-infinity and μ controller are synthesized for a reentry vehicle. The H-infinity controller was synthesized based on the nominal model (no uncertainty) while the μ controller was design taking into account the aerodynamic uncertainty. On the nominal plant model, the H-infinity controller outperforms the μ design. However, the robust performance of the μ controller, in the presence of plant uncertainty is superior to that of the H-infinity design.

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Order-Reduction of a mu controller for a Re-entry Vehicle

This can be executed after completing the `RVdesign.m` file.

CAT/MUSYN shortcourse, May 2014

Contents

Controller reduction based on BalancedRealizations

The dynamic order of `Kmu` is quite high. This is common when using `dksyn`. Usually though, significant model reduction is possible. Here use a simple balanced-reduction on `Kmu`, obtaining truncated balanced realizations from order 5 to order 12. This is an "arbitrary" choice that can be revisited, if necessary.

```
size(Kmu.A)
```

```
ans =
    28    28
```

```
stateorders = 5:12;
KB = reduce(Kmu,stateorders);
```

Assessing RobustPerformance of the various reduced-order controllers

Each of these controllers will achieve different levels of closed-loop performance (in fact, some might not even stabilize the nominal plant model). Form the closed-loop system (an array of USS objects) and assess the performance using `robustperf`.

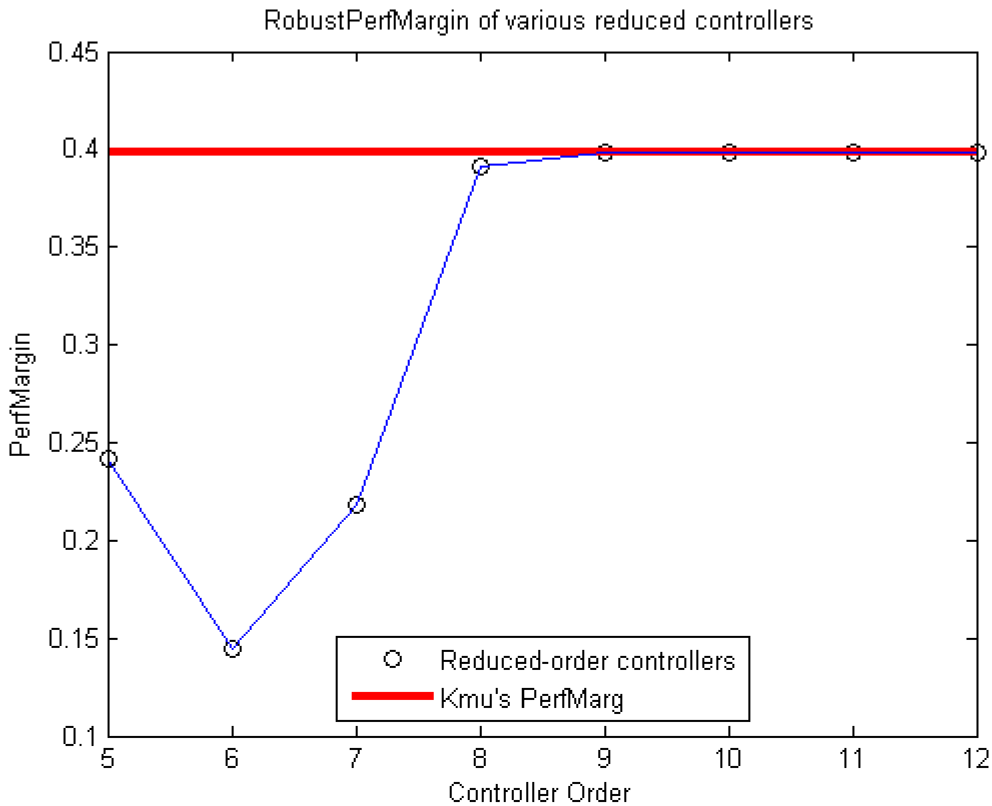
```
CLP = lft(rvdesolic,KB);
ropt = robustperfOptions('Sensi','off','Disp','on','Mussv','a');
[PM,PMU,REPORT,INFO] = robustperf(CLP,ropt);
```

```
points completed (of 150) ... 150
points completed (of 126) ... 126
points completed (of 131) ... 131
points completed (of 127) ... 127
points completed (of 115) ... 115
points completed (of 121) ... 121
points completed (of 115) ... 115
points completed (of 116) ... 116
```

Compare `PerfMargin` with 1/MUBND

The original high-order controller `Kmu` achieved a final MU-robust performance value stored in `MUBND`. The reciprical, 1/|MUBND| is the perfmargin obtained by `Kmu`. Use a simple plot to compare the performance of the original high-order controller `Kmu` with the performance of the lower-order controllers obtained via model-reduction.

```
H = plot(stateorders,[PM.LowerBound],'ko',...
   stateorders,repmat(1/MUBND,[1 numel(stateorders)]),'r',...
   stateorders,[PM.LowerBound]);
set(H(2),'linew',3)
legend('Reduced-order controllers','Kmu''s PerfMarg','Location','Best');
title('RobustPerfMargin of various reduced controllers')
xlabel('Controller Order');
ylabel('PerfMargin')
```



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Model Reduction

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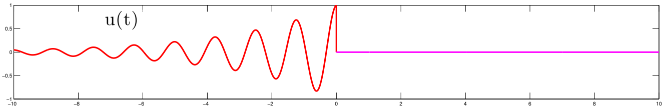
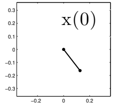
Summary of results on the accuracy with which we can approximate a transfer function, $G(s)$, of degree n, with $\hat{G}(s)$ of lower degree. Let

$$E(s) = G(s) - \hat{G}(s)$$

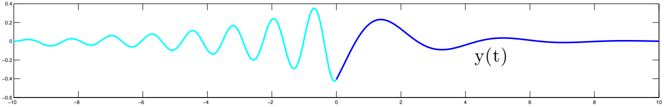
In what metrics should/can we measure $\|E\|_?$.

The Hankel Operator for $G(s) = 10/(s^2 + s + 3)$

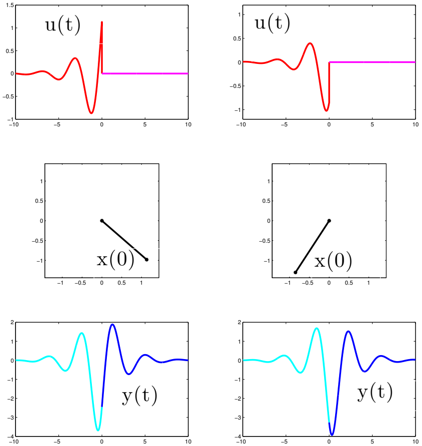




The Hankel operator maps the
past inputs
into the future outputs
via the state at t=0



The Hankel singular/Eigen values are 2.17 and 3.84 with the corresponding Eigen vectors



The rank of the Hankel operator is therefore the state dimension and is an effective object with which to consider model reduction.
It's singular values are easily computed from the controllability and observability Gramians, call them $\sigma_1 \geq \sigma_2 ... \geq \sigma_n \geq 0$.
If the degree of $\hat{G}(s)$ is $k < n$ then it can be shown that

$$\|G - \hat{G}\|_\infty \geq \sigma_{k+1}$$

The so-called truncated balanced realisation approximation satisfies

$$\|G - \hat{G}\|_\infty \leq 2 \times (\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$$

The optimal Hankel-norm approximation satisfies half this upper bound.

Other norms with upper and lower bounds are:

- Relative error $\|(G - \hat{G})G^{-1}\|_\infty$
- Gap metric
- Frequency weigted norms have lower bounds that may be far from achievable.
- Controller reduction is not clear because a low order controller might exist with similar closed-loop norm to a high order one but not close in any metric.

Demonstration program:

```
K=tf(8,[1 3]); P1=tf(2,[1 3]);
G=1/(1/K-0.5*P1*P1*P1*P1*P1*P1*P1); Gss=ss(G);
[Gbal,balinfo] = balancmr(Gss,8,'display','off');
balinfo.StabSV
k=input('pick a degree = ') [Gbal,balinfo] =
balancmr(G,k);
BalError=norm(G-Gbal,inf)
BalErrorbnd=sum(balinfo.StabSV(k+1:end)*2)
bode(Gss,Gbal) pause
[Ghank,hankinfo] = hankelmr(Gss,k,'display','off');
Dtmp=squeeze(freqresp(Gss-Ghank,0))/2;
GhankD=Ghank+Dtmp;
HankelError=norm(Gss-GhankD,inf)
HankelErrorbnd=sum(balinfo.StabSV(k+1:end)*1) pause
[Gbst,bstinfo] = bstmr(Gss,k,'display','off');
bstinfo.StabSV
```

```
[Gncf,ncfinfo] = ncfmr(Gss,k,'display','off');
ncfinfo
ncfError=gapmetric(Gss,Gncf)
bode(Gss-Gbal,Gss-GhankD,Gss-Gbst,Gss-Gncf) pause
nyquist(Gss-Gbal,Gss-GhankD,Gss-Gbst,Gss-Gncf)
```

Some robust control analysis and synthesis theory

Topics:

1. Some basic convex optimization theory (heading towards LMIs).

2. Structured singular value as an LMI problem.

3. Performance and the Bounded Real Lemma.

4. H-infinity design:
    a. State feedback;
    b. linearizing transformations.

5. H-2 design:
    a. Characterization and analysis;
    b. State feedback;
    c. linearizing transformations.

6. L-1 design:

7. Pole region constraints

8. Multi-objective analysis and synthesis.

9. Relaxations for structured and decentralised design problems.

Convex optimization problems

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & a_i^T x = b_i, & i = 1, \ldots, p \end{array}$$

The functions, $f_0, f_1, \ldots, f_m$, are convex.

The equality constraints are affine.

A problem is quasiconvex if f_0 is quasiconvex and $f_1, \ldots, f_m$, are convex.

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & Ax = b & \end{array}$$

The feasible set of a convex (or quasiconvex) optimization problem is convex.

Semidefinite program (SDP)

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \preceq 0 \\ & Ax = b \end{aligned}$$

where $F_i, G \in \mathbf{S}^k$

The matrix constraint is called a linear matrix inequality (LMI)

Multiple constraints are trivially combined into a single (larger) constraint,

$$x_1F_1 + x_2F_2 + \ldots + x_nF_n + G \preceq 0 \quad \text{and} \quad x_1H_1 + x_2H_2 + \ldots + x_nH_n + M \preceq 0$$

if and only if

$$x_1 \begin{bmatrix} F_1 & 0 \\ 0 & H_1 \end{bmatrix} + x_2 \begin{bmatrix} F_2 & 0 \\ 0 & H_2 \end{bmatrix} + \ldots + x_n \begin{bmatrix} F_n & 0 \\ 0 & H_n \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & M \end{bmatrix} \preceq 0$$

Example: matrix norm minimization (Maximum singular value)

$$\text{minimize} \quad \|A(x)\|_2 \;=\; \left(\rho(A(x)^T A(x))\right)^{1/2}$$

where $A(x)$ is an LMI: $A(x) \;=\; A_0 \,+\, x_1 A_1 \,+\, x_2 A_2 \,+\, \ldots \,+\, x_n A_n$

The equivalent SDP is:

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

The decision variables are now t and x.

The constraint equivalence follows from a Schur complement argument

$$\begin{aligned} \|A\|_2 \leq t \quad &\iff \quad A^T A \preceq t^2 I, \quad t \geq 0, \\ &\iff \quad \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

"Disciplined convex programming" `cvx`

Download `cvx` from `www.stanford.edu/ boyd/cvx/`

Example: proving the stability of a system: $\dfrac{dx(t)}{dt} = A\,x(t)$

$$\text{stable} \iff \text{there exists } P = P^T \succ 0, \quad A^T P + P A \prec 0$$

$$\iff \text{there exists } P = P^T \succeq I, \quad A^T P + P A \preceq -I$$

We can consider P as a matrix variable

```
cvx_begin sdp
  variable P(n,n) symmetric
  A'*P + P*A <= -eye(n)
  P >= eye(n)
cvx_end
```

`cvx_status` is a string returning the status of the optimization

Another example:

We want to know if the stability of two systems,

$$\frac{dx(t)}{dt} = A_1\, x(t) \qquad \text{and} \qquad \frac{dx(t)}{dt} = A_2\, x(t)$$

can be proven with a single Lyapunov function, $V(s) = x(t)^T P x(t)$

$$\frac{dx(t)}{dt} = A(t)\, x(t) \quad \text{stable for} \quad A(t) = \theta_1(t)\, A_1 + \theta_2(t)\, A_2, \quad \theta_i(t) \geq 0$$

We want to find $P = P^T \succ 0$, such that $A_1^T P + P A_1 \prec 0$, and $A_2^T P + P A_2 \prec 0$

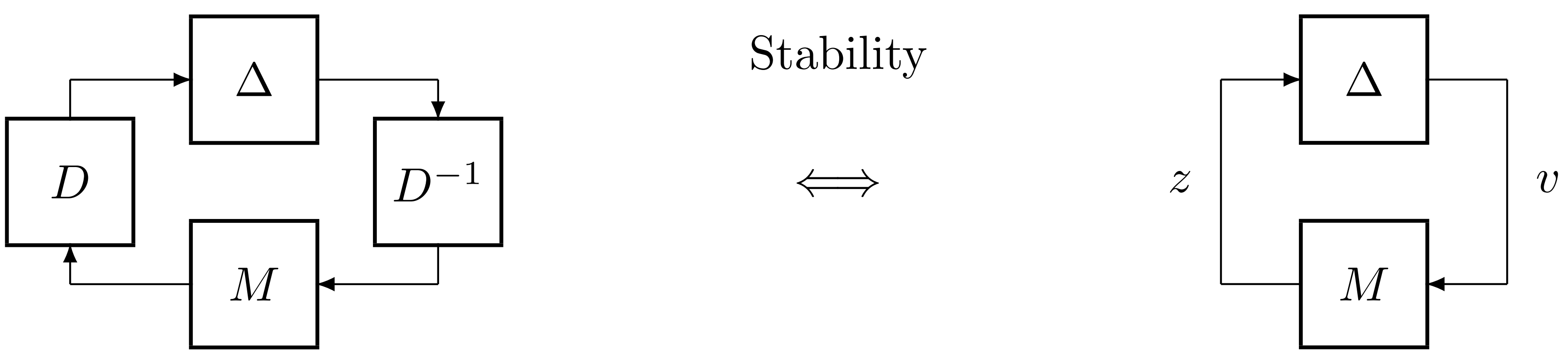
Or equivalently $P = P^T \succeq I$, such that $A_1^T P + P A_1 \preceq -I$, and $A_2^T P + P A_2 \preceq -I$

```
cvx_begin sdp
  variable P(n,n) symmetric
  A1'*P + P*A1 <= -eye(n)
  A2'*P + P*A2 <= -eye(n)
  P >= eye(n)
cvx_end
```

Upper bound calculation

Define a set of invertible matrices that commute with all $\Delta \in \mathbf{\Delta}$

$$\mathcal{D} = \{ \operatorname{diag}(D_1, \ldots, D_q, d_1 I_1, \ldots, d_m I_m,) \mid D_j = D_j^* > 0, \ \dim(I_i) = k_i, \ \ d_i \in \mathcal{R}, \ \ d_i > 0 \ \}$$



Upper bound: $\mu_{\mathbf{\Delta}}(M) \ = \ \mu_{\mathbf{\Delta}}(DMD^{-1}) \ \leq \ \inf_{D \in \mathcal{D}} \ \sigma_{\max}(DMD^{-1})$

Upper bound calculation

$$\sigma_{\max}\left(D\,M\,D^{-1}\right) < \gamma \quad\Longleftrightarrow\quad \gamma^2 I - \left(D\,M\,D^{-1}\right)^* \left(D\,M\,D^{-1}\right) \succ 0$$

$$\Longleftrightarrow\quad \gamma^2 I - D^{-1} M^*\, D^2\, M\, D^{-1} \succ 0$$

$$\Longleftrightarrow\quad \gamma^2 D^2 - M^*\, D^2\, M \succ 0$$

$$\Longleftrightarrow\quad \gamma^2 D - M^*\, D\, M \succ 0 \qquad (D \in \mathcal{D} \text{ so } D^2 \in \mathcal{D})$$

For γ fixed this is an LMI in the variables $D \in \mathcal{D}$.

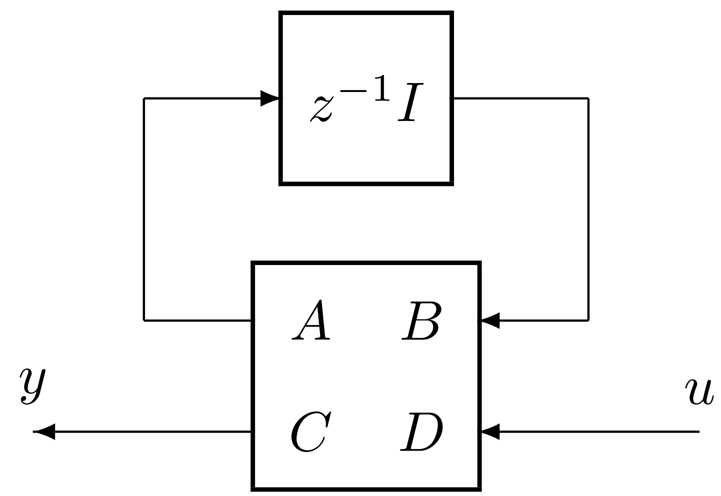
If γ varies monotomically, the feasible regions of $D \in \mathcal{D}$ are nested

$$\begin{aligned} &\underset{\eta, D \in \mathcal{D}}{\text{minimize}} && \eta \\ &\text{subject to} && \eta\, D - M^* D\, M \succ 0 \end{aligned}$$

(Quasiconvex optimization problem: generalized eigenvalue problem)

Then $\gamma = \sqrt{\eta^{\,\text{opt}}}$ is an upper bound for $\mu_{\mathbf{\Delta}}(M)$

State-space performance test (via main loop theorem)



$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

$$P(z) = F_u(P_{ss}, z^{-1}I) \qquad \text{where} \quad P_{ss} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Stability and nominal performance:

$$\mu_{\mathbf{\Delta}}(P_{ss}) < 1 \qquad \Longleftrightarrow \qquad \begin{cases} \text{P(z) is stable} \\ \text{and} \\ \|P(z)\|_\infty < 1. \end{cases}$$

$$\mathbf{\Delta} = \{\,\text{diag}(\delta_1 I_{nx}, \Delta_2) \mid \delta_1 \in \mathcal{C},\ \Delta_2 \in \mathcal{C}^{nu \times ny}\,\}$$

State-space performance test

$$\mu_{\mathbf{\Delta}}(P_{ss}) < 1 \qquad \Longleftrightarrow \qquad \begin{cases} \text{P(z) is stable} \\ \text{ and} \\ \|P(z)\|_{\infty} \; < \; 1. \end{cases}$$

$$\mathbf{\Delta} \; = \; \{\, \mathrm{diag}(\delta_1 I_{nx}, \Delta_2) \mid \delta_1 \in \mathcal{C}, \; \Delta_2 \in \mathcal{C}^{nu \times ny} \,\}$$

In this case: $\mu_{\mathbf{\Delta}}(P_{ss}) \; = \; \inf_{D \in \mathcal{D}} \sigma_{\max} \left(D \, P_{ss} \, D^{-1} \right)$

$$\mathcal{D} \; = \; \left\{ \begin{bmatrix} D_1 & 0 \\ 0 & d_2 I \end{bmatrix} \middle| \; D_1 \; = \; D_1^* \; \succ \; 0, \; d_2 \; > \; 0 \right\}$$

Consider (w.l.o.g.) finding D_1 such that: $\sigma_{\max} \left(\begin{bmatrix} D_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I \end{bmatrix} \right) \; < \; 1$

State-space performance test

This is equivalent to the LMIs: $$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\right) \prec 0,$$

$$X = X^* \succ 0 \qquad (\text{take } X = D_1^2)$$

Bounded real lemma (many equivalent expressions exist)

$P(z)$ is stable and $\|P(z)\|_{\mathcal{L}_2} < 1$

$\iff$ there exists $X = X^* \succ 0$

such that $$\begin{bmatrix} -X & 0 & A^T X & C^T \\ 0 & -I & B^T X & D^T \\ X A & X B & -X & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0$$

Bound real lemma: (discrete-time)

$P(z)$ is stable and $\|P(z)\|_\infty < \gamma$

$$\iff \quad \text{there exists } Y = Y^* \succ 0$$

$$\text{such that} \quad \begin{bmatrix} Y & AY & B & 0 \\ YA^T & Y & 0 & YC^T \\ B^T & 0 & I & D^T \\ 0 & CY & D & \gamma^2 I \end{bmatrix} \succ 0$$

