Origin of Stability Analysis: "On Governors" by J.C. Maxwell

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In 1868, James C. Maxwell published a paper, "On Governors," in *Proceedings of the Royal Society of London* [1]. This paper was overlooked for a long time because it was deemed by many to be difficult to comprehend. However, since Norbert Wiener drew attention to this paper in 1948, it has been recognized as the first significant paper on control theory; as a result, Maxwell has been regarded as the "father of control theory" [2]. The purpose of this article is to provide historical information on the origin of stability analysis in Maxwell's paper and to rederive his key equations using illustrative figures to improve the readability of that paper.

The origin of control dates to a water clock (outflow type), shown in Figure 1(a), in Egypt around 1500 BC. An improved Greek water clock (inflow type) measured time through a gradual flow of water, as shown in Figure 1(b), in which a constant flow rate was obtained from the overflowing water. However, this clock had an obvious disadvantage in that a large quantity of water was wasted by this overflow, which is this is the reason the clock was named clepsydra ("water thief" in Greek).

To resolve this problem, Greek technician Ctesibius invented a water clock with a float valve, as shown in Figure 2, around 300 BC [3]. In modern control terminology, the clepsydra in Figure 1 was an open-loop control system, whereas the water clock of Ctesibius was one of the first feedback control systems invented by humans. Ancient Chinese water clocks, such as the one built by Su Sung, and the Korean water clock Jagyeongnu (meaning "self-striking water clock," see Figure 3) built by Jang Youngsil in 1434, were based on a similar control principle (that is, the concept of open-loop control) to that of the clepsydra in Figure 1(b) [4]–[6]. One of the first significant feedback control systems in modern Europe was the flyball governor, shown in Figure 4, which was invented in 1788 by the Scottish engineer James Watt for the speed regulation of a steam engine [7]–[9]. The amount of steam (the controller output) supplied to the engine (the controlled plant) was adjusted according to the difference (the error signal) between the desired and actual speeds. If the actual speed (the controlled variable) increases beyond the desired value (the setpoint) owing to variations in the driving power or resistance (the disturbances), then the increase in the centrifugal force of the flyball governor causes a contraction of the aperture of the steam valve through a link mechanism. This results in the supply of less steam, and the speed of the steam engine decreases until the desired value is attained. On the other hand, if the engine speed drops below the desired value, then the decrease in the centrifugal force of the governor causes the steam valve to open wider, supplying more steam, and the engine speed increases until the desired value is attained [8].

The early flyball governor had the drawbacks of 1) an offset (steady-state error) resulting from proportional

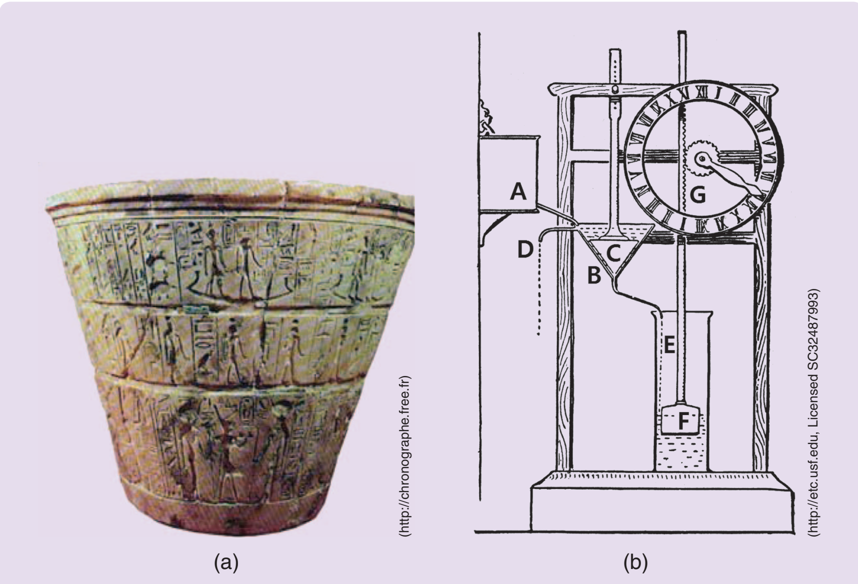

FIGURE 1 (a) An Egyptian water clock, invented around 1500 BC. This water clock (outflow type) was discovered in Karnak, Egypt (Museum of Cairo). (b) An improved Greek water clock (inflow type). The clepsydra measured time by the (gradual) flow of water, in which a constant flow rate at E was obtained by overflowing water at D.

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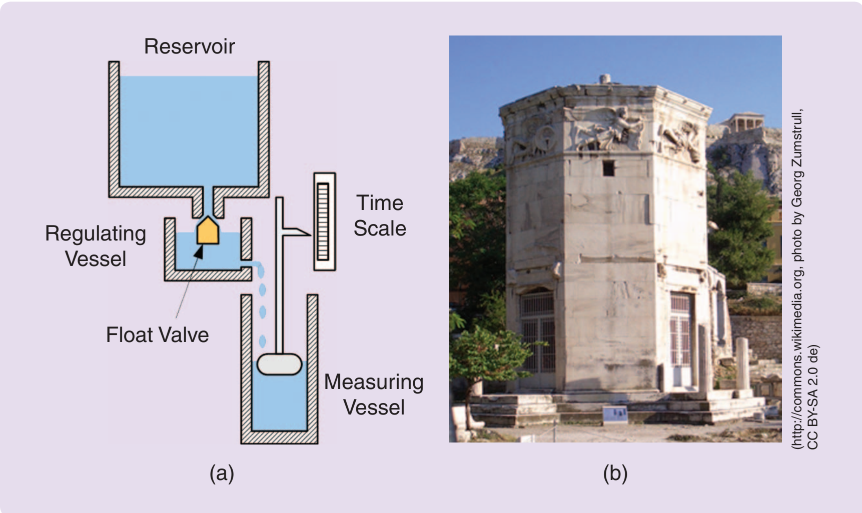

FIGURE 2 A water clock invented by Greek technician Ctesibius around 300 BC. Ctesibius's water clock is a feedback control system since the float valve works as a sensor and actuator. (a) A schematic diagram showing the principle of the water clock and (b) Tower of the Winds, inside which Ctesibius' water clock was installed.

control and 2) low power for actuating the control valve. To overcome these drawbacks, the brothers Werner and C. William Siemens (1846), Charles Porter (1858), Thomas Pickering (1862), and William Hartnel (1872) each invented devices for adding integral control action and for improving actuation power [7]. The Siemens brothers substituted integral action for proportional action and hence improved an offset inherent in the Watt's governor. Charles Porter developed a loaded governor with two small flyballs that could be run at much higher speeds and hence could generate greater forces to operate an actuator. Thomas Pickering and William Hartnell invented spring-loaded governors that could be operated at higher speeds and were smaller than the previous governors [7]. The problem of low power intrinsically originates from the one-body design of the sensor and actuator. The float valve of Ctesibius's water clock and the flyball of the Watt's governor act as an actuator as well as a sensor. Later, this problem was solved by having distinct actuators and sensors in the feedback control system.

It was reported that about 75,000 governors were used in Britain during the Industrial Revolution (around 1868). At that time, the governor system was discovered to be plagued by a hunting, or oscillation problem, and research to address this problem was started at the University of Cambridge in England [7]. The next section uses illustrative figures and comprehensible explanations to revisit Maxwell's concept of stability analysis using a differential equation. Subsequently, further developments in the area of stability analysis are presented from my perspective.

STABILITY ANALYSIS BY MAXWELL

Maxwell's interest in governors reflected, to some extent, a contemporary vogue. At the height of the Industrial Revolution, the mechanism for controlling the speed of every steam engine was plagued by problems of instability and inaccuracy that could apparently not be overcome by either theoretical or practical approaches. In those days, various governors had been newly invented. However, Maxwell's interest in governors was unrelated to their practical utility and instead originated from the desire to address the issue of their stability (see "Maxwell's Life" for further details).

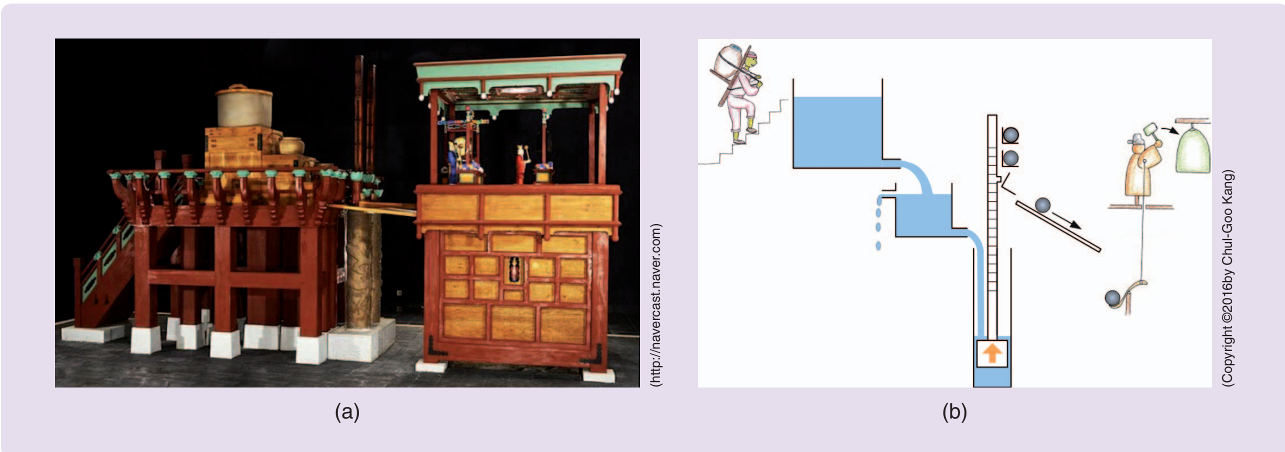

FIGURE 3 (a) The Korean water clock Jagyeongnu built in 1434. The Jagyeongnu shown in this figure was reconstructed in 2007 and is presently exhibited at the National Palace Museum of Korea in Seoul. (b) The Jagyeongnu was used to keep the standard time in the Joseon Dynasty. It marked the hour automatically with the sounds of a bell, gong, and drum.

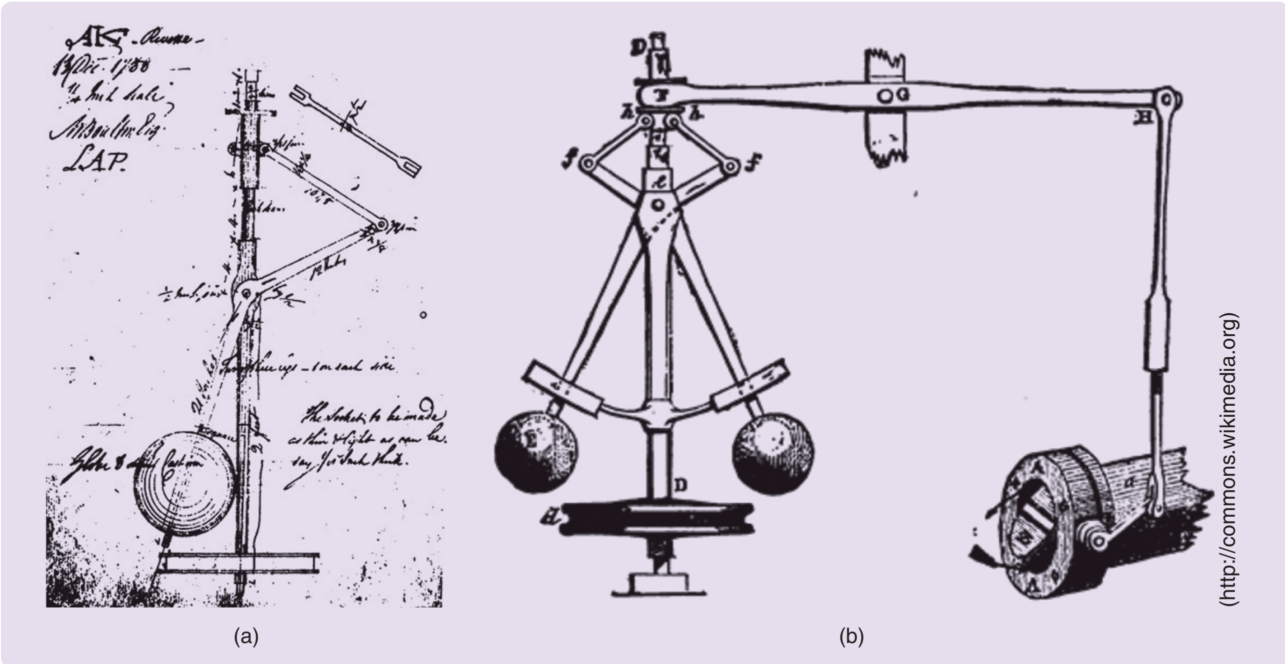

FIGURE 4 The flyball governor invented by James Watt in 1788. If the actual speed increases beyond the desired value, the increase in the centrifugal force of the flyball governor causes closing of the steam valve, resulting in the supply of less steam, and the speed of the steam engine decreases. If the engine speed drops below the desired value, the opposite action occurs. (a) The original design (reproduced by permission of the Institution of Engineering & Technology [7]), and (b) the improved design.

Maxwell described stability as [1] "the motion of a machine with its governor consists in general of a uniform motion, combined with a disturbance that may be expressed as the sum of several component motions. These components may be of four different kinds: the disturbance may (i) continually increase, (ii) continually diminish, (iii) be an oscillation of continually increasing amplitude, and (iv) be an oscillation of continually decreasing amplitude. The second and fourth kinds are admissible in a good governor, and are mathematically equivalent to the condition that all the possible roots [that is, real roots in modern terminology], and all the possible parts [that is, real parts] of the impossible roots [that is, complex roots] of a characteristic equation shall be negative."

Maxwell classified governors into two groups: moderators and genuine governors. His description of the difference between moderators and genuine governors (in modern terminology) is that moderators are controllers with only proportional control action, whereas genuine governors are controllers with both proportional and integral control actions.

Maxwell considered three kinds of governors. In the first kind, the centrifugal piece is at a constant distance from the axis of rotation; examples of this kind of governor are a friction governor [10], as shown in Figure 5, and the governor of H.C. Fleeming Jenkin (1863), shown in Figures 6 and 7. Figure 7 is a redrawing of the schematic from [7, p. 65]. In the second kind of governor, the centrifugal piece is free to move from the axis of rotation but is balanced by a centrifugal force and the force of gravity (or by the spring force, too, in some cases). Examples of this kind of governor are Watt's governor, as shown in Figure 4, and Léon Foucault's governor (1862) [11], as shown in Figure 8. In the third kind of governor, a liquid is pumped up and removed over the sides of a rotating cup, for example, the liquid governor of C. William Siemens (1866) [10], [12], [13], as shown in Figure 9.