Note that the upper left block contains the condition:

$$\begin{bmatrix} Y & AY \\ YA^T & Y \end{bmatrix} \succ 0$$

Which is equivalent to the discrete-time Lyapunov condition: $AYA^T - Y \prec 0$

Bound real lemma: (continuous-time)

$P(s)$ is stable and $\|P(s)\|_\infty < \gamma$

$$\iff \quad \text{there exists } P = P^* \succ 0$$

$$\text{such that} \quad \begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -I & D^T \\ C & D & -\gamma^2 I \end{bmatrix} \prec 0$$

The LMI contains the continuous-time Lyapunov condition: $A^T P + PA \prec 0$

An equivalent form (using $Q = P^{-1}$) :
$$\begin{bmatrix} QA^T + AQ & B & QC^T \\ B^T & -I & D^T \\ CQ & D & -\gamma^2 I \end{bmatrix} \prec 0$$

$\mathcal{H}_\infty$ Design (continuous-time)

The bounded real lemmas are used for analysis of a *closed-loop* system

State feedback:

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ I & 0 & 0 \end{array}\right] \qquad \text{with } (A, B_u) \text{ assumed to be stabilizable}$$

$$\begin{bmatrix} e \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{and for state feedback: } u = K\,x = K\,y$$

$$G(s) = \mathcal{F}_l\,(P(s), K) = \left[\begin{array}{c|c} A + B_u K & B_w \\ \hline C_e + D_{eu} K & D_{ew} \end{array}\right]$$

An equivalent LMI:

$$\begin{bmatrix} QA^T + F^T B_u^T + AQ + B_u F & B_w & QC_e^T + F^T D_{eu}^T \\ B_w^T & -I & D_{ew}^T \\ C_e Q + D_{eu} F & D_{ew} & -\gamma^2 I \end{bmatrix} \prec 0$$

(this uses the substitution: $F = K\,Q$)

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State feedback $\mathcal{H}_\infty$ Design (continuous-time)

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ I & 0 & 0 \end{array}\right] \qquad \text{with } (A, B_u) \text{ assumed to be stabilizable}$$

$$\underset{\eta, Q, F}{\text{minimize}} \quad \eta$$

subject to: $Q = Q^T \succ 0$

$$\begin{bmatrix} QA^T + F^T B_u^T + AQ + B_u F & B_w & QC_e^T + F^T D_{eu}^T \\ B_w^T & -I & D_{ew}^T \\ C_e Q + D_{eu} F & D_{ew} & -\eta I \end{bmatrix} \prec 0$$

If this has a solution ($\eta, Q,$ and F) then

$$K = F\,Q^{-1} \quad \text{gives} \quad \mathcal{F}_l\,(P(s), K) \text{ stable and } \|\mathcal{F}_l\,(P(s), K)\|_\infty \leq \sqrt{\eta}$$

State feedback $\mathcal{H}_\infty$ Design (continuous-time)

Using cvx:

```
P = ss(A, [Bw, Bu], [Ce, eye(n,n)], [Dew, Deu; zeros(n,nw+nu)]);

cvx_begin sdp
  variable Q(n,n) symmetric;
  variable F(nu,n);
  variable eta;

  minimize eta;
  subject to:
    Q > 0;
    [Q*A' + F'*Bu' + A*Q + Bu*F,  Bw,  Q*Ce' + F'*Deu';
     Bw',                        -eye(nw,nw), Dew';
     Ce*Q + Deu*F,                Dew,  -eta*eye(ne,ne)] < 0;

cvx_end
K = F*inv(Q);

Aclp = A + Bu*K;
disp(eig(Aclp));     % always check that it really is a good controller.
```

$\mathcal{H}_\infty$ **Design** (continuous-time)

The bounded real lemmas are used for analysis of a *closed-loop* system

Output feedback:

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & D_{ew} & D_{eu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

with (A, B_u) assumed to be stabilizable
and (C_y, A) assumed to be detectable

$$\begin{bmatrix} e \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix}$$

and for output feedback: $u = K(s)\, y = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] y$

$$G(s) = \mathcal{F}_l\left(P(s), K(s)\right) = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_e & D_{eu} C_k & D_{ew} \end{array}\right]$$

$\mathcal{H}_\infty$ **Design** (linearizing transformation)

$$G(s) = \mathcal{F}_l\left(P(s), K(s)\right) = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_e & D_{eu} C_k & D_{ew} \end{array}\right] = \left[\begin{array}{c|c} A_{\text{clp}} & B_{\text{clp}} \\ \hline C_{\text{clp}} & D_{\text{clp}} \end{array}\right]$$

LMI condition (to be applied to the *closed-loop*):
$$\begin{bmatrix} A_{\text{clp}}^T P + P A_{\text{clp}} & P B_{\text{clp}} & C_{\text{clp}}^T \\ B_{\text{clp}}^T P & -I & D_{\text{clp}}^T \\ C_{\text{clp}} & D_{\text{clp}} & -\gamma^2 I \end{bmatrix} \prec 0$$

Partition P as: $P = \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} X & M \\ M^T & \star \end{bmatrix}$

Define new controller variables via:

$$\begin{aligned} \hat{A} &= N A_k M^T + N B_k C_y X + Y B_u C_k M^T + Y A X \\ \hat{B} &= N B_k \\ \hat{C} &= C_k M^T \end{aligned}$$

$\mathcal{H}_\infty$ **Design** (linearizing transformation)

LMI condition (to be applied to the *closed-loop*):
$$\begin{bmatrix} A_{\rm clp}^T P + P A_{\rm clp} & P B_{\rm clp} & C_{\rm clp}^T \\ B_{\rm clp}^T P & -I & D_{\rm clp}^T \\ C_{\rm clp} & D_{\rm clp} & -\gamma^2 I \end{bmatrix} \prec 0$$

Define an inertia-preserving transform via:
$$T = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$$

Then:
$$T^T P A_{\rm clp} T = \begin{bmatrix} AX + B_u \hat{C} & A \\ \hat{A} & YA + \hat{B} C_y \end{bmatrix}$$

$$T^T P B_{\rm clp} = \begin{bmatrix} B_w \\ Y B_w + \hat{B} D_{yw} \end{bmatrix}$$

$$C_{\rm clp} T = \begin{bmatrix} C_e X + D_{eu} \hat{C} & C_e \end{bmatrix}$$

$$T^T P T = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

$\mathcal{H}_\infty$ **Design** (linearizing transformation)

Closed-loop LMI conditions:
$$\begin{bmatrix} A_{\text{clp}}^T P + P A_{\text{clp}} & P B_{\text{clp}} & C_{\text{clp}}^T \\ B_{\text{clp}}^T P & -I & D_{\text{clp}}^T \\ C_{\text{clp}} & D_{\text{clp}} & -\gamma^2 I \end{bmatrix} \prec 0, \quad \text{and} \quad P \succ 0.$$

$$\begin{bmatrix} T^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{\text{clp}}^T P + P A_{\text{clp}} & P B_{\text{clp}} & C_{\text{clp}}^T \\ B_{\text{clp}}^T P & -I & D_{\text{clp}}^T \\ C_{\text{clp}} & D_{\text{clp}} & -\gamma^2 I \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}^T B_u^T & A + \hat{A}^T & B_w & XC_e^T + \hat{C}^T D_{eu}^T \\ A^T + \hat{A} & YA + A^T Y + \hat{B} C_y + C_y^T \hat{B}^T & YB_w + \hat{B} D_{yw} & C_e^T \\ B_w^T & B_w^T Y + D_{yw}^T \hat{B}^T & -I & D_{ew}^T \\ C_e X + D_{eu}\hat{C} & C_e & D_{ew} & -\gamma^2 I \end{bmatrix}$$

$\mathcal{H}_\infty$ Design

$$\underset{\eta, X, Y, \hat{A}, \hat{B}, \hat{C}}{\text{minimize}} \quad \eta$$

$$\text{subject to:} \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}^T B_u^T & A + \hat{A}^T & B_w & XC_e^T + \hat{C}^T D_{eu}^T \\ A^T + \hat{A} & YA + A^TY + \hat{B}C_y + C_y^T\hat{B}^T & YB_w + \hat{B}D_{yw} & C_e^T \\ B_w^T & B_w^TY + D_{yw}^T\hat{B}^T & -I & D_{ew}^T \\ C_eX + D_{eu}\hat{C} & C_e & D_{ew} & -\eta I \end{bmatrix} \prec 0$$

If this has a solution $(\eta, X, Y, \hat{A}, \hat{B}$ and $\hat{C})$ then

$$P\,P^{-1} = I \quad \Longrightarrow \quad NM^T = I - YX \qquad \text{(solve for } M \text{ and } N\text{)}$$

$$\begin{aligned} \hat{A} &= NA_kM^T + NB_kC_yX + YB_uC_kM^T + YAX \\ \hat{B} &= NB_k \\ \hat{C} &= C_kM^T \end{aligned}$$

Solve for A_k, B_k and C_k from:

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] \quad \text{gives} \quad \mathcal{F}_l\left(P(s), K(s)\right) \text{ stable and } \left\|\mathcal{F}_l\left(P(s), K(s)\right)\right\|_\infty \leq \sqrt{\eta}$$

$\mathcal{H}_\infty$ Design

Using cvx:

```
P = ss(A, [Bw, Bu], [Cz; Cy], [Dzw, Dzu; Dyw, zeros(ny,nu)]);

cvx_begin sdp
  variable X(n,n) symmetric;
  variable Y(n,n) symmetric;
  variable Ah(n,n);
  variable Bh(n,ny);
  variable Ch(nu,n);
  variable eta;

  minimize eta;
  subject to:

    [X, eye(n,n);
     eye(n,n), Y] > 0;

     [A*X + Bu*Ch + X*A' + Ch'*Bu', A+Ah',  Bw, X*Ce' + Ch'*Deu';
      A'+Ah, Y*A + A'*Y + Bh*Cy + Cy'*Bh', Y*Bw + Bh*Dyw,  Ce';
      Bw', Bw'*Y + Dyw'*Bh',  -eye(nw,nw),  Dew';
      Ce*X + Deu*Ch, Ce,  Dew,  -eta*eye(ne,ne)] < 0;

cvx_end
```

$\mathcal{H}_\infty$ Design

Using the robust control toolbox:

```
P = ss(A, [Bw, Bu], [Ce; Cy], [Dew, Deu; Dew, zeros(ny,nu)]);
[K,G,clp_norm] = hinfsyn(P,ny,nu);
```

$\mathcal{H}_2$ norm characterization

$$G(s) \; = \; \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \qquad \text{is} \quad \|G(s)\|_{\mathcal{H}_2} \; < \; 1 \quad ?$$

$$\|G(s)\|_{\mathcal{L}_2}^2 \; = \; \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Trace}\left(G(j\omega)G^*(j\omega)\right) \, d\omega$$

Theorem: $G(s)$ is stable and $\|G(s)\|_{\mathcal{H}_2}^2 \; < \; \gamma$

$\iff$ there exists $X \; = \; X^T \; \succ \; 0$ such that

$$\operatorname{Trace}\left(CXC^T\right) \; < \; \gamma \qquad \text{and} \qquad AX \; + \; XA^T \; + \; BB^T \; \prec \; 0$$

$\mathcal{H}_2$ norm characterization

Continuous-time: $G(s)$ is stable and $\|G(s)\|^2_{\mathcal{L}_2} < \gamma$ $\qquad G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

$\iff$ there exists $X = X^T \succ 0$ such that:

$$AX + XA^T + BB^T \prec 0, \quad \begin{bmatrix} W & CX \\ XC^T & X \end{bmatrix} \succ 0 \quad \text{and} \quad \text{Trace}\,(W) < \gamma$$

Discrete-time: $G(z)$ is stable and $\|G(z)\|^2_{\mathcal{L}_2} < \gamma$ $\qquad G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

$\iff$ there exists $X = X^T \succ 0$ such that:

$$\begin{bmatrix} X & AX & B \\ XA^T & X & 0 \\ B^T & 0 & I \end{bmatrix} \succ 0 \quad \begin{bmatrix} W & CX \\ XC^T & X \end{bmatrix} \succ 0 \quad \text{and} \quad \text{Trace}\,(W) < \gamma$$

$\mathcal{H}_2$ **Design** (continuous-time)

State feedback:

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & 0 & D_{eu} \\ I & 0 & 0 \end{array}\right] \qquad \text{with } (A, B_u) \text{ assumed to be stabilizable}$$

$$\begin{bmatrix} e \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{and for state feedback: } u = K\,x = K\,y$$

$$G(s) = \mathcal{F}_l\,(P(s), K) = \left[\begin{array}{c|c} A + B_u K & B_w \\ \hline C_e + D_{eu} K & 0 \end{array}\right]$$

$G(s)$ is stable and $\|G(s)\|_{\mathcal{L}_2} < \gamma \iff$ there exists $X = X^T \succ 0$, and F such that:

$$AX + B_u F + XA^T + F^T B_u^T + B_w B_w^T \prec 0,$$

$$\begin{bmatrix} W & C_e X + D_{eu} F \\ X C_e^T + F^T D_{eu}^T & X \end{bmatrix} \succ 0 \qquad \text{and} \qquad \operatorname{Trace}(W) < \gamma$$

(this uses the substitution: $F = K\,X$)

$\mathcal{H}_2$ **Design** LQG problem

LQG objective: $$J = \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

Choose: $$C_e = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} \quad \text{and} \quad D_{eu} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix}$$

then, $$e(k) = \begin{bmatrix} Q^{1/2} x(k) \\ R^{1/2} u(k) \end{bmatrix} \quad \text{and} \quad e(k)^T e(k) = x(k)^T Q x(k) + u(k)^T R u(k)$$

So $$\|e(k)\|_2^2 = \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k)$$

$\mathcal{H}_2$ **Design** (continuous-time)

Output feedback:

$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_e & 0 & D_{eu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

with (A, B_u) assumed to be stabilizable

and (C_y, A) assumed to be detectable

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix}$$

and for output feedback: $u = K(s)\, y = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] y$

$$G(s) = \mathcal{F}_l\left(P(s), K(s)\right) = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_e & D_{eu} C_k & 0 \end{array}\right]$$

$\mathcal{H}_2$ **Design** (linearizing transform)

$$G(s) \;=\; \mathcal{F}_l\left(P(s), K(s)\right) \;=\; \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_e & D_{eu} C_k & 0 \end{array}\right] \;=\; \left[\begin{array}{c|c} A_{\mathrm{clp}} & B_{\mathrm{clp}} \\ \hline C_{\mathrm{clp}} & 0 \end{array}\right]$$

LMI conditions:

$$\begin{bmatrix} A_{\mathrm{clp}}^T P + P A_{\mathrm{clp}} & P B_{\mathrm{clp}} \\ B_{\mathrm{clp}}^T P & -I \end{bmatrix} \prec 0, \qquad \begin{bmatrix} W & C_{\mathrm{clp}} \\ C_{\mathrm{clp}}^T & P \end{bmatrix} \succ 0, \quad P \succ 0, \quad \mathrm{Trace}(W) < \gamma$$

Partition P as: $\quad P = \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} X & M \\ M^T & \star \end{bmatrix}$

Define new controller variables via:

$$\begin{aligned} \hat{A} &= N A_k M^T + N B_k C_y X + Y B_u C_k M^T + Y A X \\ \hat{B} &= N B_k \\ \hat{C} &= C_k M^T \end{aligned}$$

$\mathcal{H}_2$ **Design** (linearizing transform)

LMI conditions: $$\begin{bmatrix} A_{\mathrm{clp}}^T P + P A_{\mathrm{clp}} & P B_{\mathrm{clp}} \\ B_{\mathrm{clp}}^T P & -I \end{bmatrix} \prec 0, \quad \begin{bmatrix} W & C_{\mathrm{clp}} \\ C_{\mathrm{clp}}^T & P \end{bmatrix} \succ 0, \quad \text{and} \quad P \succ 0$$

Define an inertia-preserving transform via: $$T = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$$

Then:
$$\begin{aligned} T^T P A_{\mathrm{clp}} T &= \begin{bmatrix} AX + B_u \hat{C} & A \\ \hat{A} & YA + \hat{B} C_y \end{bmatrix} \\ T^T P B_{\mathrm{clp}} &= \begin{bmatrix} B_w \\ Y B_w + \hat{B} D_{yw} \end{bmatrix} \\ C_{\mathrm{clp}} T &= \begin{bmatrix} C_e X + D_{eu} \hat{C} & C_e \end{bmatrix} \\ T^T P T &= \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \end{aligned}$$

$\mathcal{H}_2$ **Design** (linearizing transform)

LMI conditions: $\begin{bmatrix} A_{\text{clp}}^T P + P A_{\text{clp}} & P B_{\text{clp}} \\ B_{\text{clp}}^T P & -I \end{bmatrix} \prec 0, \quad \begin{bmatrix} W & C_{\text{clp}} \\ C_{\text{clp}}^T & P \end{bmatrix} \succ 0, \quad \text{and} \quad P \succ 0$

$$\begin{bmatrix} T^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\text{clp}}^T P + P A_{\text{clp}} & P B_{\text{clp}} \\ B_{\text{clp}}^T P & -I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}B_u^T & A + \hat{A}^T & B_w \\ \hat{A} + A^T & YA + \hat{B}C_y + A^TY + C_y^T\hat{B} & YB_w + \hat{B}D_{yw} \\ B_w^T & B_w^TY + D_{yw}^T\hat{B}^T & -I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} W & C_{\text{clp}} \\ C_{\text{clp}}^T & P \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} W & C_eX + D_{eu}\hat{C} & C_e \\ XC_e^T + \hat{C}^TD_{eu} & X & I \\ C_e^T & I & Y \end{bmatrix}$$

$\mathcal{H}_2$ Design

minimize γ

subject to: $\text{Trace}\,(W) < \gamma,$

$$\begin{bmatrix} AX + B_u\hat{C} + XA^T + \hat{C}B_u^T & A + \hat{A}^T & B_w \\ \hat{A} + A^T & YA + \hat{B}C_y + A^TY + C_y^T\hat{B} & YB_w + \hat{B}D_{yw} \\ B_w^T & B_w^TY + D_{yw}^T\hat{B}^T & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} W & C_eX + D_{eu}\hat{C} & C_e \\ XC_e^T + \hat{C}^TD_{eu} & X & I \\ C_e^T & I & Y \end{bmatrix} \succ 0$$

$\mathcal{H}_2$ Design

If this has a solution ($\gamma, X, Y, \hat{A}, \hat{B}$ and $\hat{C}$) then

$$P\,P^{-1} = I \quad \Longrightarrow \quad N M^T = I - Y X \quad \text{(solve for } M \text{ and } N\text{)}$$

Solve for A_k, B_k and C_k from:

$$\begin{aligned} \hat{A} &= N A_k M^T + N B_k C_y X + Y B_u C_k M^T + Y A X \\ \hat{B} &= N B_k \\ \hat{C} &= C_k M^T \end{aligned}$$

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & 0 \end{array}\right] \quad \text{gives} \quad \mathcal{F}_l\left(P(s), K(s)\right) \text{ stable and } \quad \|\mathcal{F}_l\left(P(s), K(s)\right)\|_{\mathcal{H}_2} \leq \sqrt{\gamma}$$

$\mathcal{H}_2$ Design

Using CVX:

```
cvx_begin sdp

    variable X(n,n) symmetric;
    variable Y(n,n) symmetric;
    variable W(ne,ne) symmetric;
    variable Ah(n,n);
    variable Bh(n,ny);
    variable Ch(nu,n);
    variable gamma;

    minimize gamma;
    subject to

        trace(W) < gamma;

        [W, Ce*X + Deu*Ch, Ce;
         X*Ce' + Ch'*Deu', X, eye(n,n);
         Ce', eye(n,n), Y] > 0;

        [A*X + Bu*Ch + X*A' + Ch'*Bu', A+Ah', Bw;
         A'+Ah, Y*A + A'*Y + Bh*Cy + Cy'*Bh', Y*Bw + Bh*Dyw;
         Bw', Bw'*Y + Dyw'*Bh', -eye(nw,nw)] < 0;

cvx_end
```

$\mathcal{H}_2$ Design

Using the Robust Control Toolbox:

```
P = ss(A,[Bw,Bu],[Ce;Cy],[zeros(ne,nw),Deu;Dyw, zeros(ny,nu)]);
[K,G,clp_norm] = h2syn(P,ny,nu);
```

l_1 Design problems

Bounding error amplitudes for bounded amplitude inputs

$$\|M\|_\infty \;=\; \sup_{\|x\|_\infty \leq 1} \|Mx\|_\infty \;=\; \max_{1 \leq i \leq p} \sum_{j=1}^{q} |a_{ij}|$$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix} \;=\; M\,u \;=\; \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ m_2 & m_1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix}$$

Use impulse response matrices and a Youla parametrisation to set up the design problem:

$$\min_{Q} \|P + UQV\|_\infty$$

Robust problems can also be set up and solved as (large) optimisation problems.