After describing the three kinds of governors, Maxwell presented differential equations of motion for each of them without providing any detailed explanations. Maxwell approached the topic of the instability of governors by solving the differential equations of motion, and, for the first time in the history of control, partially succeeded in a stability analysis. The first mathematical investigation of governor instability was performed in 1840 by Prof. George Biddell Airy of the University of Cambridge, who also attempted to understand governor instability through differential equations of motion but failed, owing to the insolvability of the nonlinear differential equation of the form $\dot{\theta}^2 + k/(\sin\theta)^2 - (2g/a)\cos\theta = D$ that he derived [14]. Maxwell was able to obtain results by linearizing the nonlinear equations.

For Jenkin's governor, Maxwell derived differential equations of motion without any illustrative figures. In this article, the same equations of motion are rederived with clear free-body diagrams and extensive explanations. Jenkin's governor was used to regulate an experimental apparatus used to determine electrical resistance (ohms). It was essentially a friction governor and consisted of two rotating mechanisms capable of moving separately, as shown in Figure 7. If the principal axis rotates faster, the flyballs

Maxwell's Life

James Clerk Maxwell (see Figure S1) was born in Edinburgh, Scotland, in 1831, and he pursued general studies at the University of Edinburgh (age 16–19) and mathematics at the University of Cambridge (age 20–23) [S1]. An anecdote about James Maxwell and Edward Routh (another contributor to stability analysis) is well known. Maxwell and Routh were undergraduates together and appeared for the Mathematical Tripos examination at the same time in 1854. Apparently, Maxwell was so confident of achieving first place in the examination that he did not bother waking up early to hear the reading of the lists of successful candidates in the Senate House but instead sent his servant to listen for him. (Undergraduates had servants in those days!) On his return, Maxwell apparently enquired of him, "Well, tell me who's second," and was somewhat taken aback to receive the reply, "You are, sir!" for Routh had defeated him by achieving first place [S2].

At the age of 25, Maxwell became professor of physics at Marischal College, Aberdeen, and at the age of 29, he became the chair of natural philosophy at King's College, London. However, he resigned at the age of 34, returned to his hometown, Glenlair, with his wife and lived there for about six years. At the age of 40, he became the first professor of experimental physics at the University of Cambridge, where he directed the newly formed Cavendish Laboratory for eight years. He died of stomach cancer at the age of 48 [S3].

As a distinguished physicist and mathematician, Maxwell made great achievements in the theory of electromagnetism [S4] as well as in the fields of thermodynamics and optics (color vision). In particular, he made a remarkable contribution to control theory by publishing "On Governors" at the age of 37 in his hometown [10]. However, this paper was overlooked for a long time primarily because of its incomprehensibility. In this paper, the complex dynamics of the governors were described using only text, without any figures to demonstrate the operation of governors and without any free-body diagrams for deriving the equations of motion. The main reason for the lack of figures in the paper may be the lack of adequate printing technology in that era. Eighty years later (in 1948), Norbert Wiener, a professor from MIT, drew attention to this paper, and thereafter, it has been recognized as the first significant paper on control theory [2], [10].

FIGURE S1 James Clerk Maxwell (1831–1879), a Scottish mathematical physicist who is famous for formulating the theory of electromagnetism.

REFERENCES

[S1] Wikipedia. (2015). James Clerk Maxwell. [Online]. Available: http://en.wikipedia.org/wiki/James_Clerk_Maxwell

[S2] R. H. MacMillan, *Automation: Friend or Foe?*, Cambridge, MA: Cambridge Univ. Press, 1956.

[S3] James Clerk Maxwell Foundation. (2015). Who was James Clerk Maxwell? [Online]. Available: http://www.clerkmaxwellfoundation.org/html/who_was_maxwell-.html

[S4] J. C. Maxwell, *A Treatise on Electricity and Magnetism.* Oxford, UK: Clarendon Press, 1873.

are extended and will rub against the inside surface of the friction ring, which will make the friction ring begin to rotate and lift the weight. If the speed decreases, the weight causes the friction ring to rotate in the opposite direction. The weight is suspended in a hydraulic cylinder to provide viscous damping. The movement of the friction ring is used to loosen or tighten a band brake through a worm gear, which acts on the brake drum of the principal axis [7]. If the rotational speed of the principal axis varies from the nominal value, a torque proportional to the deviation of the speed is applied to the principal axis by the band brake until the speed reaches the nominal value (this is actually an integral action).

Free-body diagrams of Jenkin's governor are shown in Figure 10. Let θ be the rotation angle of the principal axis, m be the mass of a flyball, k be the spring constant, r be the distance between the flyball and the center of the axis of rotation, and V_1 be the lowest limit of the angular velocity at which the friction ring starts to rotate. At the velocity V_1, the flyballs begin to rub against the inside of the friction ring, and the centrifugal force and spring force are balanced at this speed

$$mr_1 V_1^2 = k(r_1 - r_0), \tag{1}$$

where r_0 is the radius when the principal axis is at rest. If the speed $\dot{\theta}\,(= d\theta/dt)$ increases, the centrifugal force increases

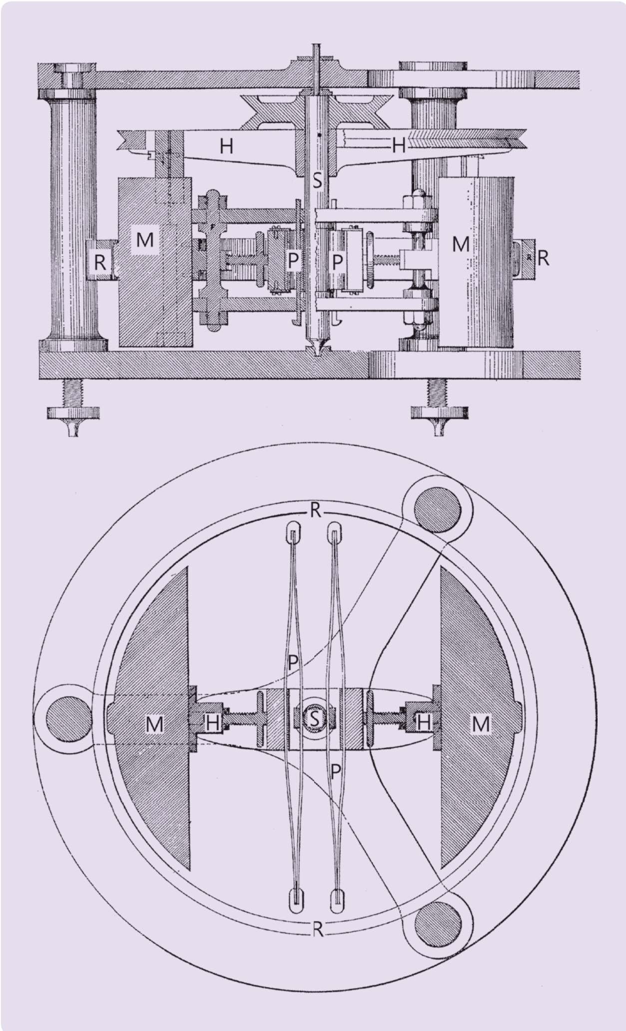

FIGURE 5 A friction governor. Two centrifugal pieces M held back by a leaf spring P are constrained to move in the horizontal plane. At overspeed, the centrifugal pieces will press outward against a stationary ring R and then produce the required braking force. (Reprinted by permission of the University of Chicago Press from [10].)

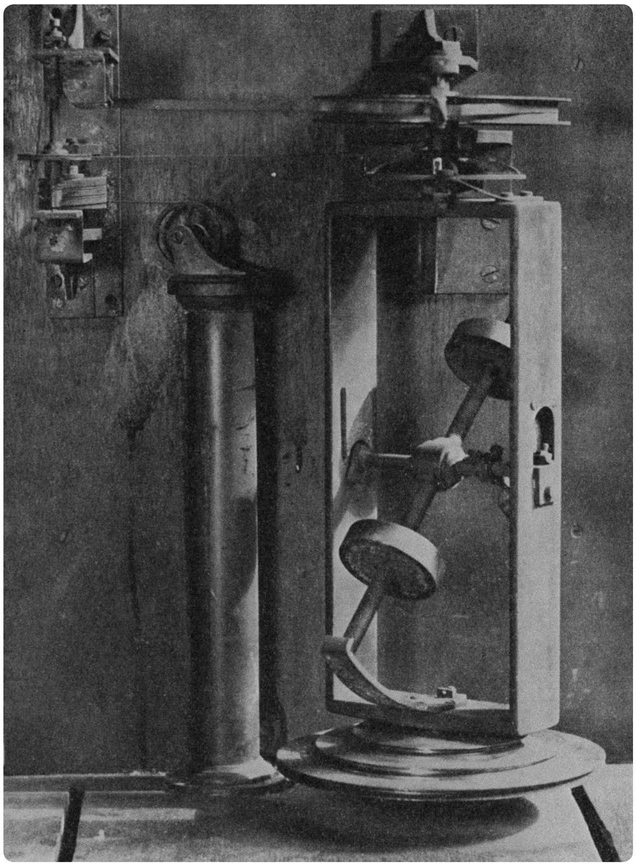

FIGURE 6 Jenkin's governor. The governor regulated an experimental apparatus used to determine electrical resistance (ohms). It is preserved in the Whipple Museum of Science at Cambridge University. (Reprinted by permission of the University of Chicago Press from [10].)

while the spring force remains constant. Thus, the friction force on the inner surface of the friction ring for one flyball is $\mu(mr_1\dot\theta^2 - mr_1V_1^2)$, where μ is the coefficient of sliding friction between the flyball and the friction ring. The torque acting on the friction ring is

$$2r_1 \cdot \mu(mr_1\dot\theta^2 - mr_1V_1^2). \tag{2}$$

This torque drives the friction ring, lifts the weight, and tightens the band brake. Unfortunately, (2) is nonlinear in $\dot\theta$, but Maxwell linearized it to be

$$F(\dot\theta - V_1), \tag{3}$$

by assuming that the velocity $\dot\theta$ varies within very narrow limits around the value V_1. That is, by assuming

$$\dot\theta \triangleq V_1 + \underset{\sim}{\dot\theta}, \tag{4}$$

where $\underset{\sim}{\dot\theta}$ is small. Substituting (4) into (2) and neglecting the term $\underset{\sim}{\dot\theta}^2$, the torque is $2r_1\mu mr_1(2V_1\underset{\sim}{\dot\theta})$, which is (3) with $F = 4r_1^2\mu mV_1$.

Next, by applying Newton's second law of motion for moments acting on the principal axis [refer to the free-body diagram in Figure 10(a)], the differential equation for the rotation θ of the principal axis is

$$M\ddot\theta = P - R - F(\dot\theta - V_1) - G\psi, \tag{5}$$

where P is the driving torque; R is the resisting torque; G is a constant; ψ is the rotation angle of the friction ring; and M is the total moment of inertia of the principal axis, brake drum, and all the rotating parts with respect to the principal axis. From the free-body diagram in Figure 10(b), the equation of motion of the friction ring is

$$B\ddot\psi = F(\dot\theta - V_1) - Y\dot\psi - W, \tag{6}$$

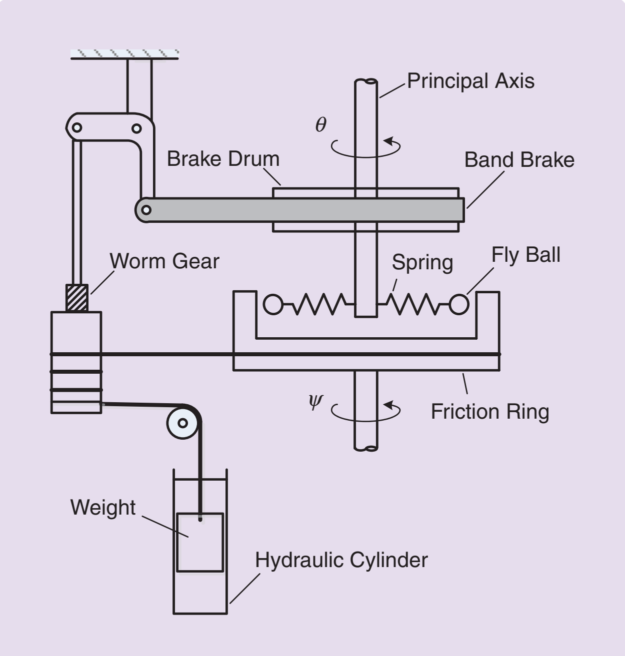

FIGURE 7 A schematic of Jenkin's governor. If the principal axis rotates faster, the flyballs are extended and will rub against the inside surface of the friction ring, which will make the friction ring begin to rotate and lift the weight. If the speed decreases, the weight causes the friction ring to rotate in the opposite direction. The movement of the friction ring acts to loosen or tighten the band brake, and thus, a torque proportional to the deviation of the speed is applied to the principal axis by the band brake.

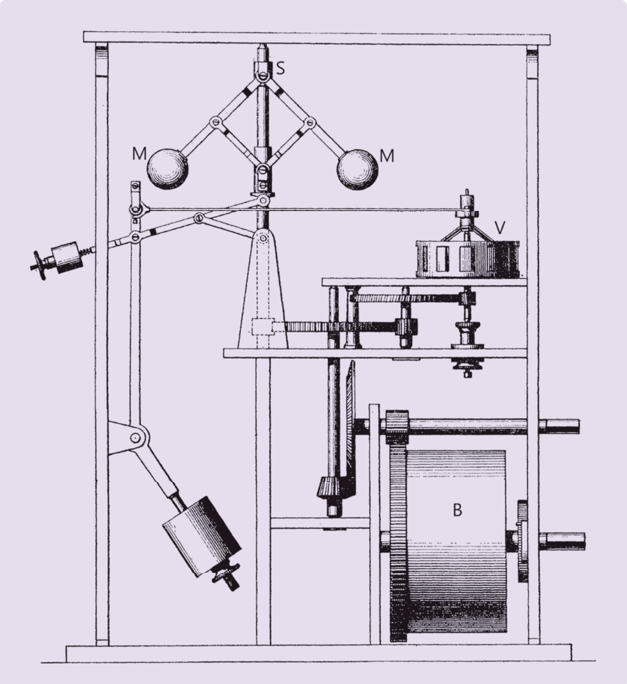

FIGURE 8 Foucault's governor. The governor regulates the motor B, which is linked to the centrifugal fan V so as to increase the flow rate with increasing speed, which increases the load resistance of the fan. The complicated arrangement of linkages and weights on the left is designed to linearize the relationship between the speed and the output motion of the governor. (Reprinted by permission of the University of Chicago Press from [10].)