Choose matrices, $L = L^T \in \mathcal{R}^{p\times p}$, and $M \in \mathcal{R}^{p\times p}$

Use these to define a function, $f_{\mathcal{D}}(z) : \mathcal{C} \longrightarrow \mathcal{S}^{p\times p}$, $\qquad f_{\mathcal{D}}(z) = L + zM + z^* M^T$

And this is used to define a region of the complex plane: $\quad \mathcal{D} = \{\, z \in \mathcal{C} \mid f_{\mathcal{D}}(z) \prec 0 \,\}$

Example: $\qquad \text{real}(z) < -\alpha$

$$f_{\mathcal{D}}(z) = 2\alpha + z + z^* \qquad (L = 2\alpha, \quad M = 1)$$

Example: $\qquad |z + q| < r \qquad$ disk or radius r centered at $(-q, 0)$

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+z^* & -r \end{bmatrix} \qquad \text{so} \quad L = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Example: $\qquad z$ within a conic sector with inner angle 2θ

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -(z+z^*)\sin\theta & (z-z^*)\cos\theta \\ (z^*-z)\cos\theta & (z+z^*)\sin\theta \end{bmatrix}$$

LMI conditions for pole region constraints

Now given $A \in \mathcal{R}^{n \times n}$ and $P = P^T \in \mathcal{R}^{n \times n}$,

define a function, $M_{\mathcal{D}}(A, P) \;=\; L \otimes P \;+\; M \otimes (AP) \;+\; M^T \otimes (PA^T)$

Theorem: $\operatorname{eig}(A) \in \mathcal{D} \quad \Longleftrightarrow \quad$ there exists $P = P^T \succ 0$, such that $M_{\mathcal{D}}(A, P) \prec 0$

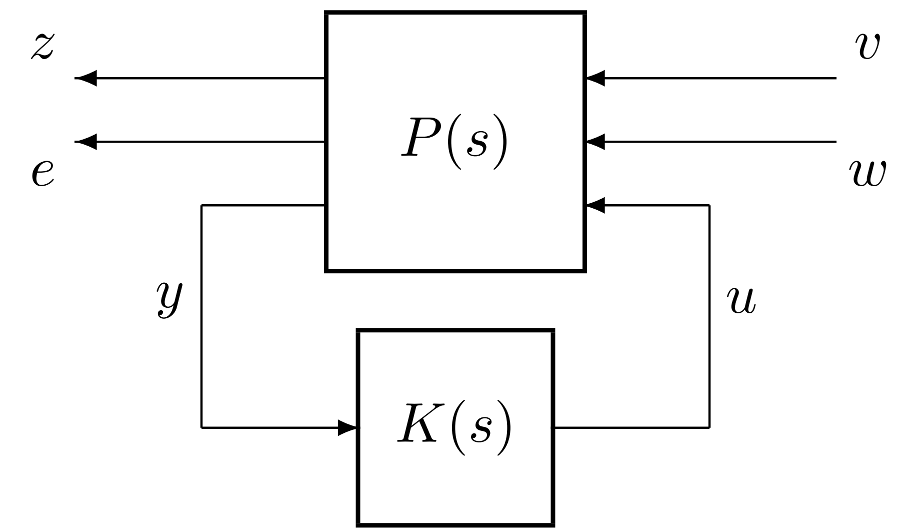
Example: All closed-loop poles have real part less than $-\alpha$

$$f_{\mathcal{D}}(z) \;=\; 2\alpha \;+\; z \;+\; z^* \qquad (L \;=\; 2\alpha, \quad M \;=\; 1)$$

$$\operatorname{eig}(A_{\text{clp}}) \in \mathcal{D} \quad \Longleftrightarrow \quad \text{there exists } P = P^T \succ 0, \quad 2\alpha P + A_{\text{clp}} P + P A_{\text{clp}}^T \prec 0$$

Multiobjective analysis

$$\mathcal{F}_l\left(P(s), K(s)\right) = G(s) = \left[\begin{array}{c|cc} A & B_z & B_w \\ \hline C_z & D_{zv} & D_{zw} \\ C_e & D_{ev} & 0 \end{array}\right]$$



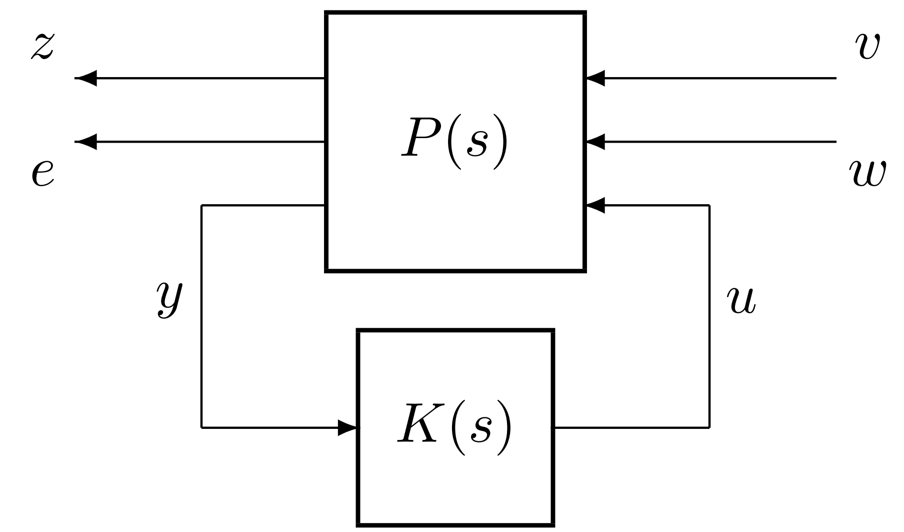
$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} G(s) \begin{bmatrix} I \\ 0 \end{bmatrix} \right\|_{\mathcal{L}_\infty} \leq \gamma \iff P_1 \succ 0, \quad \begin{bmatrix} A^T P_1 + P_1 A & P_1 B_v & C_z^T \\ B_v^T P_1 & -I & D_{zv}^T \\ C_z & D_{zv} & -\gamma^2 I \end{bmatrix} \succ 0$$

$$\left\| \begin{bmatrix} 0 & I \end{bmatrix} G(s) \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{\mathcal{L}_2} \leq \beta \iff A P_2 + P_2 A^T + B_w B_w^T \prec 0, \quad \begin{bmatrix} W & C_e P_2 \\ P_2 C_e^T & P_2 \end{bmatrix} \succ 0,$$

$$\text{Trace}(W) < \beta$$

$$\text{real}(\text{eig}(A)) < -\alpha \iff P_3 \succ 0, \quad A P_3 + P_3 A^T + 2\alpha P_3 \prec 0$$

Multiobjective design



$$\mathcal{F}_l\,(P(s), K(s)) \;=\; G(s) \;=\; \left[\begin{array}{c|cc} A & B_z & B_w \\ \hline C_z & D_{zv} & D_{zw} \\ C_e & D_{ev} & 0 \end{array}\right]$$

For synthesis we further constrain: $\quad P \;=\; P_1 \;=\; P_2 \;=\; P_3$

$$\left\| \begin{bmatrix} I & 0 \end{bmatrix} G(s) \begin{bmatrix} I \\ 0 \end{bmatrix} \right\|_{\mathcal{L}_\infty} \leq \gamma \quad \Longleftrightarrow \quad P \succ 0, \; \begin{bmatrix} A^T P + PA & PB_v & C_z^T \\ B_v^T P & -I & D_{zv}^T \\ C_z & D_{zv} & -\gamma^2 I \end{bmatrix} \succ 0$$

$$\left\| \begin{bmatrix} 0 & I \end{bmatrix} G(s) \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{\mathcal{L}_2} \leq \beta \quad \Longleftrightarrow \quad AP + PA^T + B_w B_w^T \prec 0, \; \begin{bmatrix} W & C_e P \\ P C_e^T & P \end{bmatrix} \succ 0,$$

$$\mathrm{Trace}\,(W) \;<\; \beta$$

$$\mathrm{real}(\mathrm{eig}(A)) \;<\; -\alpha \quad \Longleftrightarrow \quad P \succ 0, \quad AP + PA^T + 2\alpha P \prec 0$$

State feedback

We typically have: $A_{\mathrm{clp}} = A - BK$

Define $F = KP$

Then $A_{\mathrm{clp}} P = (A - BK)P = AP - BF$

We can express our LMI's in terms of P and F

Linearizing transformation

$$\left[\begin{array}{c|c} A_{\mathrm{clp}} & B_{\mathrm{clp}} \\ \hline C_{\mathrm{clp}} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A & B_u C_k & B_w \\ B_k C_y & A_k & B_k D_{yw} \\ \hline C_e & D_{eu} C_k & 0 \end{array}\right]$$

Partition P as: $P = \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} X & M \\ M^T & \star \end{bmatrix}$

Define new controller variables via:

$$\begin{aligned} \hat{A} &= N A_k M^T + N B_k C_y X + Y B_u C_k M^T + Y A X \\ \hat{B} &= N B_k \\ \hat{C} &= C_k M^T \end{aligned}$$

Define an inertia-preserving transform via: $T = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$

Then:

$$T^T P A_{\mathrm{clp}} T = \begin{bmatrix} AX + B_u \hat{C} & A \\ \hat{A} & YA + \hat{B} C_y \end{bmatrix}, \qquad T^T P B_{\mathrm{clp}} = \begin{bmatrix} B_w \\ Y B_w + \hat{B} D_{yw} \end{bmatrix}$$

$$C_{\mathrm{clp}} T = \begin{bmatrix} C_e X + D_{eu} \hat{C} & C_e \end{bmatrix}, \qquad T^T P T = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

Relaxations for structural controller constraints

Discrete-time formulation:
$$x(k+1) = A\,x(k) + B\,u(k)$$
$$y(k) = C\,x(k) + D\,u(k)$$

Fundamental stability result:

$$\text{There exists } P = P^T > 0, \quad \text{such that} \quad \begin{bmatrix} P & AP \\ PA^T & P \end{bmatrix} > 0$$

$$\Longleftrightarrow \quad \text{there exists} \quad P = P^T > 0 \text{ and } G \quad \text{such that} \quad \begin{bmatrix} P & AG \\ GA^T & G + G^T - P \end{bmatrix} > 0$$

Fundamental stability result:

Structured state-feedback

(an almost useless problem - only for illustrative purposes)

If there exists $P = P^T > 0,\quad , G$ (diagonal) and F (diagonal) such that

$$\begin{bmatrix} P & AG + BF \\ A^T G + F B^T & 2G - P \end{bmatrix} > 0,$$

then $K = FG^{-1}$ stabilizes $A + BK$

Note that $K = FG^{-1}$ is diagonal

Arbitrary zero structures can be imposed on K by choice of the F structure

The Lyapunov variable, P, has no structural constraints.

Extended version of the H-infinity LMI characterisation

$P(z)$ is stable and $\|P(z)\|_{\mathcal{H}_\infty} < \gamma$ if and only if

there exists $P = P^T$ and G such that,

$$\begin{bmatrix} P & AG & B & 0 \\ G^T A^T & G + G^T - P & 0 & G^T C^T \\ B^T & 0 & I & D^T \\ 0 & CG & D & \gamma I \end{bmatrix} > 0$$

The state-feedback and dynamic feedback linearising transformations can be extended to these LMI conditions.

Relaxations for structural controller constraints

Extended version of the H-2 LMI characterisation

$$P(z) \text{ is stable and } \|P(z)\|_{\mathcal{H}_2} < \gamma \quad \text{if and only if,}$$

$$\text{there exists } P = P^T \text{ and } G \text{ such that,}$$

$$\text{trace}(W) < \gamma, \quad \begin{bmatrix} W & CG \\ G^T C^T & G^T + G - P \end{bmatrix} > 0, \quad \text{and} \quad \begin{bmatrix} P & AG & B \\ G^T A^T & G^T + G - P & 0 \\ B^T & 0 & I \end{bmatrix} > 0$$

The state-feedback and dynamic feedback linearising transformations can be extended to these LMI conditions.

References

The linearizing transformation is given in:
C. Scherer, P. Gahinet and M. Chilali, "*Multiobjective output-feedback control via LMI optimization*"
IEEE Trans. Auto. Ctrl., vol. 42, no. 7, pp. 896-911, 1997.

Another version of the linearizing transformation (with a useful generalization) is found in:
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A. Megretski and A. Rantzer, "System Analysis via Integral Quadratic Constraints "
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D-stability is discussed in:
M. Chilali and P. Gahinet, " $\mathcal{H}_\infty$ *design with pole placement constraints: an LMI approach*"
IEEE Trans. Auto. Ctrl., vol. 41, no. 3, pp. 358-367, 1996.

The robust pole region derivation is given in:
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IEEE Trans. Auto. Ctrl., vol. 44, no. 12, pp. 2257-2270, 1999.

The multi-objective design approach (including pole regions) is given in:
C. Scherer, P. Gahinet and M. Chilali, "*Multiobjective output-feedback control via LMI optimization*"
IEEE Trans. Auto. Ctrl., vol. 42, no. 7, pp. 896-911, 1997.

References

This approach to $\mathcal{H}_2$ synthesis is similar to the one given in:
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The l1 design problem is introduced and studied in:
M. Dahleh and J.B. Pearson, "l1-optimal feedback controllers for MIMO discrete-time systems," *IEEE Trans. Automatic Control*, vol. 32, no. 4, pp. 314–322, Apr. 1987.

Robust l1 synthesis is studied in:
M. Khammash and J.B. Pearson, "Performance robustness of discrete-time systems with structured uncertainty," *IEEE Trans. Automatic Control*, vol. 36, no. 4, pp. 398–412, 1991.

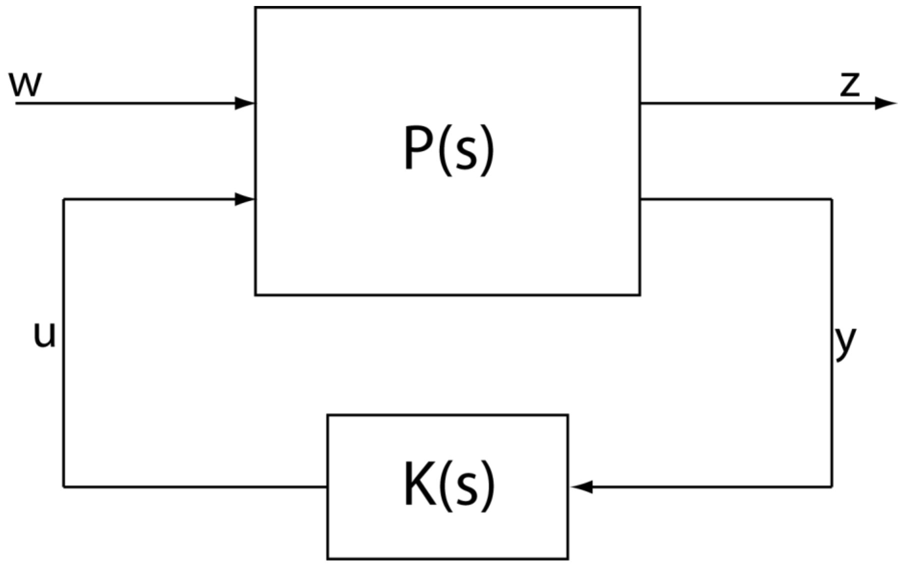
SYSTUNE:

Robust Control for the Masses

Pascal Gahinet
MathWorks, USA

Pierre Apkarian
ONERA, France

Robust Control is a beautiful theory



$$A^T X_\infty + X_\infty A + X_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty + C_1^T C_1 = 0$$

$$\cdots$$

$$\rho(X_\infty Y_\infty) < \gamma^2$$

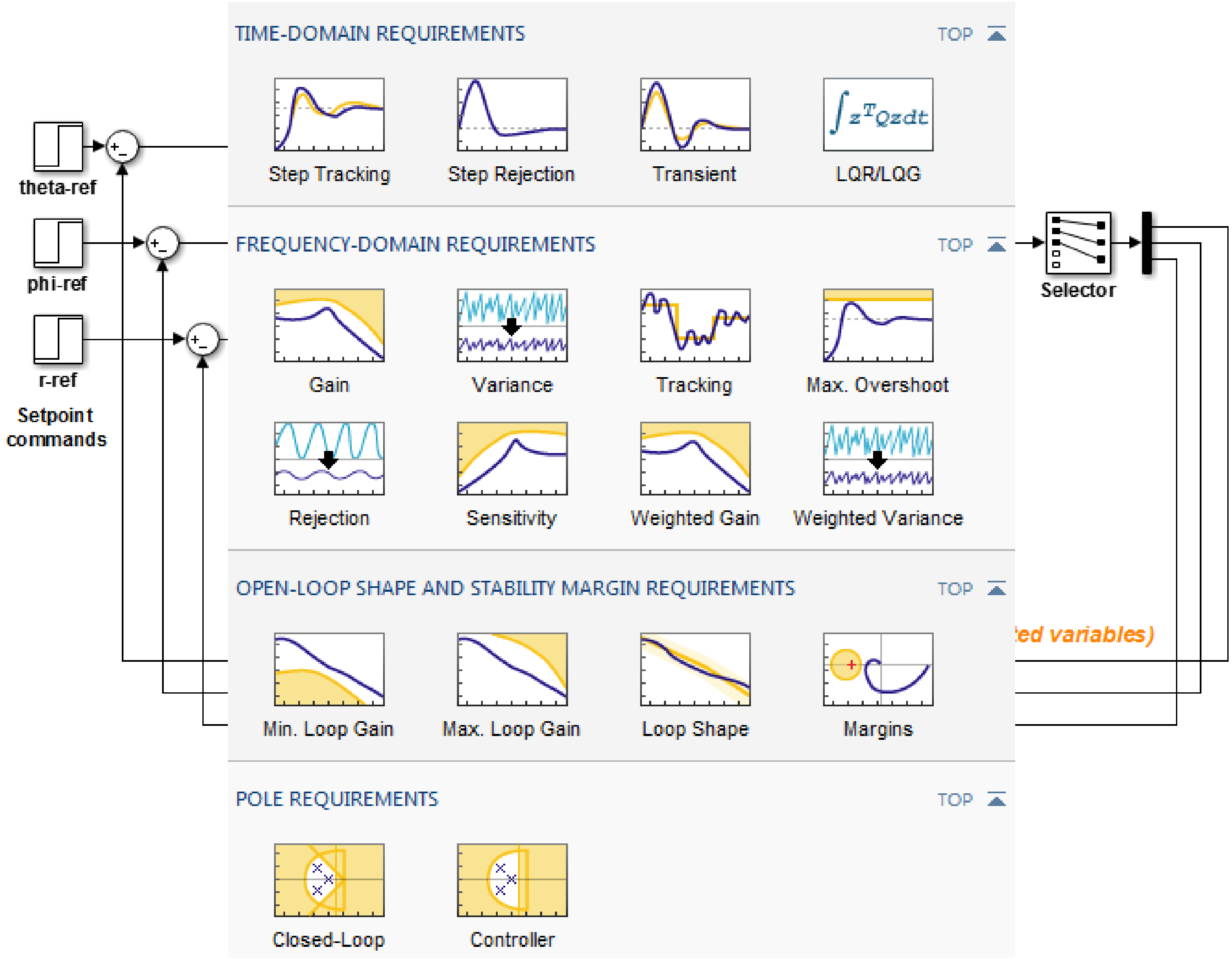
… based on solid principles:

- *Think in terms of gains and loop shapes*
- *Take holistic approach to MIMO control*
- *Pay attention to the Gang of Four (or Six?)*
- *Use disk margins rather than gain/phase margins*
- *Account for plant uncertainty*

Yet it has practical limitations

- *Hard to distill **all** design goals into **one** frequency-weighted H∞ criterion*
- *Produces monolithic, black-box controllers*
- *Controller complexity tends to be high*
- *Convexity often comes at the price of conservatism*

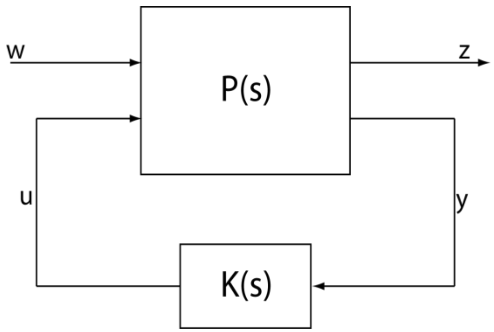
… that make it difficult to apply:



SYSTUNE is a bridge

SYSTUNE is a bridge

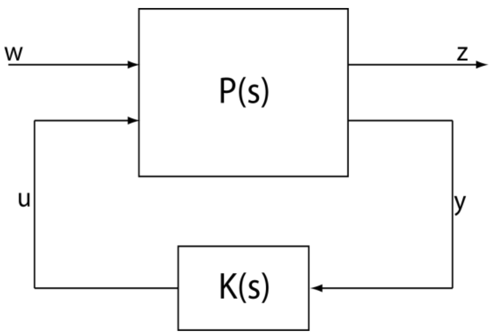
between Robust Control theory…



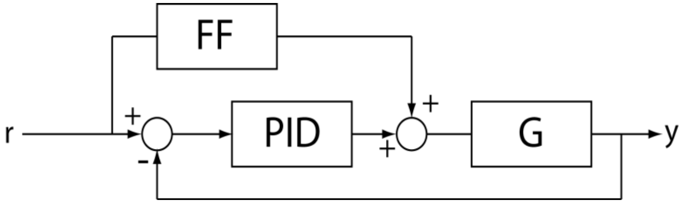
$$\|T_{wz}\| < \gamma$$

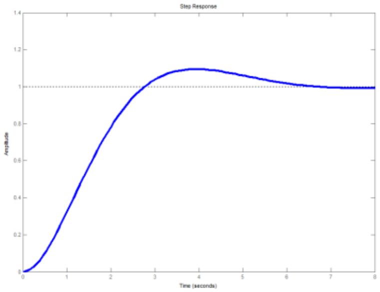
SYSTUNE is a bridge

between Robust Control theory…



$$\|T_{wz}\| < \gamma$$

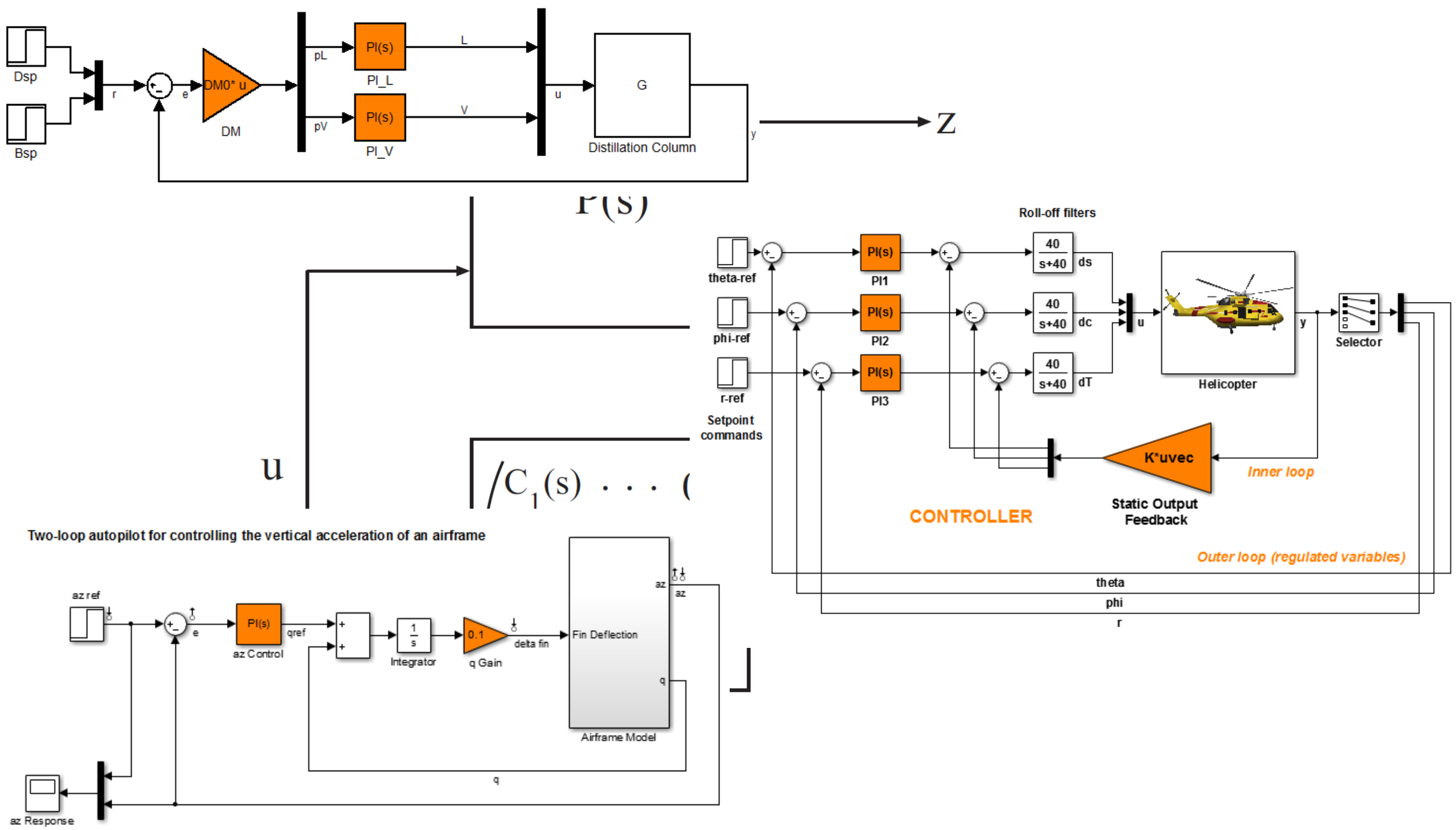




… and Control Engineering practice

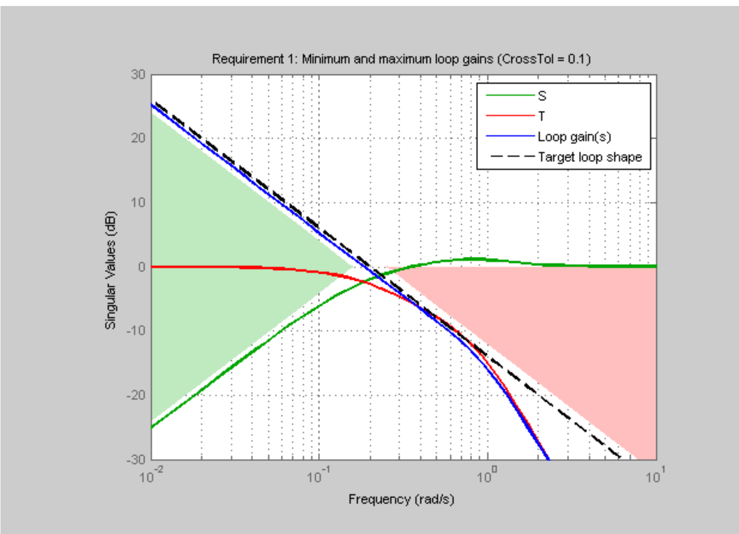
SYSTUNE in a Nutshell

1. Turn any control structure into Standard Form with structured C(s)



SYSTUNE in a Nutshell

2. Automatically turn design goals into $\mathrm{H}_2/\mathrm{H}_\infty$ cost functions

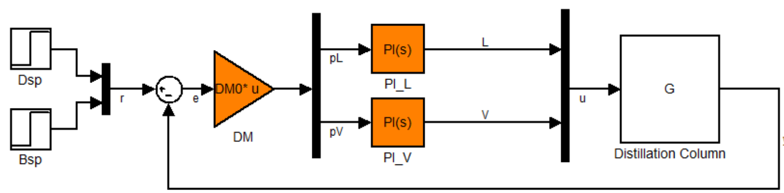


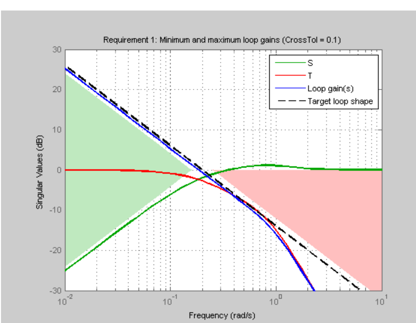
$$f(x) = \frac{\left\| (T(s,x) - T_{ref}(s))/s \right\|_2}{\left\| T_{ref}(s)/s \right\|_2}$$

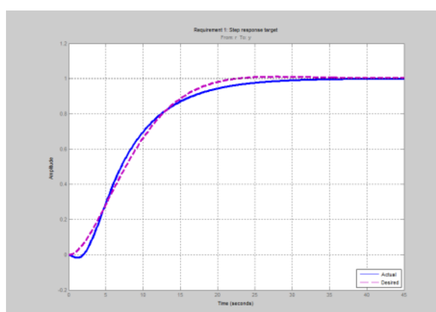
$$f(x) = \left\| \begin{pmatrix} w_S S(s,x) \\ w_T T(s,x) \end{pmatrix} \right\|_\infty$$

SYSTUNE in a Nutshell

3. Use optimization to accommodate the demands of multi-objective, fixed-structure synthesis







$$\min_x \max_i f_i(x)$$

subject to

$$\max_j g_j(x) < 1$$

SYSTUNE in a Nutshell

4. Use specialized solvers that exploit problem nature and structure to solve it efficiently

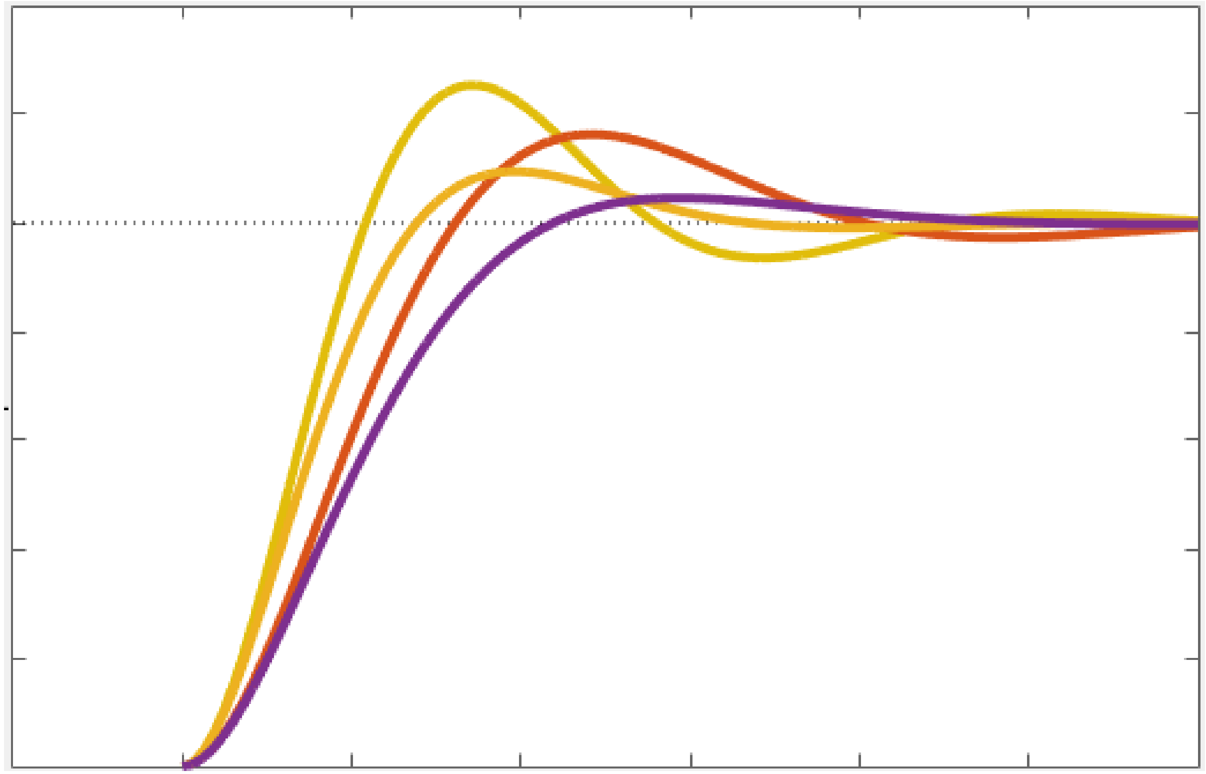
Nonsmooth minimax optimization:

Stabilization: $\min_{x} \max_{i} \operatorname{Re} \lambda_i \left(A(x)\right)$

H_∞ Optimization: $\min_{x} \max_{\omega,i} \sigma_i \left(T(j\omega, x)\right)$

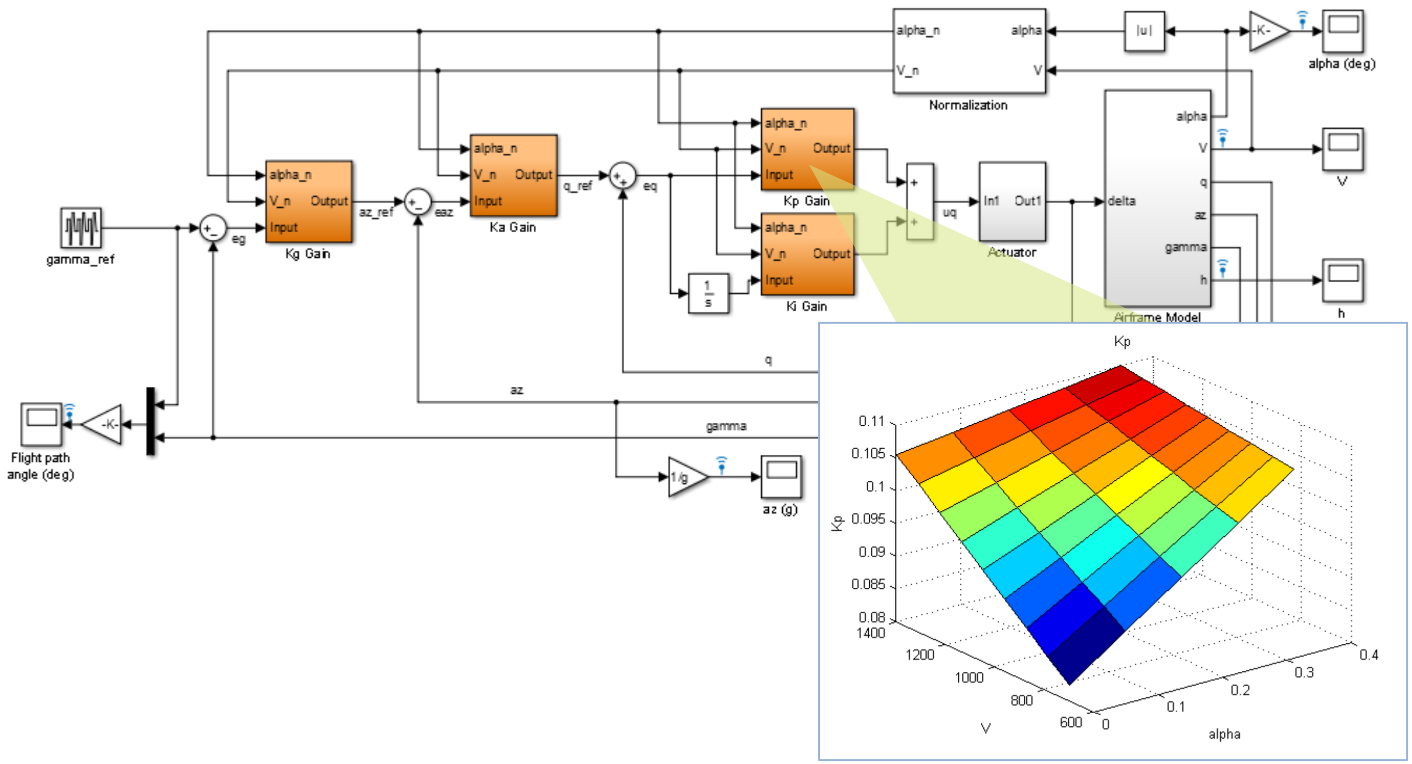
SYSTUNE in a Nutshell

5. Tune controller against multiple models of the plant



SYSTUNE in a Nutshell

5. Tune controller against multiple models of the plant



Pros and Cons

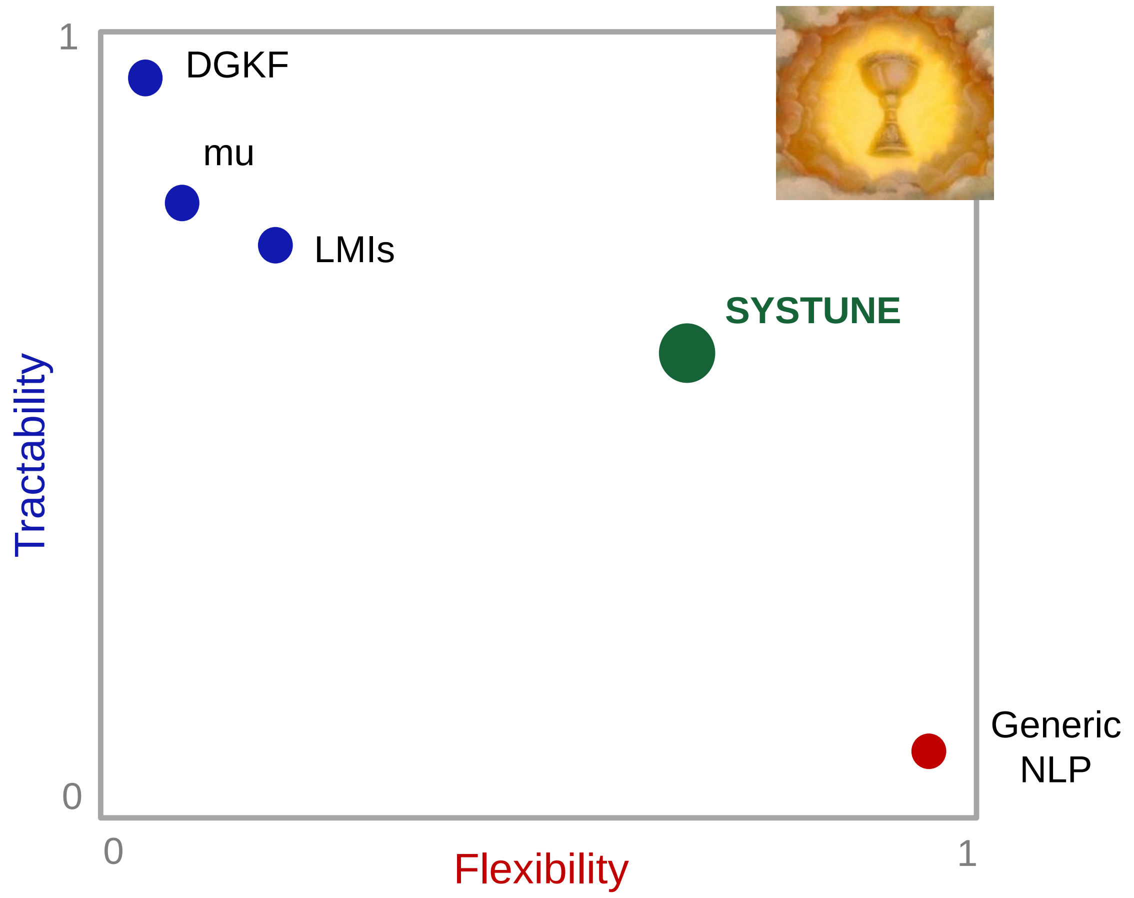
Nonlinear + nonsmooth + nonconvex = **hopeless?**

No, as long as:

- You can live with a **satisfactory** design that is not necessarily globally **optimal**
- Solver is **fast** and gives **coherent** answers (to support iterative design)