where B is the total moment of inertia of the friction ring and the attached parts, Y is a coefficient corresponding to viscous friction torque due to the hydraulic cylinder, and W is a constant torque acting on the friction ring owing to the weight. Equations (5) and (6) are the same equations of motion derived by Maxwell, except that Maxwell called θ and ψ, x and y, respectively. Combining (5) and (6) leads to a linear differential equation that is third order in the velocity $\omega\,(=\dot{\theta})$,

$$MB\dddot{\omega} + (MY + FB)\ddot{\omega} + FY\dot{\omega} + FG\omega = u(t), \tag{7}$$

where input $u(t)$ is

$$u(t) = B(\ddot{P} - \ddot{R}) + Y(\dot{P} - \dot{R}) + GFV_1 + GW. \tag{8}$$

For constant P and R, Maxwell obtained a solution of the form

$$\omega(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t} + V, \tag{9}$$

where V is the nominal velocity given by

$$V = V_1 + W/F, \tag{10}$$

from $GFV = GFV_1 + GW$ at the steady state of (7) and (8). Note that Maxwell expressed the solution in terms of $\theta(t)$ instead of $\omega(t)$ by integrating the solution (9), but in this case, a constant term must be added to his solution. In (9), s_1, s_2, s_3 are the roots of the cubic characteristic equation

$$MBs^3 + (MY + FB)s^2 + FYs + FG = 0, \tag{11}$$

although Maxwell used the variable "n" instead of "s." Maxwell obtained the stability condition that the real roots and the real parts of the complex conjugate roots of the characteristic equation (11) must all be negative. He presented the stability condition as

$$\left(\frac{F}{M} + \frac{Y}{B}\right)\frac{Y}{B} - \frac{G}{B} = \text{ a positive quanitity}, \tag{12}$$

without a detailed derivation. This condition is identical to the condition obtained using the Routh stability criterion.

$$\begin{array}{lll} \text{Routh array} & & \\ s^3: & MB & FY \\ s^2: & MY + FB & FG \\ s: & \dfrac{(MY+FB)FY - (MB)(FG)}{MY+FB} & \\ 1: & FG. & \end{array}$$

For stability, all elements of the first column of the Routh array must be positive, and also all coefficients in (11) must

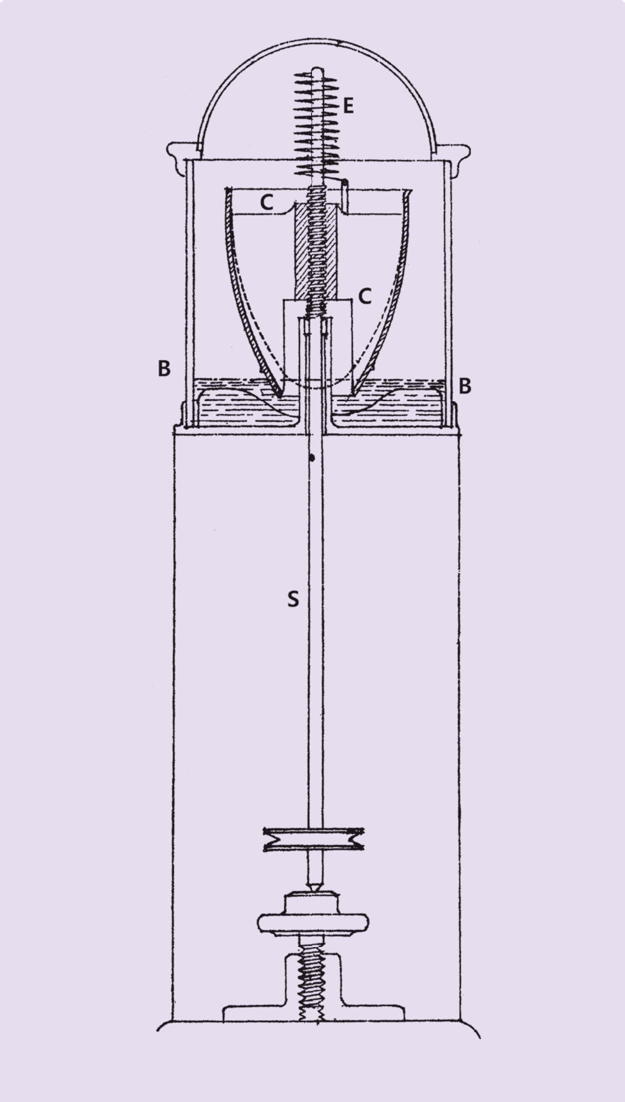

FIGURE 9 Siemens' liquid governor. The speed of a drive shaft S is controlled according to the depth of immersion of a rotating cup C connected to the shaft by a screw and a spring E. For overspeed, the rotation of the cup C falls behind that of the shaft S. Forced downward by the thread, the cup C is immersed deeper into the liquid, thus pumping at a higher rate and exerting an increasing resistance torque on the drive shaft. [Reproduced with permission of W. Bowyer and J. Nichols for Lockyer Davis, printer to the Royal Society from [12] (CCC Licensed 3811700428077 and 3834140751528).]

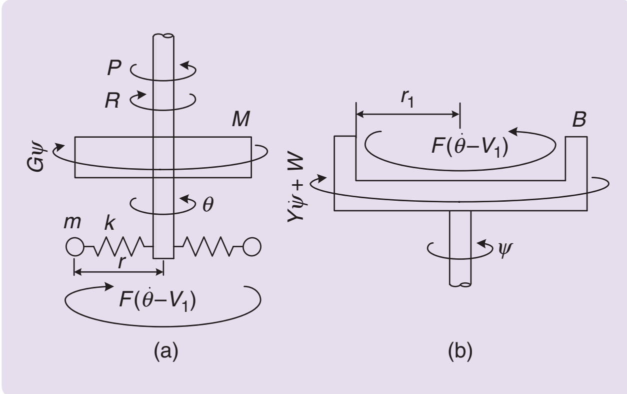

FIGURE 10 A free-body diagram of Jenkin's governor. The friction torque $F(\dot{\theta} - V_1)$ acting on the friction ring is obtained by linearization about a constant speed V_1.

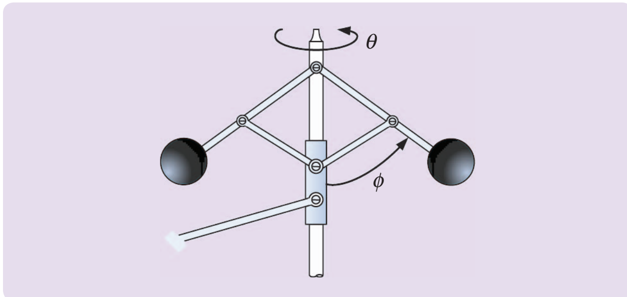

FIGURE 11 The centrifugal pieces (that is, flyballs) of Foucault's governor. A is the moment of inertia of a revolving apparatus for θ motion, and B is the moment of inertia of flyballs for ϕ motion.

be positive. All coefficients in (11) are actually positive since they are physical quantities, and thus the system is stable if the third row value of the Routh array is positive, which gives the condition (12). However, since there was no Routh or Hurwitz stability criterion at that time, the next section of this article presents an alternative derivation of Maxwell's stability criterion.

Next, Maxwell considered the dynamic equations of motion for the governors of Sir William Thomson and Léon Foucault. For the centrifugal pieces of Foucault's governor, shown in Figure 11, Maxwell expressed the equations of motion using the angular momentum $A\dot{\theta}$,

$$\frac{d}{dt}(A\dot{\theta}) = L, \tag{13}$$

where θ is the angle of revolution about the vertical axis, A is the moment of inertia of a revolving apparatus for θ motion, and L is the total torque acting on the axis. Let B be the moment of inertia of the flyballs in Figure 11 for ϕ motion. Then, the sum of the kinetic and potential energies of Foucault's governor is

$$E = \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}B\dot{\phi}^2 + P = \int L d\theta, \tag{14}$$

where P is the potential energy of the apparatus, which is a function of the divergence angle ϕ of the centrifugal piece. Here, A and B are both functions of the angle ϕ. Differentiating (14) with respect to time t and using (13) gives

$$\left(\frac{1}{2}A_\phi\dot{\theta}^2 + \frac{1}{2}B_\phi\dot{\phi}^2 + P_\phi\right)\dot{\phi} + A\dot{\theta}\ddot{\theta} + B\dot{\phi}\ddot{\phi} = L\dot{\theta} = (A_\phi\dot{\phi}\dot{\theta} + A\ddot{\theta})\dot{\theta}, \tag{15}$$

where the subscript ϕ indicates $d(\cdot)/d\phi$. If the apparatus is arranged such that $P = 0.5AV^2 + \text{constant}$, where V is a

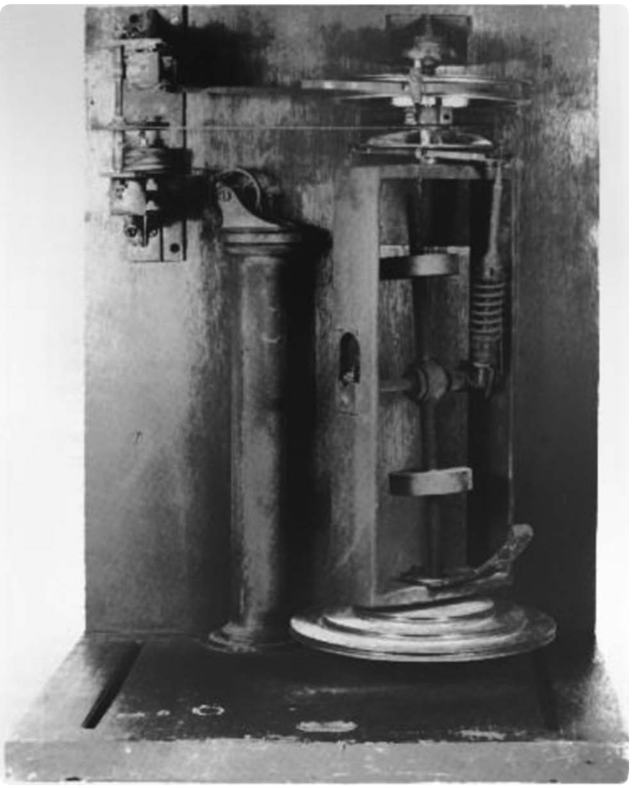

FIGURE 12 A compound governor with a spring-loaded rod. The brake of Thomson's governor is applied to a movable wheel, as is the case in Jenkin's governor, and this wheel works a more powerful brake. (Reprinted by permission of Cambridge University Press from [15].)

constant angular velocity (the nominal angular velocity of the shaft), then $P_\phi = 0.5A_\phi V^2$. Thus, rearranging (15), gives

$$\frac{d}{dt}(B\dot{\phi}) = \frac{1}{2}A_\phi(\dot{\theta}^2 - V^2) + \frac{1}{2}B_\phi\dot{\phi}^2. \tag{16}$$

Equations (13) and (16) are the nonlinear differential equations of motion of Foucault's governor. To linearize (13) and (16), consider small deviations $\underset{\sim}{\omega}$ and $\underset{\sim}{\phi}$ around the nominal values V and ϕ_1. That is,

$$\dot{\theta} \triangleq V + \underset{\sim}{\omega}, \quad \phi \triangleq \phi_1 + \underset{\sim}{\phi}. \tag{17}$$

Substituting (17) into (13) and (16) and noting that $A_\phi = A_{\underset{\sim}{\phi}}$ and $B_\phi = B_{\underset{\sim}{\phi}}$, yields the linear differential equations corresponding to (7) and (8) of Maxwell's paper

$$A\dot{\underset{\sim}{\omega}} + A_{\underset{\sim}{\phi}}V\dot{\underset{\sim}{\phi}} = L, \tag{18}$$

$$B\ddot{\underset{\sim}{\phi}} - A_{\underset{\sim}{\phi}}V\underset{\sim}{\omega} = 0, \tag{19}$$

where $\underset{\sim}{\omega}^2 = 0$, $\dot{\underset{\sim}{\phi}}\underset{\sim}{\omega} = 0$, $\ddot{\theta} = \dot{\underset{\sim}{\omega}}$, $\ddot{\phi} = \ddot{\underset{\sim}{\phi}}$, and $\dot{\phi} = \dot{\underset{\sim}{\phi}}$.

To expand the flyball apparatus into a governor, Maxwell added the term $G\underset{\sim}{\phi}$, which represents the control action (of the friction brake in Thomson's governor or of the air brake in Foucault's governor), to (18), and he added viscous friction terms $X\underset{\sim}{\omega}, Y\dot{\underset{\sim}{\phi}}$ to (18) and (19). Through substitution of $A_\phi V = K$ and letting $\dot{\underset{\sim}{\phi}}\dot{\underset{\sim}{\phi}} = 0$, (18) and (19) become

$$A\dot{\underset{\sim}{\omega}} + X\underset{\sim}{\omega} + K\dot{\underset{\sim}{\phi}} + G\underset{\sim}{\phi} = L, \tag{20}$$

$$B\ddot{\underset{\sim}{\phi}} + Y\dot{\underset{\sim}{\phi}} - K\underset{\sim}{\omega} = 0. \tag{21}$$

Combining (20) and (21) gives a third-order linear differential equation in $\underset{\sim}{\phi}$,

$$AB\dddot{\underset{\sim}{\phi}} + (AY + BX)\ddot{\underset{\sim}{\phi}} + (XY + K^2)\dot{\underset{\sim}{\phi}} + GK\underset{\sim}{\phi} = L. \tag{22}$$

Maxwell obtained the stability condition of the motion represented by (22) by considering that the real parts of the roots must be negative, yielding the condition

$$\left(\frac{X}{A} + \frac{Y}{B}\right)(XY + K^2) > GK, \tag{23}$$

which can also be confirmed by the Routh stability criterion.

After describing the stability conditions for two third-order linear differential equations, Maxwell focused on a more complicated device, a compound governor (a combination of Thomson's and Jenkin's governors) composed of three pieces, in which the brake of Thomson's governor was applied to a movable wheel, as was the case in Jenkin's governor, and this wheel worked a more powerful brake, as shown in Figure 12. Maxwell added a spring-loaded rod, which was not included in Jenkin's governor (see Figure 6), to the compound governor. Since Maxwell's description of the compound governor in his paper was ambiguous and short, visualization of its schematic is not easy; however, a conjectured reconstruction performed by A.A. Andronov is available [10]. Another pointer for the compound governor and its differential equations of motion has been provided in Maxwell's 1863 letter to Thomson [15].