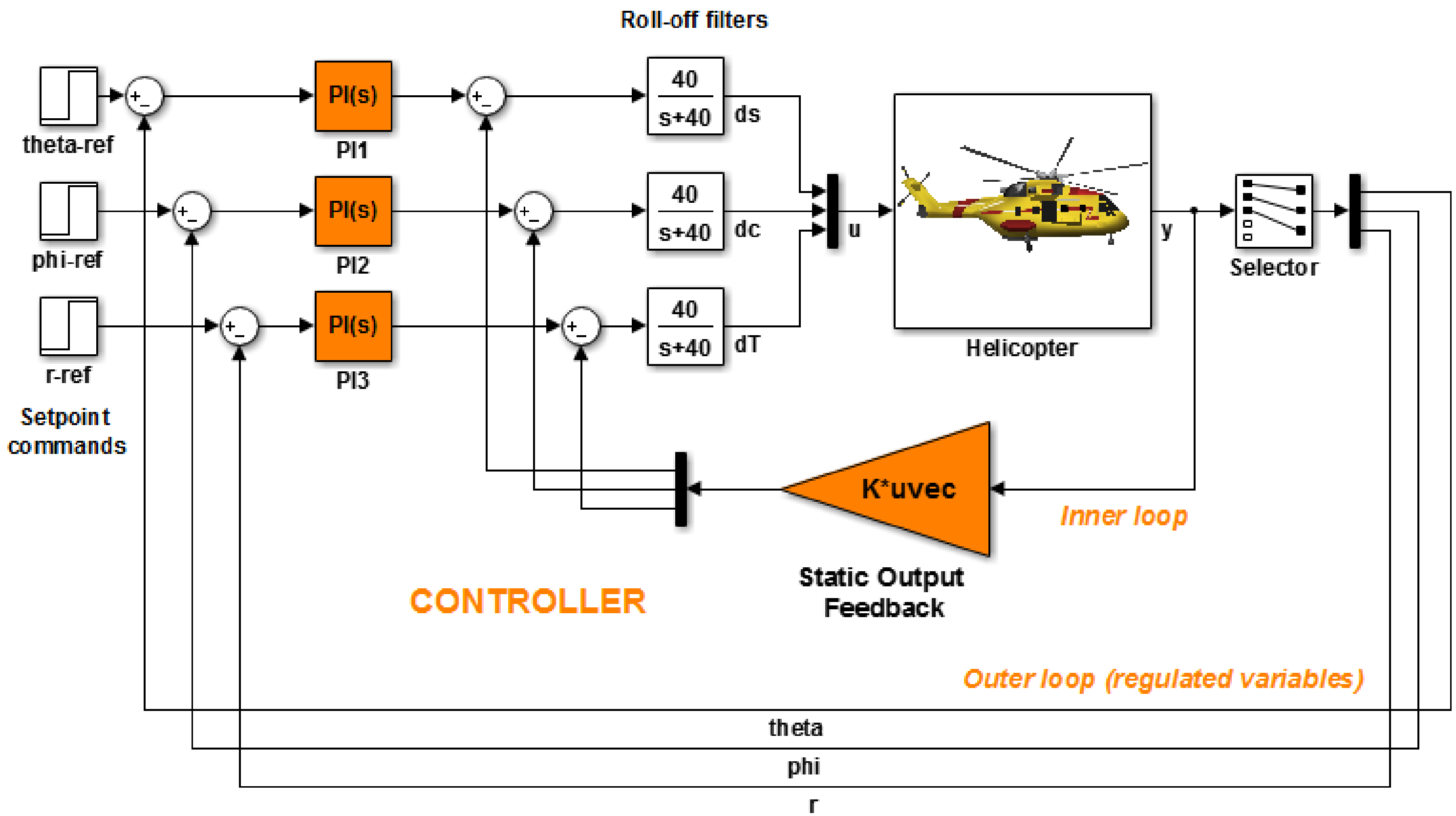
Pros and Cons

- The simpler the controller, the smaller the search space
- No auxiliary variables or Lyapunov matrices
- More constraints tend to make problem easier to solve



Tractability vs. generality tradeoff

Demo: Helicopter Flight Control



8+6 states, 21 tunable parameters

SYSTUNE Software

- SYSTUNE and the Control System Tuner app live in Robust Control Toolbox
- Interface with Simulink (slTuner) lives in Simulink Control Design
- Contact: Pascal.Gahinet@mathworks.com

Conclusion

- Robust Control is not just for PhDs and academics
- You don’t have to go back to manual gain tuning once you leave the classroom
- Tools are available to apply Robust Control methodology to real-world applications

Intro to IQCs:
A simple State-Space Approach

ACC, June 2014

Uncertainty Quantification (UQ) analysis in control

Components

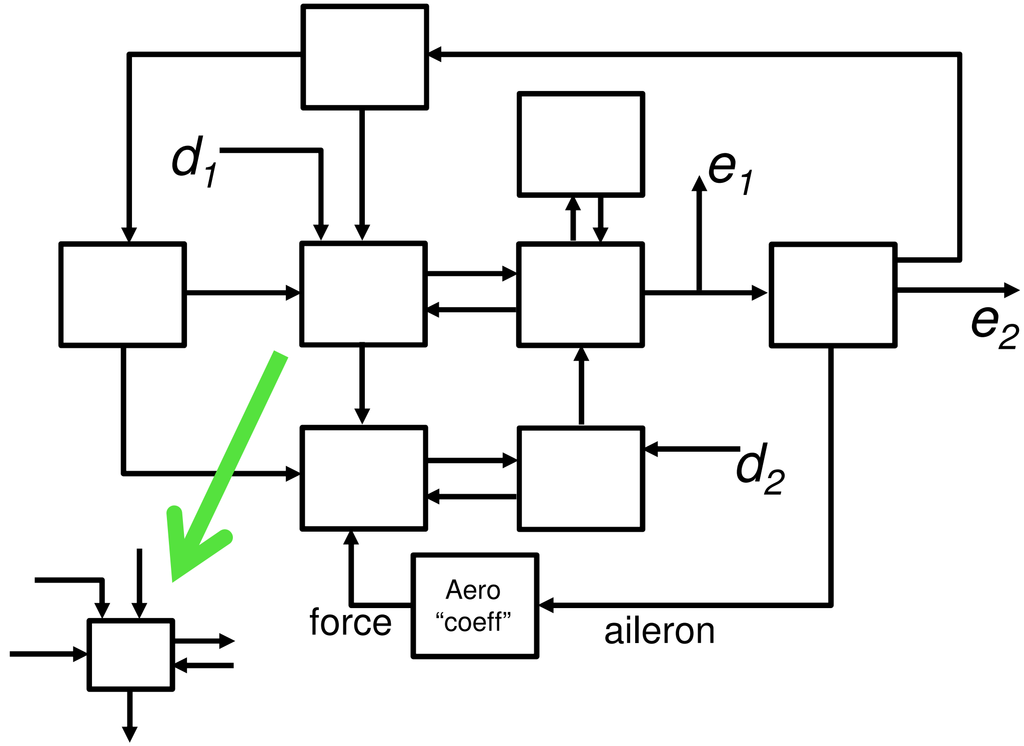
- Relations among variables
  - Here, drawn as input/output

External variables (*d*)

Selected internal variables (*e*)

Interconnection

- Equates variables of "communicating" components
- Implicitly gives (*d*/*e*) relation



UQ question

- Uncertain components
  - Uncertainty is quantified at component level
- Quantify uncertainty in (*d*/*e*) relation

How is uncertainty in a component quantified?

- List of quadratic (in)equalities that variables it relates are **guaranteed** to satisfy
- "Certain": just a special case of uncertain
- Uncertainty in (*d*/*e*) is quantified in same manner – certify that (*d*/*e*) relation always satisfies specific quadratic inequalities

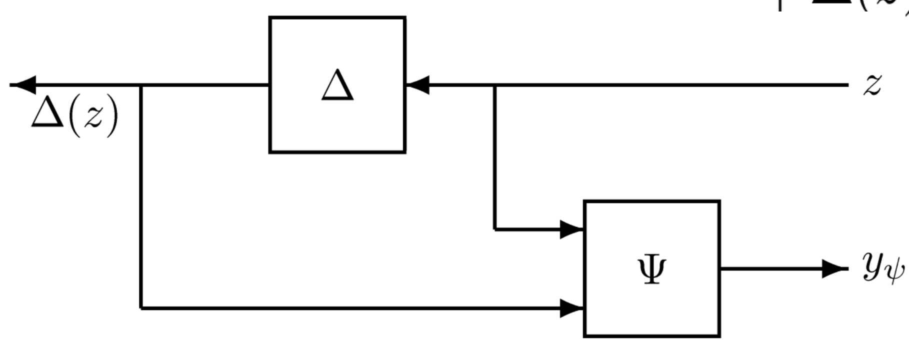
Hard IQCs (dynamic supply rates)

Definition: Suppose Ψ is a stable linear system and M is a symmetric matrix. A bounded operator Δ satisfies the *hard IQC defined by* (Ψ, M) if

$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0$$

for all T and all signals $z \in \mathbf{L}_2^e[0\ \infty)$, with

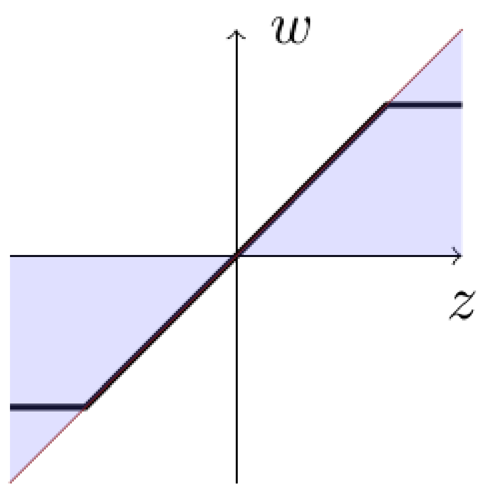
$$y_\psi := \Psi \begin{bmatrix} z \\ \Delta(z) \end{bmatrix}$$

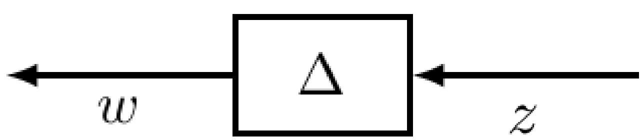


If Δ is an ODE-based model, then this integral constraint must hold for all initial conditions set to 0.

Remark: IQCs generalize the dissipative systems model, allowing for supply rates, q, which are themselves linear dynamical systems

Example 1: IQC For Saturation





Δ is a saturation

$\Downarrow$

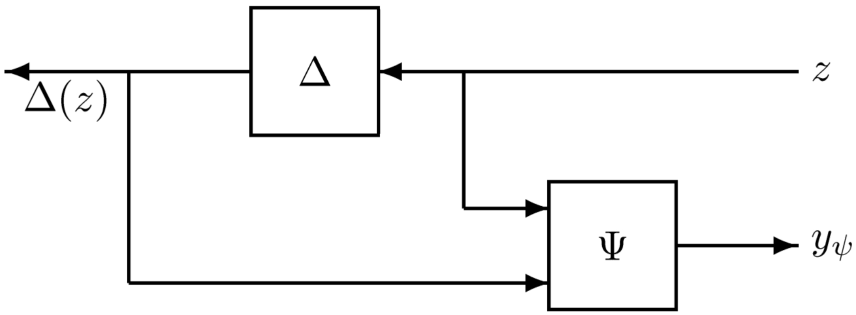
$$2\,(z(t) - w(t))\, z(t) \geq 0 \;\forall t$$

$\Downarrow$

$$\int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} dt \geq 0$$

for all $z \in L_2[0, \infty)$ and $w = \Delta(z)$.

Example 2: IQC For Norm-Bounded LTI Uncertainty



Δ stable, SISO, and LTI with $\|\Delta\| \leq 1$

$\Downarrow$

For any stable, LTI system $d(s)$,

$$\int_{-\infty}^{\infty} d(j\omega)\left(|\hat{z}(j\omega)|^2 - |\hat{w}(j\omega)|^2\right) d\omega \geq 0$$

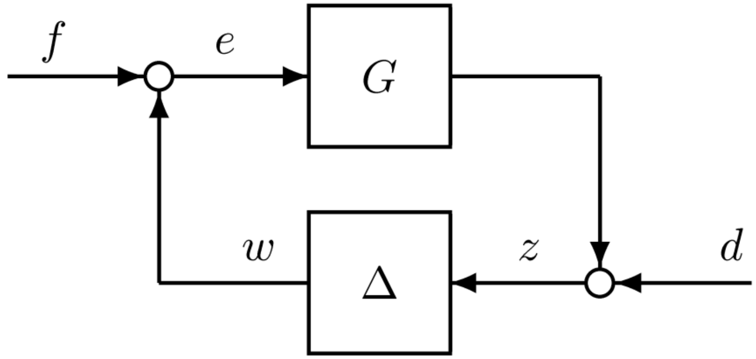
holds for all $z \in L_2[0,\infty)$ and $w = \Delta(z)$.

$\Downarrow$

Δ satisfies the (finite-horizon) hard IQC defined by $\Psi = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

1. Causality is used to show that constraint holds over all finite time horizons.
2. Equivalent to D-scales in robust control

Using IQCs to prove I/O stability



G and Δ in feedback. Δ is unknown, but satisfies the IQC defined by (Ψ, M).

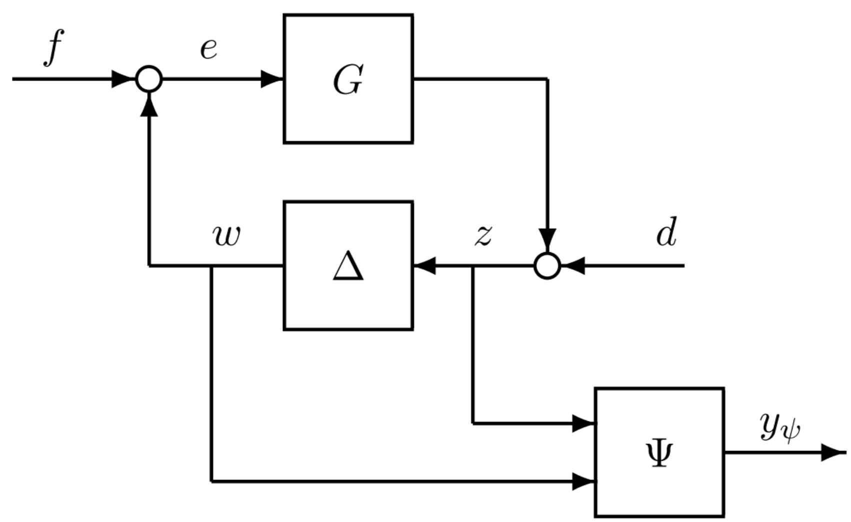
Define interconnection *stable* if $\mathbf{L}_2$-gain from (f, d) to (e, z) is bounded

Does there exist $\gamma > 0$ such that for all $d, f \in \mathbf{L}_2^e[0\ \infty)$ and all T

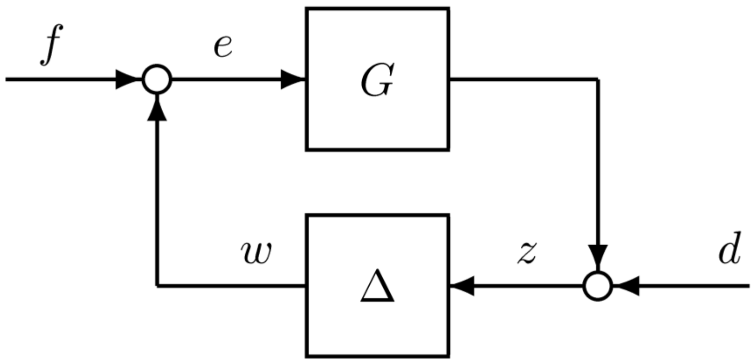
$$\int_0^T e^T(t)e(t) + z^T(t)z(t)dt \leq \gamma^2 \int_0^T f^T(t)f(t) + d^T(t)d(t)dt$$

Augment with Ψ. Since Δ satisfies the IQC defined by (Ψ, M), regardless of external signals f and d, the signal y_ψ is guaranteed to satisfy a constraint, namely, for all T

$$\int_0^T y_\psi^T(t) M y_\psi(t)dt \geq 0$$

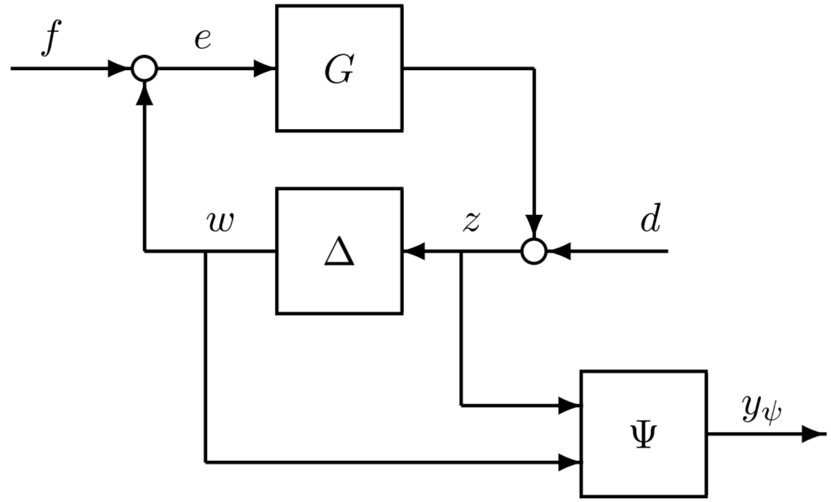


Three different systems



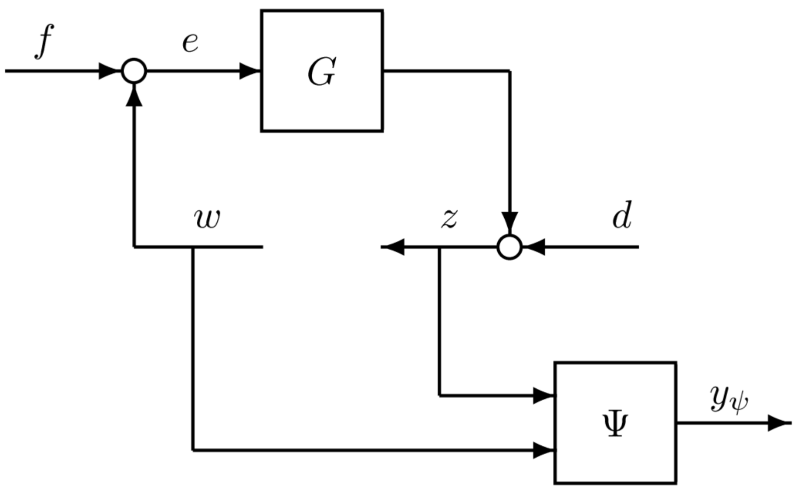
System under consideration, with "unknown" or "troublesome" Δ

System under consideration, augmented with known Ψ, which captures "correlations" in input/output of Δ



… to reach conclusions here

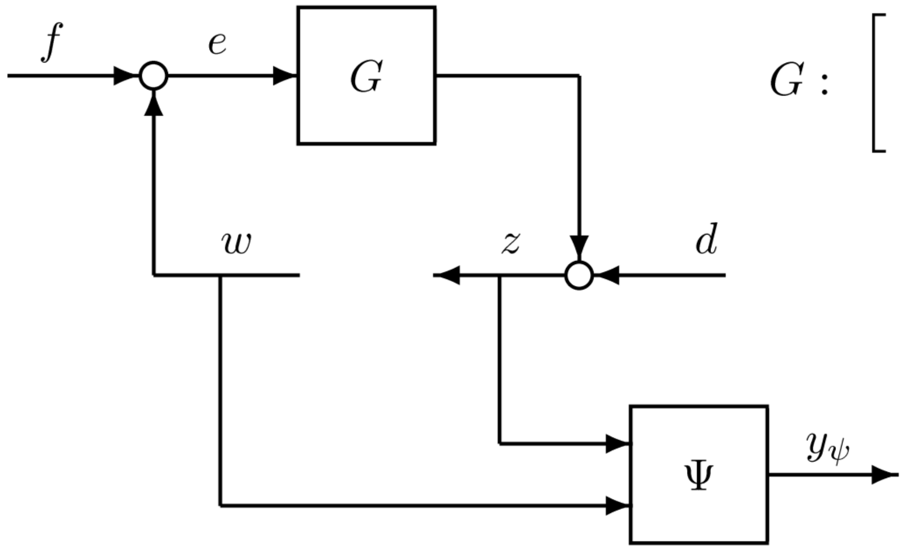
Analyze this… (system model and signal constraint)



System under consideration, with "unknown" or "troublesome" Δ removed, but known augmented correllator Ψ (implicitly) providing information about signals

and $$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0$$

What are the known constraints?



$$G: \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\Psi: \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0 \quad \forall T$$

Equality constraints

- Summing junctions, eg., $e = f + w$
- ODE models of G and Ψ

Inequality constraint

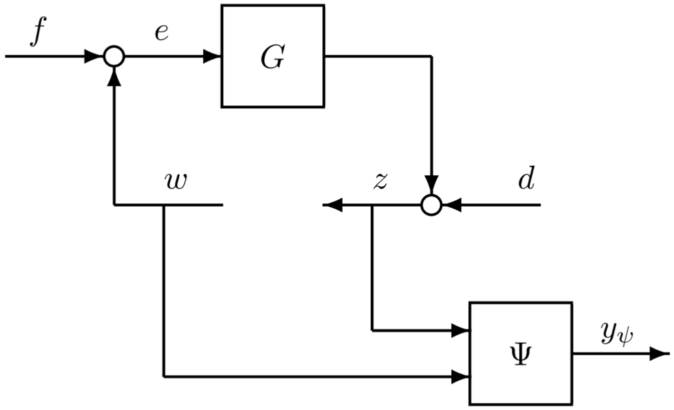
- $\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0$
- extra information about z and w

Under what conditions do these constraints actually imply a constraint between (f, d) and (e, z)? Specifically,

$$\int_0^T e^T(t)e(t) + z^T(t)z(t)dt \leq \gamma^2 \int_0^T f^T(t)f(t) + d^T(t)d(t)dt$$

Use Lyapunov-like construction and S-procedure...

Look for a generalized storage function



$$G: \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\Psi: \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0 \quad \forall T$$

Under what conditions is there a finite $\gamma > 0$ such that the system and signal constraints actually imply that

$$\int_0^T e^T(t)e(t) + z^T(t)z(t)dt \leq \gamma^2 \int_0^T f^T(t)f(t) + d^T(t)d(t)dt \quad \forall T$$

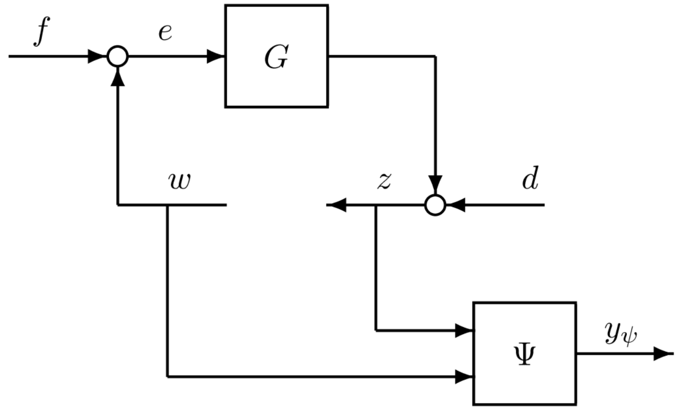
Lyapunov + S-procedure: If there exists a positive, semidefinite function $V(x, \eta)$ and $\lambda \geq 0$ such that

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2(f^T f + d^T d) - (e^T e + z^T z)$$

$$\left(\dot{V} := \nabla_x V \cdot (Ax + Be) + \nabla_\eta V \cdot (\bar{A}\eta + \bar{B}_1 z + \bar{B}_2 w)\right)$$

for **all** values of x, η, d, f, w, e, z and y_ψ, constrained only by the interconnection, then the desired relation holds.

Combining integrated inequality with IQC



$$G : \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

$$\Psi : \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$

Integrate along solutions

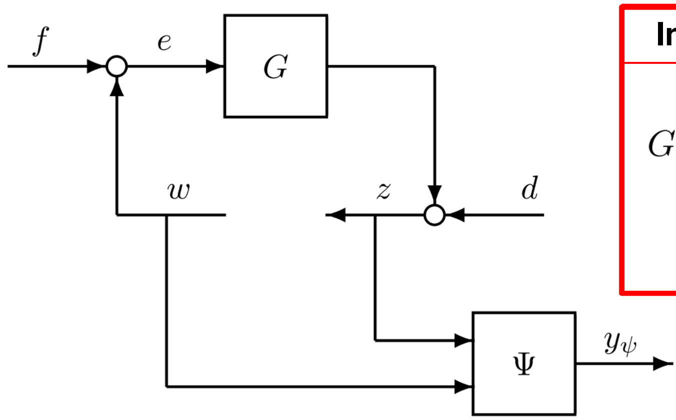
$$(\|e\|_{2,T}^2 + \|z\|_{2,T}^2) \leq \gamma^2 (\|f\|_{2,T}^2 + \|d\|_{2,T}^2) \underbrace{- V(x(T),\eta(T))}_{\leq 0} \underbrace{- \lambda \int_0^T y_\psi^T(t) M y_\psi(t) dt}_{\leq 0}$$

$$(\|e\|_{2,T}^2 + \|z\|_{2,T}^2) \leq \gamma^2 (\|f\|_{2,T}^2 + \|d\|_{2,T}^2)$$

Invoke hard-IQC on y_ψ

$$\int_0^T y_\psi^T(t) M y_\psi(t) dt \geq 0 \quad \forall T$$

Solving the inequality: Finding V and λ



Interconnection equations

$$G: \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \Psi: \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$e = f + w \qquad z = d + y$$

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$

Inequality in 8 variables $(x, \eta, f, w, d, e, z, y_\psi)$, but only (x, η, f, w, d) are independent

Use interconnection equations, recast as…

Inequality in 5 variables (x, η, f, w, d), to hold for all values

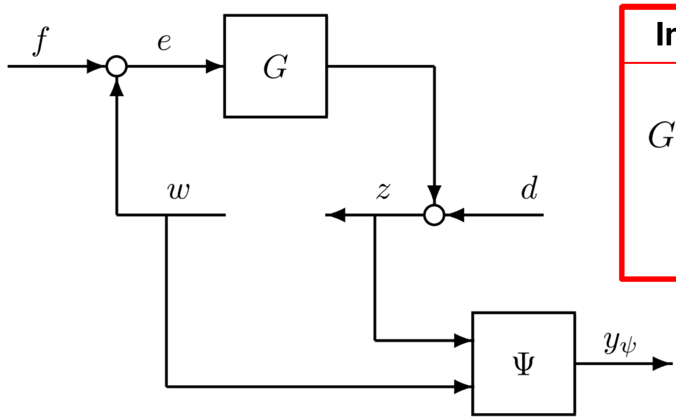
For notational purposes, define

$$s := \begin{bmatrix} x \\ \eta \\ f \\ d \\ w \end{bmatrix} \in \mathbf{R}^{n_x + n_\eta + n_f + n_d + n_w}$$

Step 1: Restrict attention to quadratic $V(x,\eta)$, so $V(x,\eta) := \begin{bmatrix} x \\ \eta \end{bmatrix}^T P \begin{bmatrix} x \\ \eta \end{bmatrix}$ for some $P = P^T \succeq 0$. Then

$$\begin{aligned} \dot{V}(x,\eta) &= 2 \begin{bmatrix} x \\ \eta \end{bmatrix}^T P \begin{bmatrix} Ax + B(f+w) \\ \bar{A}\eta + \bar{B}_1 (Cx + D(f+w) + d) + \bar{B}_2 w \end{bmatrix} \\ &= s^T \, [\text{Linear in } P] \, s \end{aligned}$$

Solving the inequality: SDP to find V and λ



Interconnection equations

$$G: \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad \Psi: \begin{bmatrix} \dot{\eta} \\ y_\psi \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \eta \\ z \\ w \end{bmatrix}$$

$$e = f + w \qquad z = d + y$$

$$\dot{V}(x,\eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$

Inequality in 5 variables (x, η, f, w, d) $\qquad s := \begin{bmatrix} x \\ \eta \\ f \\ d \\ w \end{bmatrix}$

$$\dot{V}(x,\eta) = s^T \,[\text{Linear in } P]\, s \qquad \gamma^2 \left(f^T f + d^T d\right) = s^T \left[\gamma^2 \cdot \text{Simple Matrix}\right] s$$

$$\lambda y_\psi^T M y_\psi = s^T [\text{Linear in } \lambda] s \qquad e^T e = s^T \,[\text{Simple Matrix}]\, s$$

$$z^T z = s^T \,[\text{Simple Matrix}]\, s$$

$$M(P, \lambda, \gamma^2)$$

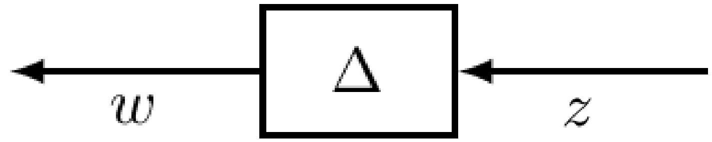
Inequality becomes: $s^T \left[\text{Linear in } P, \lambda, \gamma^2 \right] s \leq 0 \quad \forall s \in \mathbf{R}^{n_x + n_\eta + n_f + n_d + n_w}$

IQC Robust Stability Analysis: Does there exist $P = P^T \succeq 0$, $\lambda \geq 0$, and (representing γ^2) $\gamma_s > 0$ such that

$$M(P, \lambda, \gamma_s) \preceq 0,$$

which is yet another (important) example of a semidefinite program.

IQCs in the Frequency Domain



Let $\Pi : j\mathbb{R} \to \mathbb{C}^{m\times m}$ be Hermitian-valued.

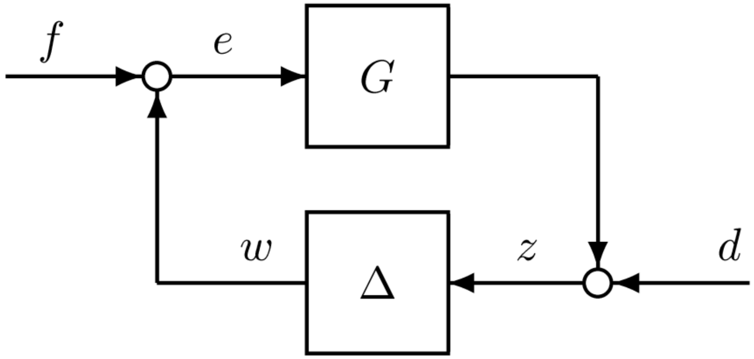
Def.: Δ satisfies IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $z \in L_2[0,\infty)$ and $w = \Delta(z)$.

Ref: Megretski and Rantzer, "System Analysis via Integral Quadratic Constraints", TAC, 1997.

Frequency Domain Stability Condition



Thm: Assume:

1. Interconnection of G and $\tau\Delta$ is well-posed $\forall\tau \in [0,1]$
2. $\tau\Delta \in$ IQC(Π) $\forall\tau \in [0,1]$.
3. $\exists\ \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \, \forall\omega$$

Then interconnection is stable.

Connection Between Time and Frequency Domain

1. Hard Time Domain IQC (TD IQC) defined by (Ψ, M):

$$\int_0^T y_\psi(t)^T M y_\psi(t)\, dt \geq 0$$

for all $T \geq 0$ where $y_\psi = \Psi \left[\begin{smallmatrix} z \\ \Delta(z) \end{smallmatrix} \right]$.

2. Frequency Domain IQC (FD IQC) defined by Π:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

where $w = \Delta(z)$.

A non-unique factorization $\Pi = \Psi^\sim M \Psi$ connects the approaches but there are two technical issues.

“Soft” Infinite Horizon Constraint

Freq. Dom. IQC: $$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

"Soft" Infinite Horizon Constraint

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$\Updownarrow$ **Factorization** $\Pi = \Psi^{\sim} M \Psi$

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Psi(j\omega)^* M \Psi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega = \int_{-\infty}^{\infty} \hat{y}_\psi^*(j\omega) M \hat{y}_\psi(j\omega) \geq 0$$

“Soft” Infinite Horizon Constraint

Freq. Dom. IQC: $\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$

$\Updownarrow$ **Factorization $\Pi = \Psi^{\sim} M \Psi$**

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Psi(j\omega)^* M \Psi(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega = \int_{-\infty}^{\infty} \hat{y}_\psi^*(j\omega) M \hat{y}_\psi(j\omega) \geq 0$$

$\Updownarrow$ Parseval's Theorem

"Soft" IQC: $: \int_0^{\infty} y_\psi(t)^T M y_\psi(t) dt \geq 0$

Issue # 1: DI stability test requires “hard” finite-horizon IQC

Sign Indefinite Quadratic Storage

Factorize $\Pi = \Psi^{\sim} M \Psi$ and define $\Psi \begin{bmatrix} G \\ I \end{bmatrix} := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

Sign Indefinite Quadratic Storage

Factorize $\Pi = \Psi^{\sim} M \Psi$ and define $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right] := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

(*) KYP LMI: $\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0$

Sign Indefinite Quadratic Storage

Factorize $\Pi = \Psi^{\sim} M \Psi$ and define $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix} \right] := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$.

$$\text{(*) KYP LMI: } \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0$$

KYP Lemma: $\exists \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I$$

iff $\exists$ $P = P^T$ satisfying the KYP LMI (*).

Lemma: $V = \left[\begin{smallmatrix} x \\ \eta \end{smallmatrix} \right]^T P \left[\begin{smallmatrix} x \\ \eta \end{smallmatrix} \right]$ satisfies

$$\dot{V}(x, \eta) + \lambda y_\psi^T M y_\psi \leq \gamma^2 (f^T f + d^T d) - (e^T e + z^T z)$$

for some finite $\gamma > 0$ iff $\exists$ $P \geq 0$ satisfying the KYP LMI (*).

Issue # 2: DI stability test requires $P \geq 0$

Equivalence of Approaches

Def.: $\Pi = \Psi^{\sim} M \Psi$ is a **J-Spectral factorization** if $M = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ and Ψ, Ψ^{-1} are stable.

Equivalence of Approaches

Def.: $\Pi = \Psi^{\sim} M \Psi$ is a **J-Spectral factorization** if $M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and Ψ, Ψ^{-1} are stable.

Thm.: If $\Pi = \Psi^{\sim} M \Psi$ is a J-spectral factorization then:

1. If $\Delta \in$IQC(Π) then $\Delta \in$ IQC(Ψ, M)
   (FD IQC $\Leftrightarrow$ Finite Horizon Time-Domain IQC)
2. All solutions of KYP LMI satisfy $P \geq 0$.

Proof: 1. follows from Megretski (Arxiv, 2010)
2. follows from LQ results by Willems (TAC, 1972) and game theory results by Engwerda (2005). ■

Ref: Seiler, "Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints", Submitted to TAC, 2014.

Equivalence of Approaches

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Proof: 1. follows from Megretski (Arxiv, 2010)
2. follows from LQ results by Willems (TAC, 1972) and game theory results by Engwerda (2005). ■

Thm.: Partition $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{21}^* \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$. Π has a J-spectral factorization if $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ $\forall \omega \in \mathbb{R} \cup \{+\infty\}$.
Proof: Use equalizing vectors thm. of Meinsma (SCL, 1995) ■.

Ref: Seiler, "Stability Analysis with Dissipation Inequalities and Integral Quadratic Constraints", Submitted to TAC, 2014.

Robust Model Predictive Control
A Short Introduction

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† Ecole Polytechnique Federale de Lausanne, Switzerland

June 2014

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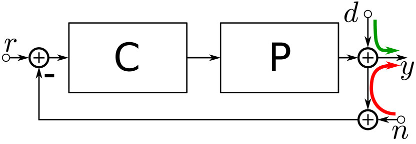
Two Different Perspectives

Classical design: design C

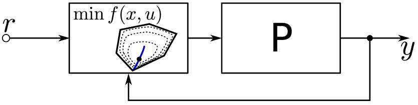


Dominant issues addressed

- Disturbance rejection $(d \rightarrow y)$
- Noise insensitivity $(n \rightarrow y)$
- Model uncertainty

(usually in *frequency domain*)

MPC: real-time, repeated optimization to choose $u(t)$



Dominant issues addressed

- Control constraints (limits)
- Process constraints (safety)

(usually in *time domain*)

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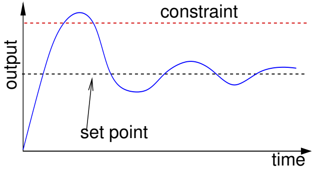
Constraints in Control

All physical systems have **constraints**:

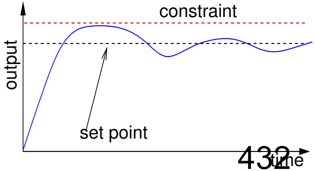
- Physical constraints, e.g. actuator limits
- Performance constraints, e.g. overshoot
- Safety constraints, e.g. temperature/pressure limits

Optimal operating points are often near constraints.



Predictive control:

- Constraints included in the design
- Optimal plant operation



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Main Idea

Objective:

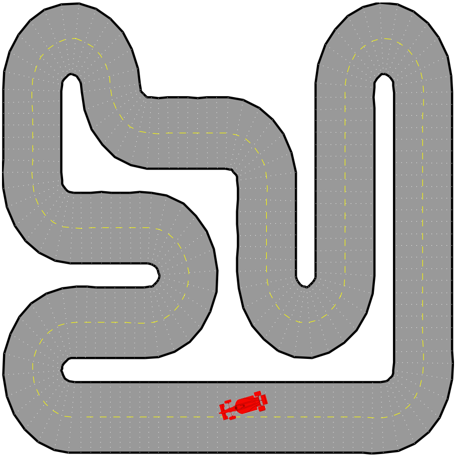
- Minimize lap time

Constraints:

- Avoid other cars
- Stay on road
- Don't skid
- Limited acceleration

Intuitive approach:

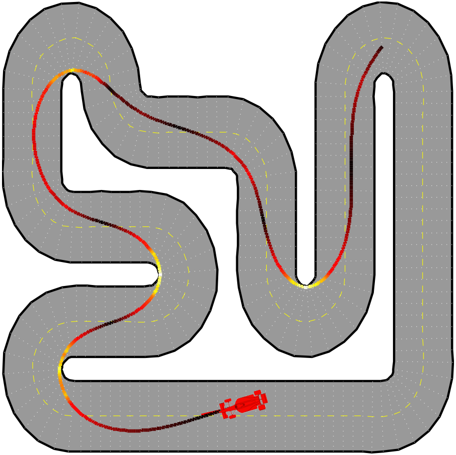
- Look forward and plan path based on
  - Road conditions
  - Upcoming corners
  - Abilities of car
  - etc...



Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

- Solve **optimization problem** to compute minimum-time path



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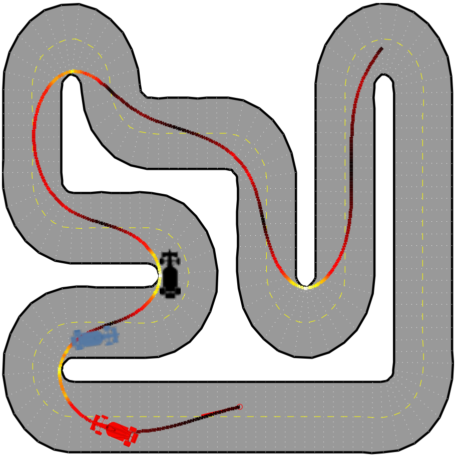
Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

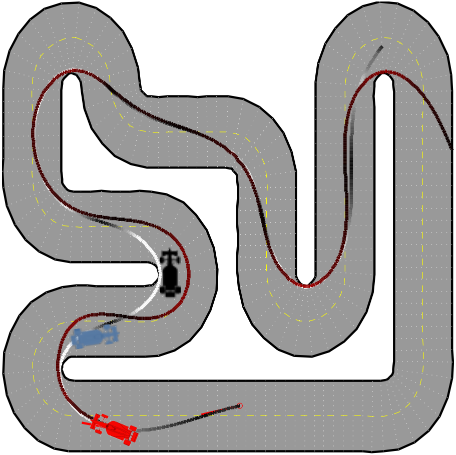
- Solve **optimization problem** to compute minimum-time path
- What to do if something unexpected happens?
  - We didn't see a car around the corner!
  - Must introduce *feedback*



Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

- Solve **optimization problem** to compute minimum-time path
- Obtain series of planned control actions
- Apply *first* control action
- Repeat the planning procedure



Model Predictive Control

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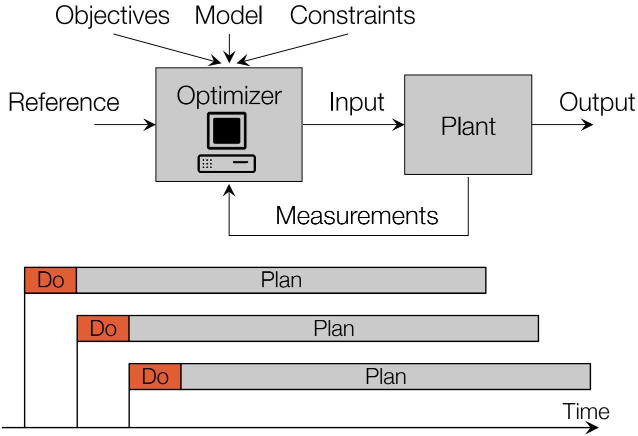

Receding horizon strategy introduces **feedback**.

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MPC: Mathematical Formulation

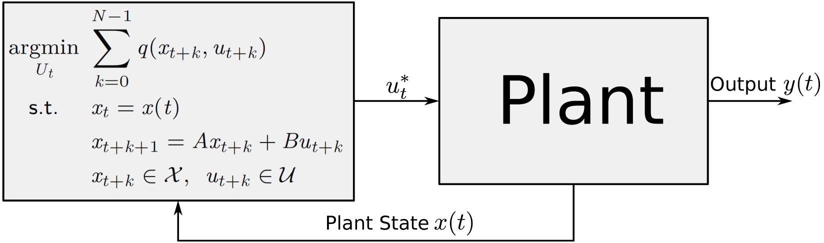
$$U_t^*(x(t)) := \underset{U_t}{\operatorname{argmin}} \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k})$$

subj. to	$x_t = x(t)$	measurement
	$x_{t+k+1} = Ax_{t+k} + Bu_{t+k}$	system model
	$x_{t+k} \in \mathcal{X}$	state constraints
	$u_{t+k} \in \mathcal{U}$	input constraints
	$U_t = \{u_t, u_{t+1}, \ldots, u_{t+N-1}\}$	optimization variables

Problem is defined by

- **Objective** that is minimized,
  e.g., distance from origin, sum of squared/absolute errors, economic,...
- Internal **system model** to predict system behavior
  e.g., linear, nonlinear, single-/multi-variable, ...
- **Constraints** that have to be satisfied
  e.g., on inputs, outputs, states, linear, quadratic,...