Without performing a calculation of the equations of motion of the three pieces of the compound governor, Maxwell expressed the resulting equations as

$$\begin{aligned} &A\ddot{\underset{\sim}{\theta}} + X\dot{\underset{\sim}{\theta}} + K\dot{\underset{\sim}{\phi}} + T\underset{\sim}{\phi} + J\underset{\sim}{\psi} = L, \\ &B\ddot{\underset{\sim}{\phi}} + Y\dot{\underset{\sim}{\phi}} - K\dot{\underset{\sim}{\theta}} = 0, \\ &C\ddot{\underset{\sim}{\psi}} + Z\dot{\underset{\sim}{\psi}} - T\underset{\sim}{\phi} = 0, \end{aligned} \tag{24}$$

where $\underset{\sim}{\theta}$, $\underset{\sim}{\phi}$, and $\underset{\sim}{\psi}$ are the angles of small disturbances of the main shaft, centrifugal arm, and movable wheel, respectively; A, B, C are their respective moments of inertia; X, Y, Z are the respective viscosities of their connections; K is $A_\phi V$ as described earlier; and T and J are the respective powers of Thomson's and Jenkin's brakes.

Eliminating $\underset{\sim}{\theta}$ and $\underset{\sim}{\phi}$ in (24) and assuming small motions, yields a fifth-order linear differential equation with a quintic characteristic equation

$$s^5+\left(\frac{X}{A}+\frac{Y}{B}+\frac{Z}{C}\right)s^4+\left[\frac{XYZ}{ABC}\left(\frac{A}{X}+\frac{B}{Y}+\frac{C}{Z}\right)+\frac{K^2}{AB}\right]s^3 +\left(\frac{XYZ+KTC+K^2Z}{ABC}\right)s^2+\frac{KTZ}{ABC}s+\frac{KTJ}{ABC}=0. \quad (25)$$

Maxwell was not successful in completely determining the stability conditions from (25), but he found two necessary conditions: $a_1a_2 > a_3$ and $a_1a_4 > a_5$ for

$$s^5+a_1s^4+a_2s^3+a_4s^2+a_4s+a_5=0, \text{ all } a_i>0. \quad (26)$$

He left this unsolved problem to other mathematicians, and finally, Edward J. Routh solved this problem in 1876 to obtain a necessary and sufficient condition of stability for an nth order characteristic equation, which is known as the Routh stability criterion. The necessary and sufficient condition of stability of (26) on the basis of the Routh stability criterion is

$$\begin{aligned}&a_1a_2>a_3,\\&(a_1a_2-a_3)a_3>a_1(a_1a_4-a_5),\\&[(a_1a_2-a_3)a_3-a_1(a_1a_4-a_5)](a_1a_4-a_5)>(a_1a_2-a_3)^2a_5.\end{aligned}$$

PROOF OF MAXWELL'S STABILITY CONDITIONS

Maxwell only succeeded in determining a necessary and sufficient condition of stability of the linear differential equation of the third order. However, he neither proved nor explained his results (12) and (23) in his paper. He only said that one root of (11) is "evidently a real negative quantity." Thus, this article presents a proof for (12) and (23) using methods Maxwell could have used.

Maxwell's Result

For a cubic equation $s^3+a_1s^2+a_2s+a_3=0$ with positive real coefficients a_1, a_2, and a_3,

$$\begin{pmatrix}\text{All roots have}\\\text{negative real parts}\end{pmatrix} \text{ if, and only if, } a_1a_2>a_3. \quad (27)$$

If the coefficient of s^3 is not one but a_0, the equation can always be converted into the form (27) by dividing by a_0. Note that a_1, a_2, a_3 being positive real coefficients is a necessary condition for all roots to have negative real parts. That is, if any one of a_1, a_2, or a_3 is zero or negative, then some roots will have zero or positive real parts.

Here, we consider a polynomial equation with real coefficients. A linear equation $s+a_1=0$ has a negative real root if, and only if, $a_1>0$. A quadratic equation $s^2+a_1s+a_2=0$ has two roots with negative real parts if, and only if, $a_1>0, a_2>0$. A cubic polynomial $s^3+a_1s^2+a_2s+a_3$ can always be factored as

$$s^3+a_1s^2+a_2s+a_3=(s+a)(s^2+bs+c), \quad (28)$$

where a,b,c are real numbers. Therefore, a cubic equation $s^3+a_1s^2+a_2s+a_3=0$ has three negative real roots or one negative root and two complex roots with negative real parts if, and only if, $a>0$, $b>0$, $c>0$. Thus, instead of proving (27), we prove that

$$a>0, b>0, c>0 \text{ if, and only if, } a_1a_2>a_3. \quad (29)$$

Comparing the coefficients on both sides of (28), we obtain $a_1=a+b$, $a_2=ab+c$, $a_3=ac$.

i) The if part of (29) is obvious since $a_1a_2=(a+b)(ab+c)>ac=a_3$.
ii) To show that $a_1a_2>a_3$ implies $a>0, b>0, c>0$, note that, from the positive real coefficients $a_1, a_2, a_3, a+b>0$, $ab+c>0$, and $ac>0$. Given $a_1a_2>a_3$, $(a+b)(ab+c)>ac$. Thus, $(a^2+ab+c)b>0$ and $b>0$. Note that $ac>0$ implies that $a>0, c>0$ or $a<0, c<0$, but the case $a<0, c<0$ cannot occur because then $ab+c<0$ by $b>0$, which is not true. Therefore, $a>0$, $c>0$. □

In his book, E.J. Routh made an interesting comment on Maxwell's result [16], explained as follows. If the roots of $s^3+a_1s^2+a_2s+a_3=0$ with positive a_1,a_2,a_3 are $s=\alpha\pm\beta i, \gamma$ (α,β,γ real, $i=\sqrt{-1}$), then $a_1=-(2\alpha+\gamma)$, $a_2=\alpha^2+\beta^2+2\alpha\gamma$, $a_3=-(\alpha^2+\beta^2)\gamma$. Since $a_1a_2-a_3=-2\alpha[(\alpha+\gamma)^2+\beta^2]$, we obtain the condition $a_1a_2-a_3>0$ is obtained if, and only if, $\alpha<0$. Furthermore, it follows that $\gamma<0$ since $a_3=-(\alpha^2+\beta^2)\gamma>0$.

FURTHER DEVELOPMENTS

This section is not a comprehensive survey on the developments of stability analysis but, rather, an attempt is made to briefly describe the history of stability analysis. The basic concept of Maxwell's studies on the stability analysis of motion was adopted by E.J. Routh, who completed a condition of stability, known as the Routh stability criterion, for a general characteristic equation

$$s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{n-1}s+a_n=0 \quad (30)$$

in his Adams Prize essay, "A Treatise on the Stability of a Given State of Motion, Particularly Steady Motion" in 1877 [17]. The Routh stability criterion specifies that for the stability of a dynamic system: 1) all the coefficients a_i in (30) must be positive, and 2) all the first-column elements of the Routh array must be positive, which is constructed using coefficients a_i of the characteristic equation. Routh proved this criterion by way of Cauchy's index theorem [17], [18].