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(t)$
- Find the optimal input sequence for the entire planning window N: $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the *first* control action u_t^*

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System Model

- consider generic discrete–time dynamical system:

$$x_{k+1} = g(x_k, u_k, w_k), \tag{1}$$

where the state and control vector are subject to constraints:

$$(x_k, u_k) \in \mathcal{X} \times \mathcal{U} \tag{2}$$

and the perturbation vector w_k assumes its values in a set $\bar{\mathcal{W}}$:

$$w_k \in \bar{\mathcal{W}} \tag{3}$$

- the set $\bar{\mathcal{W}}$ is evaluation of a set–valued function $\mathcal{W}(\cdot)$, which can be:
  - simply a constant set: $\bar{\mathcal{W}} = \mathcal{W} = \text{const}$
  - time varying: $\bar{\mathcal{W}} = \mathcal{W}_k$,
  - a mapping of the state vector x_k, control u_k or any other information pattern: $\bar{\mathcal{W}} = \mathcal{W}(x_k, u_k, x_{k-1})$,

Examples of Uncertain Models

Linear Additive Uncertainty

$$x_{k+1} = Ax_k + Bu_k + Gw_k,$$

$$(x_k, u_k) \in \mathcal{X} \times \mathcal{U},$$

$$w_k \in \mathcal{W}$$

- Offset w_k unknow at time k. Bounds $\mathcal{W}$ known.
- $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ are polytopes.

Examples of Uncertain Models

Linear Parameter Varying (LPV) / Polytopic Uncertainty

$$x_{k+1} = A(w_k^p)x_k + B(w_k^p)u_k + Ew_k^a$$

$$A(w^p) = A^0 + \sum_{i=1}^{n_p} A^i w_c^{p,i},\ \ B(w^p) = B^0 + \sum_{i=1}^{n_p} B^i w_c^{p,i}$$

$$x_k \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \forall t \geq 0.$$

- Vectors $w_k^a \in \mathbb{R}^{n_a}$ and $w_k^p \in \mathbb{R}^{n_p}$ are unknown additive disturbances and parametric uncertainties, respectively.
- The disturbance vector is $w = [w^{a\prime},\ w^{p\prime}]' \in \mathcal{W} \subset \mathbb{R}^{n_w}$
- $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ are polytopes.
- Results can be extended to PieceWise Affine LPV

Constrained Robust Control

We will discuss two main goals:

Robust reachability/controllablity

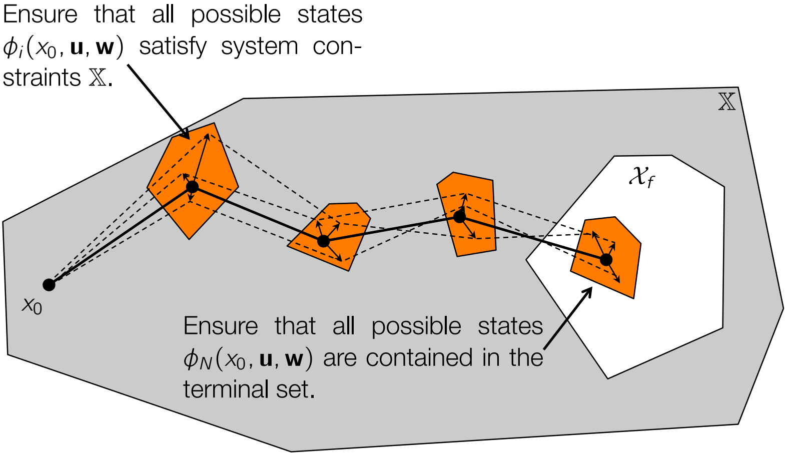
- For which initial conditions $x_0 \in \mathcal{X}$ can the state vector be "steered" into a given target set $\mathcal{X}_0$?

Robust control synthesis

- Select appropriate control laws $\pi(\cdot)$ using a suitable optimality criteria
  (min–max, max–min, time–optimal, mean value, ...)

Some classical references: [14, 8, 4, 3, 7, 1]

Constrained Robust Control



The idea: Compute a set of tighter constraints such that if *the nominal system* meets these constraints, then the uncertain system will too.
We then design control law with robustified constraints

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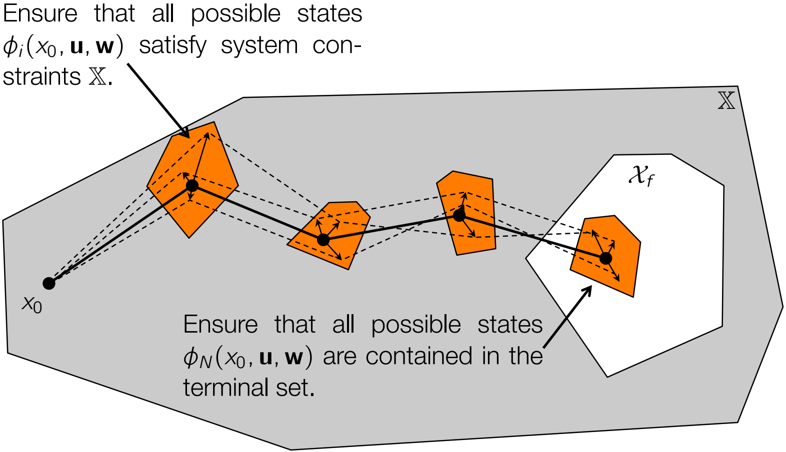
Robust Constraint Satisfaction



The idea: Compute a set of tighter constraints such that if *the nominal system* meets these constraints, then the uncertain system will too.

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Robust Controllability

- for a given target set $\mathcal{S}$ we define:

One step controllable sets

$$\mathrm{Pre}(\mathcal{S}, \mathcal{W}) \triangleq \{x \in \mathbb{R}^n \;:\; \exists u \in \mathcal{U} \text{ s.t. } g(x,u,w) \subseteq \mathcal{S},\ \forall w \in \mathcal{W}\}.$$

- $\mathrm{Pre}(\mathcal{S}, \mathcal{W})$ is the set of states which can be robustly driven into the target set $\mathcal{S}$ in one time step for all admissible disturbances.

Robust Controllability: Example

- Consider the second order unstable system[1]

$$\left\{ \; x(t+1) \;\; = \;\; \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + w(t) \right.$$

subject to the input and state constraints

$$u(t) \in \mathcal{U} = \{u \;:\; -5 \le u \le 5\}\,,\ \forall t \ge 0$$
$$x(t) \in \mathcal{X} = \left\{x \;:\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}\right\},\ \forall t \ge 0,$$

[1] Click here to download the Matlab© code.

Robust Controllability: Example

- where

$$w(t) \in \mathcal{W} = \{w \;:\; -1 \le w \le 1\},\; \forall t \ge 0.$$

The set $\text{Pre}(\mathcal{X}, \mathcal{W})$ is computed as follows

$$\mathcal{X} = \{x \;:\; Hx \le h\}, \quad \mathcal{U} = \{u \;:\; H_u u \le h_u\},$$

to obtain

$$\begin{aligned}
\text{Pre}(\mathcal{X}, \mathcal{W}) &= \left\{x \in \mathbb{R}^2 \;:\; \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + w \in \mathcal{X},\; \forall\; w \in \mathcal{W}\right\} \\
&= \left\{x \in \mathbb{R}^2 \;:\; \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h - Hw \\ h_u \end{bmatrix}, \right. \\
&\qquad \left. \forall\; w \in \mathcal{W}\right\}.
\end{aligned}$$

Robust Controllability: Example

- The set $\mathrm{Pre}(\mathcal{X}, \mathcal{W})$ can be compactly written as

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \;=\; \left\{ x \in \mathbb{R}^2 \;:\; \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} \tilde{h} \\ h_u \end{bmatrix} \right\}, \quad (5)$$

  where

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w).$$

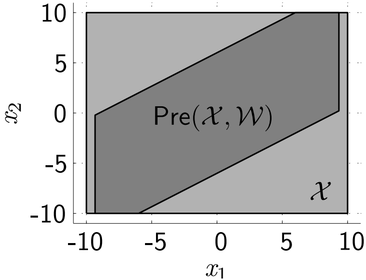
  A linear program is required to solve the above. In this example H_i and $\mathcal{W}$ have simple expressions and we get $\tilde{h} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$.

- This is called "Robustification"

Robust Controllability: Example

- The halfspaces in (5) define a polytope in the state-input space
- $\exists\, u$ is a projection operation and the set $\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$ is

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1.5 \\ -1 & 1.5 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 9.3 \\ 9.3 \\ 9 \\ 9 \\ 10 \\ 10 \end{bmatrix}.$$



Robust Controllability: constrained LTI systems

For constrained LTI sytems:

$$x_{k+1} = Ax_k + Bu_k + Gw_k,$$
$$(x_k, u_k) \in \mathcal{X} \times \mathcal{U}$$
$$w \in \mathcal{W}$$

the one–step controllable sets for a given $\mathcal{S} \subseteq \mathcal{X}$ can be expressed as:

One–step Robust Controllable Sets, linear case

$$\operatorname{Pre} \mathcal{S}, \mathcal{W} := \{(\mathcal{S} \ominus G\mathcal{W}) \oplus (-B\mathcal{U})\} \circ A$$

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N-Steps Robust Controllable Set

Definition (N-Step Robust Controllable Set $\mathcal{K}_N(\mathcal{S}, \mathcal{W})$)

For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step robust controllable set $\mathcal{K}_N(\mathcal{S}, \mathcal{W})$ is defined recursively as:

$$\mathcal{K}_j(\mathcal{S}, \mathcal{W}) \triangleq \mathrm{Pre}(\mathcal{K}_{j-1}(\mathcal{S}, \mathcal{W}), \mathcal{W}) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{S}, \mathcal{W}) = \mathcal{S}, \quad j \in \{1, \ldots, N\}.$$

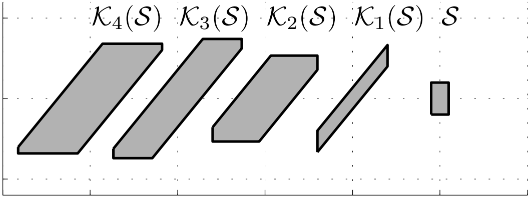

Figure: One-step controllable sets $\mathcal{K}_j(\mathcal{S})$ for N=1,2,3,4. The sets are shifted along the x-axis for a clearer visualization. Download code.

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Robust Control Invariance

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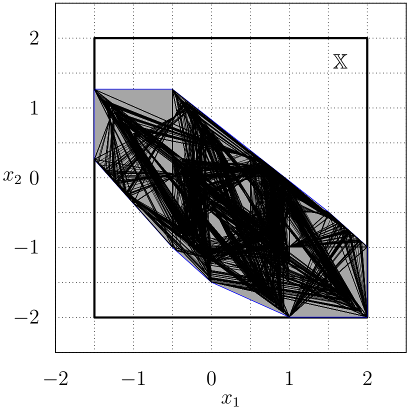
We define robust control invariant (RCI) sets as :

Robust Control Invariant Sets

A set $\mathcal{R} \subseteq \mathcal{X}$ is a robust control invariant (RCI) set if for all $x \in \mathcal{R}$ there exists an input $u \in \mathcal{U}$ such that $g(x, u, w) \in \mathcal{R}$ *for all* $w \in \mathcal{W}$.



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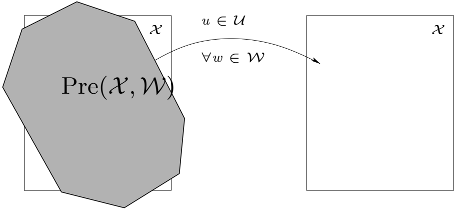
Control Invariant Computation

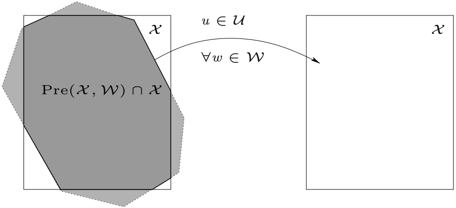
- Mapping $\mathrm{Pre}(\cdot)$: $\mathrm{Pre}(\mathcal{X})$ is the set of states robustly controllable into $\mathcal{X}$.



- Repeat this until:
- Fixed point of $\mathrm{Pre}(\cdot,\cdot)$: **Robust Control Invariant Set** (RCI set).

Control Invariant Computation

- Mapping $\mathrm{Pre}(\cdot)$: $\mathrm{Pre}(\mathcal{X})$ is the set of states robustly controllable into $\mathcal{X}$.



- Repeat this until:
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Control Invariant Computation

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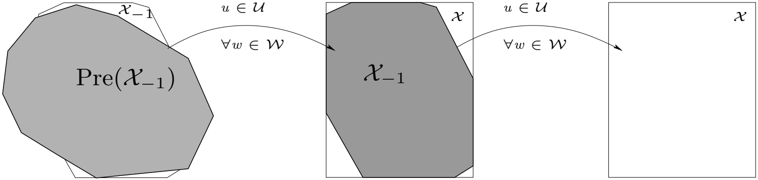

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Control Invariant Computation

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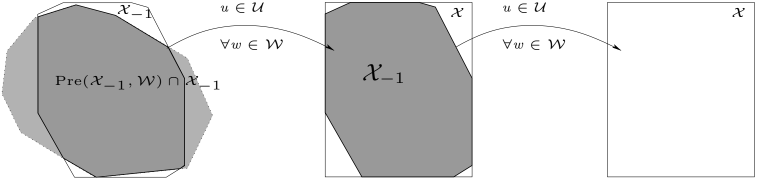

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Control Invariant Computation

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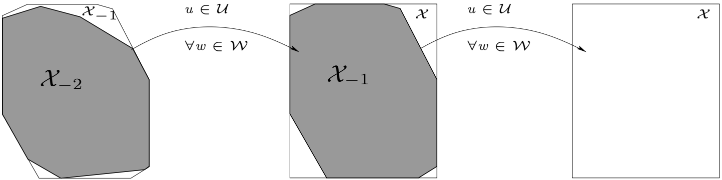

- Repeat this until:
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Control Invariant Computation

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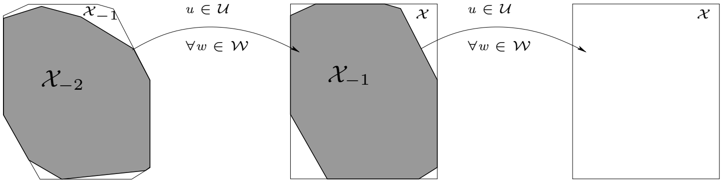
- Mapping $\mathrm{Pre}(\cdot)$: $\mathrm{Pre}(\mathcal{X})$ is the set of states robustly controllable into $\mathcal{X}$.



- Repeat this until:
- Fixed point of $\mathrm{Pre}(\cdot, \cdot)$: **Robust Control Invariant Set** (RCI set).

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Constrained Robust Control: Goals

Design control law $u = \pi(x)$ such that the closed-loop system:

1. Satisfies constraints : $x_k \in \mathcal{X}, u_k \in \mathcal{U}$ for all admissible disturbance realizations
2. Convergence: to a terminal set $\mathcal{X}_f$,
3. Optimizes: "performance"
4. Maximizes the set of x_0 for which Conditions 1-3

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Robust CFTOC: Ingredients

- consider robust optimal control over a finite time horizon N (robust Constrained Finite Time Optimal Control - rCFTOC).
- define, at time instance 0:
  - n-step controllable sets:

$$\mathcal{X}_N = \mathcal{X}_f, \quad \mathcal{X}_{j-1} = \text{Pre}(\mathcal{X}_j, \mathcal{W}), \quad j \in \{N-1, \ldots, 0\}$$

  - a control policy set Π_0:

$$\pi_0 := \{\pi_0(\cdot), \ldots, \pi_{N-1}(\cdot)\},$$

    where $\pi_j(\cdot)$ are control laws, $u_k = \pi_k(x_k)$ where $U_0 := \{u_0, \ldots, u_{N-1}\}$,
  - a sequence of possible disturbances $\mathbf{w}_0$:

$$\mathbf{w}_0 := \{w_0, \ldots, w_{N-1}\}, \quad w_j \in \mathcal{W}$$

  - the cost functional:

$$J_0(x_0, U_0)$$

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Defining a Cost to Minimize

Several common options: Given

$$J_{\mathcal{W}}(x_0, U_0, \mathbf{w}_0) := \left[p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)\right]$$

- Minimize the expected value (requires some assumption on the distribution)

$$J_0(x_0, U_0) := \mathbf{E}\left(J_{\mathcal{W}}(x_0, U_0, \mathbf{w}_0)\right)$$

- Minimize the variance (requires some assumption on the distribution)

$$J_0(x_0, U_0) := \mathrm{Var}\left(J_{\mathcal{W}}(x_0, U_0, \mathbf{w}_0)\right)$$

- Take the worst-case

$$J_0(x_0, U_0) := \max_{\mathbf{w}_0 \in \mathcal{W}^N} J_{\mathcal{W}}(x_0, U_0, \mathbf{w}_0)$$

- Take the nominal case

$$J_0(x_0, U_0) := J_{\mathcal{W}}(x_0, U_0, 0)$$

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Robust CFTOC

- The general rCFTOC problem is formally stated as follows:

$$J_0^*(x_0) = \min_{\pi_0 \in \Pi_0} J_0(x_0, \pi_0),$$

where:

  - Π_0 is the set of admissible control policies:

$$\Pi_0 := \{\{\pi_0, \dots, \pi_{N-1}\} \colon \pi_j(x) \subseteq \mathcal{U} \text{ and } g(x, \pi_j(x), w) \in \mathcal{X}_{j+1}, \\ \forall (x, w) \in \mathcal{X}_j \times \mathcal{W},\ j \in \{0, \dots, N-1\}\}$$

- In general, NP-hard
- Several options for Π_0 and $J_0(x_0, \pi_0)$ are used to trade off conservatism and complexity

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rCFTOC with open loop predictions and nominal cost

- Π_0: optimize over one sequence U_0 of admissible control inputs
- $J_0(x_0, \pi_0)$: nominal

Robust Open-Loop MPC

$$\begin{aligned} \min_{U_0} \quad & \sum_{i=0}^{N-1} x_N' P x_N + \sum_{k=1}^{N} x_k' Q x_k + u_k' R u_k. \\ & x_{i+1} = A x_i + B u_i \\ & x_i \in \mathcal{X} \ominus \mathcal{A}_i \mathcal{W}^i \\ & u_i \in \mathcal{U} \\ & x_N \in \mathcal{X}_f \ominus \mathcal{A}_N \mathcal{W}^N \end{aligned}$$

where $\mathcal{A}_i := \begin{bmatrix} A^0 & A^1 & \dots & A^i \end{bmatrix}$

- We do *nominal optimal control*, but with tighter constraints on the states and inputs.
- if the nominal system satisfies the tighter constraints, then the uncertain system will satisfy the real constraints.

rCFTOC with open loop predictions and nominal cost

- Π_0: optimize over one sequence U_0 of admissible control inputs:
- $J_0(x_0, \pi_0)$: nominal

Robust Open-Loop MPC

$$\begin{aligned} \min_{U_0} \quad & \sum_{i=0}^{N-1} x_N' P x_N + \sum_{k=1}^{N} x_k' Q x_k + u_k' R u_k. \\ & x_{i+1} = A x_i + B u_i \\ & x_i \in \mathcal{X} \ominus \mathcal{A}_i \mathcal{W}^i \\ & u_i \in \mathcal{U} \\ & x_N \in \mathcal{X}_f \ominus \mathcal{A}_N \mathcal{W}^N \end{aligned}$$

where $\mathcal{A}_i := \begin{bmatrix} A^0 & A^1 & \dots & A^i \end{bmatrix}$

- *All we're doing is tightening the constraints on the nominal system*
- Two issues: open-loop MPC has a *very* small region of attraction! + Need online optimization !