In continental Europe, Ivan A. Vyshnegradsky, a Russian professor at the St. Petersburg Technological Institute and later the Russian Minister of Finance, in 1877 independently obtained a stability condition together with a stability diagram (that is, the Vyshnegradsky criterion) similar to that of Maxwell for Watt's governor and steam engine [19], [20]. Vyshnegradsky's stability diagram showed the nature of the transient response according to typical pole

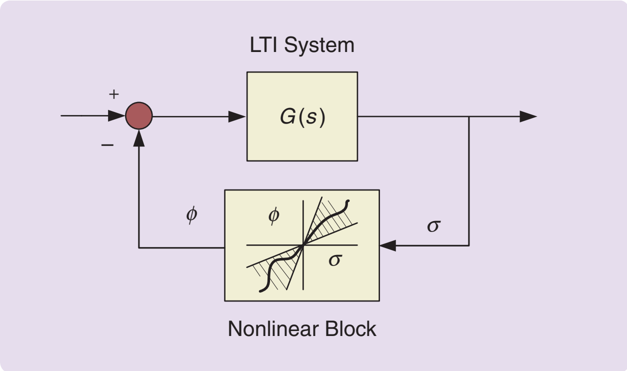

FIGURE 13 A nonlinear system. The dynamics of the forward path are linear time invariant, and the feedback path contains a memoryless sector-bounded nonlinearity.

constellations for various regions in the s-plane. The work of Vyshnegradsky was continued, independently of Maxwell and Routh, by two professors at ETH Zürich, an engineer Aurel Stodola and a mathematician Adolph Hurwitz, whose collaboration led to the Hurwitz stability criterion (1895) [21], [22]. This criterion specifies that all the leading principal minors $\Delta_1, \Delta_2, \ldots, \Delta_n$ (that is, the Hurwitz determinants) of the Hurwitz matrix constructed using coefficients a_i of the characteristic equation must be positive for stability. Later, in 1911, Italian mathematician Enrico Bompiani showed the equivalence of the Routh and Hurwitz stability criteria [23]–[25].

Alexandr Mikhailovich Lyapunov (1857–1918), professor of mechanics at Kharkov University in Russia and a former student of P.L. Chebyshev, made a breakthrough in stability analysis when he presented his doctoral dissertation, "The General Problem of the Stability of Motion," in 1892 [26]. His work (Lyapunov stability theory) was seminal in control theory because his methods were applicable to nonlinear systems. His idea was a generalization of the concept that a system is stable if the total mechanical energy of it is decreasing. Lyapunov was aware of the work by Routh, but his own work remained largely unknown in the English-speaking world until after World War II [27].

With the development of electricity and various electric appliances in the early 20th century, telephones and electronic feedback amplifiers were invented in the United States, particularly at the Bell Telephone Laboratories [28], [29]. In relation to feedback amplifiers, the stability problem came into focus again at this time, and several graphical techniques for stability analysis were developed, for example, by Harry Nyquist (the Nyquist stability criterion, 1932) [30] and Hendrik Bode (Bode plot, gain margin, phase margin, 1940) [31], [32]. In this period, proportional-integral-derivative control was developed and analyzed mainly by Elmer Sperry (1910) [33], Nicolas Minorsky (1922) [34], John G. Ziegler, and Nathaniel B. Nichols (1942) [35]-[38]. Furthermore, the concepts of transfer function, block diagram, and servomechanism were defined clearly and used successfully for stability analysis at the Massachusetts Institute of Technology [2], [39], [40]. This graphical and symbolic representation of a complex dynamic system characterized the concept of feedback clearly and made it easy to understand complex connections between the component dynamics of the overall feedback control system.

During World War II, demands for military systems based on feedback control, such as automatic airplane pilots, radar control systems, and gun-positioning systems, provided a large impetus to the development of control theory and practice and resulted in significant growth of automatic control. Control engineering then became an independent discipline. In 1948, Walter Evans developed another graphical method for stability analysis, termed the root-locus method, using the transfer function of a feedback control system [41]–[44].

After World War II, stability conditions for a class of nonlinear systems, as shown in Figure 13, were developed mainly in the time domain and also in the frequency domain by Vasile M. Popov (the Popov criterion, 1961) [45] and George Zames (the circle criterion, 1966) [46]–[48]. The nonlinear system with linear time-invariant feedforward dynamics and a sector-bounded nonlinearity, as shown in Figure 13, was originally introduced from the "companding" problem in communications. Furthermore, Zames, Irwin W. Sandberg, and coworkers developed a small-gain theorem for a more general nonlinear system from an input–output stability point of view [49], [50].

Later, Rudolf E. Kalman and coworkers (1960) [51]–[53] analyzed the stability of control systems in state space using eigenvalues of system matrices. For a linear discrete-time system, Eliahu I. Jury developed the Jury stability criterion (1961), which is a method for determining stability by analyzing the coefficients of its characteristic equation. It is a discrete-time version of the Routh stability criterion [54]–[56]. More recently, Vladimir L. Kharitonov derived a stability condition for interval polynomials with real coefficients, such as (31) (Kharitonov's theorem, 1978) [57], and a stability condition for multivariate polynomials [58]

$$p(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n,$$
$$l_i \le a_i \le u_i, \quad a_0 \ne 0, \tag{31}$$

where l_i and u_i are the upper limit and lower limit, respectively, of the coefficients a_i, $i = 0, 1, 2, \ldots, n$.

CONCLUSION

Maxwell's paper "On Governors" was motivated by a hunting problem reported in industry during the Industrial Revolution. Maxwell 1) successfully analyzed, for the first time, the stability of a dynamic system by using a linear differential equation of motion, 2) obtained a specific stability condition $a_1 a_2 > a_3$ for a linear differential equation of the third order from the solutions of its characteristic equation $s^3 + a_1 s^2 + a_2 s + a_3 = 0, a_i > 0$, 3) introduced the linearization of a nonlinear differential equation, 4) classified the motion of a machine into four kinds in view of the stability of a linear system, and 5) proposed an unsolved problem to determine stability conditions for a linear differential equation of the nth order.

However, Maxwell's paper was incomprehensible because it lacked descriptions and figures. Thus, this article presents illustrative figures and free-body diagrams of the considered governors to aid in gaining a better understanding of Maxwell's seminal paper. Most of the equations in Maxwell's paper have been rederived using present-day stability terminology and comprehensible explanations. Furthermore, a proof of Maxwell's result is also presented. Finally, the history of further developments in the stability analysis of dynamic systems is briefly summarized.

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The Maximum Principle

The physical processes which take place in technology arc, as a rule, controllable, i.e., they can be realized by various means depending on the will of man. In this connection, there arises the question of finding the very best (in one sense or another) or, as is said, the optimal control of the process. For example, one can speak about optimality in the sense of rapidity of action, i.e., about achieving the aim of the process in the shortest time; about achieving this aim with a minimum expenditure of energy, etc. Mathematically formulated, these are problems in the calculus of variations, which in fact owes its origin to these problems. However, the solution of a whole range of variational problems, which are important in contemporary technology, is outside the classical calculus of variations... In its essential features, this solution is unified in one general mathematical method, which we call the maximum principle. It should be noted that all the fundamental necessary conditions in the classical calculus of variations (with ordinary derivatives) follow from the maximum principle.

—L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*, authorized translation from the Russian translator K. N. Trirogoff, Aerospace Corporation, El Segundo, California, Interscience Publishers, 1962.