Solution

$$J_0^*(x(0)) = \min_{U_0} \quad [U_0' \ x_0'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix} \right] [U_0{}' \ x_0']'$$
$$\text{such that} \quad G_0 U_0 \le w_0 + E_0 x_0$$

- For a given $x_0 = x(t)$, U_0^* can be found via a QP solver.
- Example

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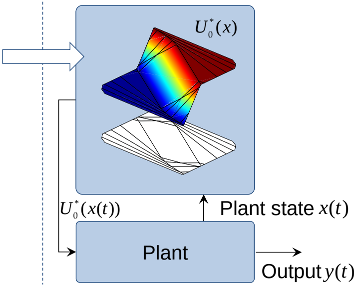
Robust MPC through Explicit Solution

OFFLINE

$$U_0^*(x(t)) = \text{argmin} \;\; x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$$

$$\begin{aligned} \text{subj. to} \;\; & x_0 = x(t) \\ & x_{k+1} = Ax_k + Bu_k, \\ & x_k \in \mathcal{X} \ominus \mathcal{A}_i \mathcal{W}^i, \;\; u_k \in \mathcal{U}, \\ & x_N \in \mathcal{X}_f \ominus \mathcal{A}_N \mathcal{W}^N \end{aligned}$$

ONLINE



- Optimization problem is parameterized by state
- Pre-compute control law as function of state x
- Control law is piecewise affine for linear system/constraints

Result: Online computation dramatically reduced and *real-time*
Tool: *Parametric programming* [5]

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Parametrization of the Control Policies

- rCFTOC problem is, in general, intractable:

$$J_0^*(x_0) = \min_{\pi_0 \in \Pi_0} J_0(x_0, \pi_0),$$

- one "reasonable" parametrization of the predicted control inputs:

$$u_k = \sum_{i=0}^{k} L_{k,i} x_i + g_i, \quad k \in \mathbb{N}_{[0,N-1]}$$

- compact notation:

$$U_0 = \mathbf{L}x + \mathbf{g}, \quad \text{where}$$

$$U_0 = \left[u_0', u_1', \dots, u_{N-1}'\right]', \quad x = \left[x_0', x_1', \dots, x_N'\right]',$$

$$\mathbf{L} = \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_0 \\ \vdots \\ g_{N-1} \end{bmatrix}$$

Parametrizations of the Control Policies

- consider the set of admissible parameters:

$$\begin{aligned}\mathcal{P}_0^{Lg}(x_0) = \quad & \{L, g : x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1,\ x_N \in \mathcal{X}_f \\ & \forall\ w_k^a \in \mathcal{W}^a\ k = 0, \ldots, N-1, \\ & \text{where } x_{k+1} = Ax_k + Bu_k + Ew_k^a,\ u_k = \textstyle\sum_{i=0}^{k} L_{k,i} x_i + g_i\}\end{aligned}$$

- feasible set:

$$\mathcal{X}_0^{Lg} = \left\{ x_0 \in \mathbb{R}^n :\ \mathcal{P}_0^{Lg}(x_0) \neq \emptyset \right\}$$

Bad news

For a given $x_0 \in \mathcal{X}_0^{Lg}$ the set $\mathcal{P}_0^{Lg}(x_0)$ is non–convex, in general.

- therefore: finding $(\mathbf{L}, \mathbf{g})$ for a given x_0 may be difficult

"Magic" Convex Parametrization

- consider parametrization of the predicted control in past disturbances:

$$u_k = \sum_{i=0}^{k-1} M_{k,i} w_i + v_i, \quad k \in \mathbb{N}_{[0,N-1]}$$

- since we implicitly assumed the full state information:

$$w_k = x_{k+1} - Ax_k - Bu_k, \quad k \in \mathbb{N}_{0,N-1}.$$

- compact notation:

$$U_0 = \mathbf{M}\mathbf{w} + \mathbf{v}, \quad \text{where}$$

$$\mathbf{w} = \begin{bmatrix} w_0' & w_1' & \dots & w_{N-1}' \end{bmatrix}',$$

$$\mathbf{M} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_{N-1} \end{bmatrix}.$$

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Convex Parametrization of Control Policies

- also define:

$$\begin{aligned}\mathcal{P}_0^{Mv}(x_0) = \ & \{M, v : x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1,\ x_N \in \mathcal{X}_f \\ & \forall\ w_k \in \mathcal{W}^a\ k = 0, \ldots, N-1,\ \text{where } x_{k+1} = Ax_k + Bu_k + Ew_k, \\ & u_k = \textstyle\sum_{i=0}^{k-1} M_{k,i} w_i + v_i\}\end{aligned}$$

$$\mathcal{X}_0^{Mv} = \left\{x_0 \in \mathbb{R}^n :\ \mathcal{P}_0^{Mv}(x_0) \neq \emptyset\right\}$$

How is that different form the Lg–parametrization ?

Convex Parametrization of Control Policies

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- well, it isn't in the sense that

$$\mathcal{X}_0^{Lg} = \mathcal{X}_0^{Mv}$$

- except, for a given $x_0 \in \mathcal{X}_0^{Mv}$:

$\mathcal{P}_0^{Mv}(x_0)$ is CONVEX.

- therefore: for a given $x_0 \in \mathcal{X}_0^{Mv}$ the computation of $(\mathbf{M}, \mathbf{v})$ reduces to a convex optimization problem

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Convex Parametrization of Control Policies

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- well, it isn't in the sense that

$$\mathcal{X}_0^{Lg} = \mathcal{X}_0^{Mv}$$

- except, for a given $x_0 \in \mathcal{X}_0^{Mv}$:

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Convex Parametrization of Control Policies

- well, it isn't in the sense that

$$\mathcal{X}_0^{Lg} = \mathcal{X}_0^{Mv}$$

- except, for a given $x_0 \in \mathcal{X}_0^{Mv}$:

$\mathcal{P}_0^{Mv}(x_0)$ is CONVEX.

- therefore: for a given $x_0 \in \mathcal{X}_0^{Mv}$ the computation of $(\mathbf{M}, \mathbf{v})$ reduces to a convex optimization problem

Historical Notes

Or who thought of it first?

- essentially Youla parametrization for discrete–time linear systems,
- apparently, the idea appears in the work of Gartska & Wets in 1974. in the context of stochastic optimization [6],
- recently, it re-appeared in robust optimization work by Guslitzer and Ben-Tal (2002 and 2004) [10, 2],
- in the context of robust MPC: van Hessem & Bosgra 2002, Löfberg 2003, Goulart & Kerrigan 2006 [13, 12, 9]

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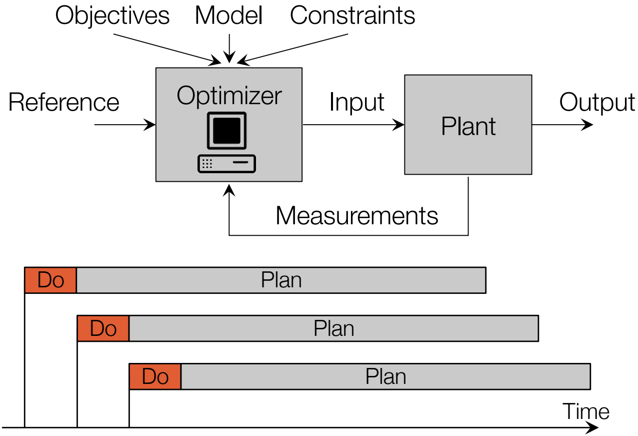
Model Predictive Control



Receding horizon strategy introduces **feedback**.

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Robust RHC Synthesis: Main Challenges

- Computational issues
  - on–line evaluation of the robust MPC law through optimization in space of feedback policies in general intractable, or computationally demanding,
  - explicit computation of optimal control policy limited to few classes of systems/problems and small dimensions of the state space,
- Stability and feasibility
  - how ensures stability of the robust MPC controller?
  - how ensures persistent feasibility of the robust MPC controller?

Robust RHC - Stability and Feasibility (for completness only)

- we can stabilize the system to a set $\mathcal{O} \subseteq \mathcal{X}_f$ (for the concept of set stabilization see [11]),
- Result: $\lim_{k\to\infty} d(x(k), \mathcal{O}) = 0$ for all $x \in \mathcal{X}_0$, if

(A0) There exist constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 \, d(x, \mathcal{O}) \le p(x) \le c_2 \, d(x, \mathcal{O}) \quad \forall x \in \mathcal{X}_0 \tag{6}$$

$$c_3 \, d(x, \mathcal{O}) \le q(x, u) \le c_4 \, d(x, \mathcal{O}) \quad \forall (x, u) \in \mathcal{X}_0 \times \mathcal{U} \tag{7}$$

(A1) The sets $\mathcal{X}$, $\mathcal{X}_f$, $\mathcal{U}$, $\mathcal{W}$ are compact.

(A2) $\mathcal{X}_f$ and $\mathcal{O}$ are robust control invariants, $\mathcal{O} \subseteq \mathcal{X}_f \subseteq \mathcal{X}$.

(A3) $J^p(x) \le 0 \; \forall x \in \mathcal{X}_f$ where

$$J^p(x) = \min_{u\in\mathcal{U}} \; \max_w \, p(x^+) - p(x) + q(x, u)$$

$$\text{subj. to} \begin{cases} w \in \mathcal{W} \\ x^+ = A(w^p)x + B(w^p)u + Ew \end{cases} \tag{8}$$

Topic not Discussed worth Listing

- Stochastic MPC
- Closed-Loop vs Open-Loop Predictions
- Interpretation as games
- References in [5]

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40 Years of Robust Control: 1978–2018

Recent advances in decentralized/distributed control

Laurent Lessard, UC Berkeley

American Control Conference
Portland, Oregon
June 3, 2014

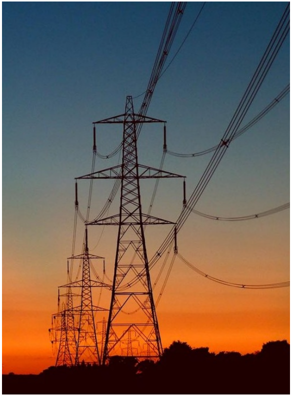
Why decentralization?

We have no choice

- complexity
- delays
- intermittency

By design

- efficiency
- robustness
- scalability



- Some decentralized problems are **as easy** to solve as their centralized counterparts!
- The **optimal** controller can be explicitly computed, and there is a nice separation structure.

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Sparsity and delays

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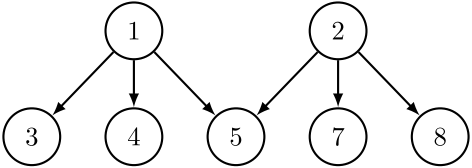

Performance under information constraints

- sparsity: some links are missing
- delays: transmission is not instantaneous

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Outline

- **Quadratic invariance and convexification**
- The two-player problem
- More general problems

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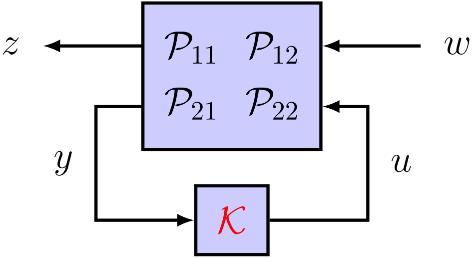
A useful abstraction



- $\mathcal{K}$ belongs to the constraint set $\mathcal{S}$. e.g.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- We care about the map $w \to z$.

$$z = \Big(\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}\big(I - \mathcal{P}_{22}\mathcal{K}\big)^{-1}\mathcal{P}_{21}\Big)\, w$$

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General optimization form (centralized)

$$\begin{array}{rl} \text{minimize} & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}\left(I - \mathcal{P}_{22}\mathcal{K}\right)^{-1}\mathcal{P}_{21} \right\| \\ \text{subject to} & \mathcal{K} \text{ stabilizes } \mathcal{P} \end{array}$$

Simple case: $\mathcal{P}_{22}$ is stable. Define $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$.

Fact: $\mathcal{K}$ stabilizes $\mathcal{P}$ if and only if $\mathcal{Q}$ is stable (Youla).

$$\begin{array}{rl} \text{minimize} & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \right\| \\ \text{subject to} & \mathcal{Q} \text{ is stable} \end{array}$$

This is a convex problem!

General optimization form (decentralized)

$$\begin{array}{ll} \text{minimize} & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}\left(I - \mathcal{P}_{22}\mathcal{K}\right)^{-1}\mathcal{P}_{21} \right\| \\ \text{subject to} & \mathcal{K} \text{ stabilizes } \mathcal{P} \\ & \mathcal{K} \in \mathcal{S} \end{array}$$

Quadratic Invariance (Rotkowitz/Lall '06)
The following are equivalent.

1) $\mathcal{K}P_{22}\mathcal{K} \in \mathcal{S}$ for all $\mathcal{K} \in \mathcal{S}$

2) $\mathcal{K}\left(I - \mathcal{P}_{22}\mathcal{K}\right)^{-1} \in \mathcal{S}$ for all $\mathcal{K} \in \mathcal{S}$

$$\begin{array}{ll} \text{minimize} & \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \right\| \\ \text{subject to} & \mathcal{Q} \text{ is stable} \\ & \mathcal{Q} \in \mathcal{S} \end{array}$$

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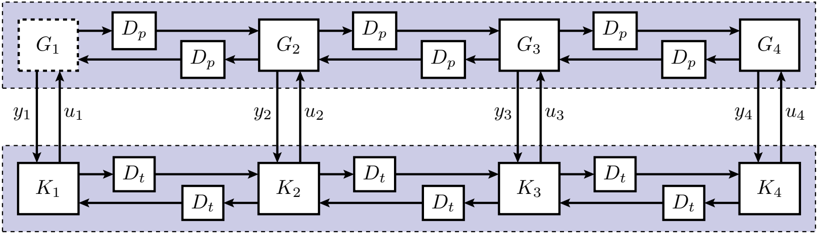
Example



Quadratic invariance if:

1) no delays and plant/controller have same architecture.

2) controller communication is faster than plant interaction ($D_t < D_p$).

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Outline

- Quadratic invariance and convexification
- **The two-player problem**
- More results

Two-player state-feedback

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_+ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w$$

with a standard infinite-horizon LQR cost

- $u_1[k]$ only measures $x_1[0:k]$
- $u_2[k]$ measures both $x_1[0:k]$ and $x_2[0:k]$

Centralized:

$$\begin{aligned} u_1 &= K_{11}x_1 + K_{12}x_2 \\ u_2 &= K_{21}x_1 + K_{22}x_2 \end{aligned}$$

First guess:

$$\begin{aligned} u_1 &= K_{11}x_1 \\ u_2 &= K_{21}x_1 + K_{22}x_2 \end{aligned}$$

Second guess:

$$\begin{aligned} \eta &= \mathbf{E}(x_2 \,|\, x_1) \\ u_1 &= K_{11}x_1 + K_{12}\eta \\ u_2 &= K_{21}x_1 + K_{22}x_2 \end{aligned}$$

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None of these methods work!

Two-player state-feedback

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_+ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w$$

with a standard infinite-horizon LQR cost

- $u_1[k]$ only measures $x_1[0:k]$
- $u_2[k]$ measures both $x_1[0:k]$ and $x_2[0:k]$

K is the LQR gain $u \to x$

J is the LQR gain $u_2 \to x_2$

Optimal Controller:

- Estimator: $\eta = \mathbf{E}(x_2 \,|\, x_1)$
- Controller:

$$u_1 = K_{11}x_1 + K_{12}\eta$$
$$u_2 = K_{21}x_1 + K_{22}\eta + J(x_2 - \eta)$$

J. Swigart and S. Lall, ACC'10

Proof (column decomposition)

1) Youla parameterization is only one-sided

$$\min_{\mathcal{Q}\in\mathcal{S}} \left\| \mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \right\|^2$$

2) Separate by columns

$$\min_{\mathcal{Q}\in\mathcal{S}} \left\| \mathcal{P}_{11} \begin{bmatrix} I \\ 0 \end{bmatrix} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} \right\|^2 + \left\| \mathcal{P}_{11} \begin{bmatrix} 0 \\ I \end{bmatrix} + \mathcal{P}_{12} \begin{bmatrix} 0 \\ I \end{bmatrix} \mathcal{Q}_{22} \right\|^2$$

3) Solve separate centralized problems

Two-player output-feedback

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_+ = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

- $u_1[k]$ only measures $y_1[0:k]$
- $u_2[k]$ measures both $y_1[0:k]$ and $y_2[0:k]$

Optimal Controller:

- Estimator:

$$\zeta = \mathbf{E}(x \,|\, y_1)$$
$$\xi = \mathbf{E}(x \,|\, y_1, y_2)$$

- Controller:

$$u_1 = K_{11}\zeta_1 + K_{12}\zeta_2$$
$$u_2 = K_{21}\zeta_1 + K_{22}\zeta_2 + J_1(\xi_1 - \zeta_1) + J_2(\xi_2 - \zeta_2)$$

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Two-player output-feedback

Centralized estimate update (Kalman filter)

$$\begin{aligned} \hat{x}_+ &= A\hat{x} + Bu - L(y - C\hat{x}) \\ u &= K\hat{x} \end{aligned}$$

$$\hat{x} = \mathbf{E}\left(x \mid \mathcal{Y}_1\right)$$

L, K are found by solving separate AREs

Two-player estimate update

$$\begin{aligned} \zeta_+ &= A\zeta + B\hat{u} - \hat{L}(y - C\zeta) \\ \hat{u} &= K\zeta \end{aligned}$$

$$\zeta = \mathbf{E}\left(x \mid \mathcal{Y}_1\right)$$

$$\begin{aligned} \xi_+ &= A\xi + Bu - L(y - C\xi) \\ u &= K\zeta + \hat{K}(\xi - \zeta) \end{aligned}$$

$$\xi = \mathbf{E}\left(x \mid \mathcal{Y}_{1,2}\right)$$

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L, K are the same as before, $\hat{L}, \hat{K}$ are computed jointly (easily).

Optimal decentralized cost

$$\mathcal{J}_{\text{opt}}^2 = \left\| \left[\begin{array}{c|c} A + BK & B_1 \\ \hline C_1 + D_{12}K & 0 \end{array} \right] \right\|^2$$

centralized cost

$$+ \left\| \left[\begin{array}{c|c} A + LC & B_1 + LD_{21} \\ \hline D_{12}K & 0 \end{array} \right] \right\|^2$$

cost of decentralization

$$+ \left\| \left[\begin{array}{c|c} A + B\hat{K} + \hat{L}C & (\hat{L} - L)D_{21} \\ \hline D_{12}(\hat{K} - K) & 0 \end{array} \right] \right\|^2$$

Proof (person-by-person approach)

1) Youla parameterization is two-sided

$$\min_{\mathcal{Q}\in\mathcal{S}} \left\| \mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right\|^2$$

2) Fix $\mathcal{Q}_{11}$, solve remaining centralized problem

$$\min_{\mathcal{Q}_{21},\mathcal{Q}_{22}\in\mathcal{S}} \left\| \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix} \mathcal{P}_{21} \right) + \left(\mathcal{P}_{12} \begin{bmatrix} 0 \\ I \end{bmatrix} \right) \begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right\|^2$$

3) Repeat with $\mathcal{Q}_{22}$ fixed instead

4) Enforce consistency

Outline

- Quadratic invariance and convexification
- The two-player problem
- **More results**

More results

Sparsity structures

- State-feedback for arbitrary graphs (Shah/Parrilo and Swigart/Lall)
- Two-player Finite-horizon output feedback (Lessard/Nayyar)
- Output-feedback for broadcast structures (Lessard)
- Output-feedback for chain structures (Tanaka/Parrilo)

Delay structures

- state-feedback with delays only (Lamperski/Doyle)
- output-feedback with delays only (Lamperski/Doyle)
- state-feedback with delays and sparsity (Lamperski/Lessard)

Other cost functions

- Two-player $\mathcal{H}_\infty$ output-feedback via entropy minimization (Lessard)
- Chain structure $\mathcal{H}_\infty$ output-feedback via LMI approach (Scherer)

Meta-theorem 1

For decentralized control with nested information,

$$u = \left(\begin{array}{c} \text{centralized control based} \\ \text{on common information} \end{array} \right) + \left(\begin{array}{c} \text{correction} \\ \text{terms} \end{array} \right)$$

Meta-theorem 2

For decentralized control with nested information, there is an estimation-control separation structure in the person-by-person sense.

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- Some decentralized problems are **as easy** to solve as their centralized counterparts!
- The **optimal** controller can be explicitly computed, and there is a nice separation structure.

Looking ahead

Shortcomings

- The real world isn't: linear, quadratically invariant, etc.
- Can't estimate the global state in practice.

What we can do

- Inspire new control algorithms (e.g. EKF)
- Inform better network design
- Inspire tighter relaxation techniques
- Efficiently compute performance bounds

Example: block diagonal plant, with control structure

$$\begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix} \le \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ 0 & \times & \times \end{bmatrix} \le \min \left\{ \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & \times & \times \end{bmatrix}, \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{bmatrix} \right\}$$

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Thank you!

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